MATH 1A MIDTERM 2 (8 AM VERSION) SOLUTION

(*Last edited November 15, 2013 at 8:38pm.*)

Problem 1. (i) *State Rolle's Theorem.*
(ii) *State the Mean Value Theorem.*
(iii) *Prove the Mean Value Theorem. You may assume Rolle's Theorem.*

Solution. (i) Let f be a function continuous on $[a, b]$ and differentiable on (a, b). Suppose $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

(ii) Let f be a function continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

(iii) Let f be a real-valued function that is continuous on $[a, b]$ and differentiable on (a, b). Define $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$. Note that g is a real-valued function that is continuous on $[a, b]$ and differentiable on (a, b). Note that $g(a) = g(b)$. Hence, by Rolle's Theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$. But $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. Hence $f'(c) = \frac{f(b)-f(a)}{b-a}$.

□

Problem 2. *In each of the following cases, evaluate* $\frac{dy}{dx}$.

(i) $y = \sin(x) \ln |x|$.
(ii) $y = \frac{x^2}{\cos x}$
(iii) $y = (10x^4 - 5)^{20}$
(iv) $y = \sin(x \sin x)$
(v) $y = \tan(x^2)e^{x^3}$
(vi) $y^4x - 3y = e^{2x}$ *(you should leave your answer in terms of y and x)*

Solution. (i) Use the product rule:

$$\frac{dy}{dx} = \cos(x) \ln |x| + \sin(x)\frac{1}{x} .$$

(A common mistake was to say that $\frac{d}{dx}(\ln |x|) = \frac{1}{|x|}$ instead of $\frac{1}{x}$. Notice that the domain of definition for the function $\sin(x) \ln |x|$ is $(-\infty, 0) \cup (0, \infty)$, so $\cos(x) \ln |x| + \sin(x)\frac{1}{x}$ and $\cos(x) \ln |x| + \sin(x)\frac{1}{|x|}$ are actually different functions on $(-\infty, 0) \cup (0, \infty)$. A sanity check that you can use is that the derivative of an odd function should be even, and vice versa.)

(ii) Use the quotient rule:

$$\frac{dy}{dx} = \frac{(\cos x)(2x) - (-\sin x)(x^2)}{(\cos x)^2} .$$

(iii) Use the chain rule and the power rule:

$$\frac{dy}{dx} = 20(10x^4 - 5)^{19}(40x^3) .$$

(iv) Use the chain rule and the product rule:

$$\frac{dy}{dx} = \cos(x \sin x) \cdot (\sin x + x \cos x)$$

(v) Use the product and chain rules:

$$\frac{dy}{dx} = \sec^2(x^2)(2x)e^{x^3} + \tan(x^2)e^{x^3}(3x^2)$$

(vi) Differentiate implicitly:

$$4y^3\frac{dy}{dx}x + y^4 - 3\frac{dy}{dx} = 2e^{2x} \tag{1}$$

which implies

$$\frac{dy}{dx}\cdot(4y^3x-3) = 2e^{2x} - y^4$$

so

$$\frac{dy}{dx} = \frac{2e^{2x}-y^4}{4y^3x-3}$$

(A common mistake was to forget the "x" hiding behind the "$\frac{dy}{dx}$" in the first term of (1), so that the final answer became $\frac{2e^{2x}-y^4}{4y^3-3}$ instead.)

□

Problem 3. *Showing your working carefully, calculate* $\frac{dy}{dx}$ *when* $y = x^{e^x}$. *Give your answer in terms of just* x.

Solution. We have $y = x^{e^x}$. Then $\ln(y(x)) = \ln(x^{e^x}) = e^x \ln x$. Differentiating both sides with respect to x gives $\frac{1}{y}\frac{dy}{dx} = e^x \ln x + e^x\frac{1}{x}$. Thus

$$\frac{dy}{dx} = x^{e^x}\left(e^x \ln x + e^x\frac{1}{x}\right).$$

□

Problem 4. *I start with* 5 *ties to wear on special occasions. My hair-dresser tells me that my separation anxiety is because I don't have enough nice clothes. Taking the advice to heart, I start shopping and my tie collection starts to grow exponentially. After* 3 *days I have* 12 *ties. How long will it be before I have* 100 *ties? You do not need to simplify or evaluate your answer.*

