MATH 1A MIDTERM 1 (8 AM VERSION) SOLUTION

(*Last edited October 18, 2013 at 5:06pm.*)

Problem 1. (i) *State the Squeeze Theorem.*

(ii) *Prove the Squeeze Theorem.*

(iii) *Using a carefully justified application of the Squeeze Theorem, find*

$$\lim_{x\to 0} x^7 \sin\left(\frac{1}{x}\right).$$

Solution. (i) If f, g, h are real-valued functions such that

$$f(x) \le g(x) \le h(x) \qquad (1)$$

when x is near a (except possibly at a) and if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

The precise meaning of "near a (except possibly at a)" is: "there exists some $\delta_1 > 0$ such that (1) holds for all x satisfying $0 < |x-a| < \delta_1$".

(Remark: f, g, h do not need to be continuous.)

(ii) Suppose f, g, h satisfy (1) when $0 < |x-a| < \delta_1$ for some fixed $\delta_1 > 0$. Also suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$.

Let $\varepsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$, then $|f(x) - L| < \varepsilon$. Also, since $\lim_{x\to a} h(x) = L$, there exists $\delta_3 > 0$ such that if $0 < |x-a| < \delta_3$, then $|h(x) - L| < \varepsilon$.

Put $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Suppose $0 < |x-a| < \delta$. Then (1), $L - \varepsilon < f(x) < L + \varepsilon$ and $L - \varepsilon < h(x) < L + \varepsilon$ hold. Hence $g(x) \le h(x) < L + \varepsilon$. Also $L - \varepsilon < f(x) \le g(x)$. Thus $L - \varepsilon < g(x) < L + \varepsilon$, so $|g(x) - L| < \varepsilon$. Hence $\lim_{x\to a} g(x) = L$.

(iii) Set $f(x) = -|x^7|$, $g(x) = x^7 \sin\frac{1}{x}$, and $h(x) = |x^7|$. (Note that g is only defined for $x \ne 0$.) Note that

$$\left|\sin\frac{1}{x}\right| \le 1 \qquad (2)$$

for all $x \ne 0$. Multiplying both sides of (2) by $|x^7|$, we have

$$\left|x^7 \sin\frac{1}{x}\right| = |x^7| \cdot \left|\sin\frac{1}{x}\right| \le |x^7|$$

which implies

$$-|x^7| \le x^7 \sin\frac{1}{x} \le |x^7|$$

which is just (1). (Here we can take δ_1 to be any positive real number.) Note that $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$. Hence, by the Squeeze Theorem, $\lim_{x\to 0} g(x) = 0$.

(Remark: it was commonly claimed that

$$-x^7 \le x^7 \sin\frac{1}{x} \le x^7$$

for all $x \neq 0$, but it isn't true if $x < 0$; we have instead

$$x^7 \leq x^7 \sin \frac{1}{x} \leq -x^7$$

so the Squeeze Theorem does not apply.)

$\square$

Problem 2. *Evaluate*

$$\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \, .$$

You should show your reasoning carefully, however you may use any of the limit laws without explanation or proof.

Solution. We have

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} &= \lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \cdot \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1-x} + \sqrt{1+x}} \\
&= \lim_{x \to 0} \frac{(\sqrt{1+x} - \sqrt{1-x})(\sqrt{1+x} - \sqrt{1-x})}{x(\sqrt{1+x} - \sqrt{1-x})} \\
&= \lim_{x \to 0} \frac{(1+x) - (1-x)}{x(\sqrt{1+x} + \sqrt{1-x})} \\
&= \lim_{x \to 0} \frac{2x}{x