Solution. We have $T(t) = T(0)e^{kt}$ where k is constant. We know $T(0) = 3$. So $T(t) = 3e^{kt}$. Also, $T(3) = 12$. Hence $12 = T(3) = 3e^{k3}$, which implies $k = \frac{1}{3}\ln 4$. So $T(t) = 3e^{(\frac{1}{3}\ln 4)t}$. Let t_0 be the time at which I have 100 ties, i.e. $T(t_0) = 100$. Then $100 = T(t_0) = 3e^{(\frac{1}{3}\ln 4)t_0}$. Solving for t_0 gives $t_0 = \frac{3\ln(\frac{100}{3})}{\ln 4}$ days. □

Problem 5. *Which point on the graph of* $y = \sqrt{x}$ *is closest to the point* $(4, 0)$*?*

Solution. Let $D(x)$ be the distance between $(4, 0)$ and $(x, \sqrt{x})$. We have $(D(x))^2 = (x-4)^2 + (\sqrt{x}-0)^2 = (x-4)^2 + x$. Differentiating with respect to x gives

$$2D(x)D'(x) = 2(x-4)+1$$

so

$$D'(x) = \frac{2(x-4)+1}{\sqrt{(x-4)^2+x}}.$$

Suppose that $D(x)$ takes a minimum at $x = x_0$. Then $D'(x_0) = 0$. This requires $2(x_0 - 4) + 1 = 0$, or $x_0 = \frac{7}{2}$. We have

$$\begin{aligned} D''(x) &= \frac{2\sqrt{(x-4)^2+x} - (2(x-4)+1)\frac{1}{2\sqrt{(x-4)^2+x}}(2(x-4)+1)}{(\sqrt{(x-4)^2+x})^2} \\ &= \frac{4((x-4)^2+x) - (2(x-4)+1)^2}{2(\sqrt{(x-4)^2+x})^{3/2}} \\ &= \frac{17}{2(\sqrt{(x-4)^2+x})^{3/2}} \end{aligned}$$

which is positive at $x_0 = \frac{7}{2}$, so by the Second Derivative Test (page 295) we have that $D(x_0)$ is a local minimum. It's a global minimum since $D'(x) > 0$ on $(\frac{7}{2}, \infty)$ and $D'(x) < 0$ on $(0, \frac{7}{2})$. So the closest point is $(\frac{7}{2}, \sqrt{\frac{7}{2}})$. □

Problem 6. *A salt crystal is growing in a super-saturated solution of salt. It is a perfect cube and its length, width and height are all growing at a rate of* 1 *mm per day. What is the rate of increase of the volume of the cube when its length, width and height all equal* 10 *mm?*

Solution. Let $V(t)$ and $w(t)$ be the volume and width of the cube at time t, respectively. We have $V(t) = (w(t))^3$. So $V'(t) = 3(w(t))^2 w'(t)$, and $w'(t) = 1$ for all t. So when $w(t) = 10$, we have $V'(t) = 3(10^2)(1) = 300\ \text{mm}^3/\text{day}$. □

Problem 7. *Showing your working carefully, evaluate*

$$\lim_{x\to 0} \frac{\sin^2(x)}{4x^2}\ .$$

If you use a rule to change the limit, then you should name that rule each time you use it.

Solution. We have

$$\begin{aligned} \lim_{x\to 0} \frac{\sin^2(x)}{4x^2} &= \lim_{x\to 0} \frac{2(\sin x)(\cos x)}{8x} && \text{by L'Hospital's Rule} \\ &= \lim_{x\to 0} \frac{2\cos^2(x) - 2\sin^2(x)}{8} && \text{by L'Hospital's Rule} \\ &= \frac{1}{4}\ . \end{aligned}$$

□

Problem 8. *What is* $9^{\log_{49} 7}$*?*

Solution. We have $\log_{49} 7 = \frac{\ln 7}{\ln 49} = \frac{\ln 7}{2 \ln 7} = \frac{1}{2}$ so $9^{\log_{49} 7} = 9^{1/2} = \pm 3$. □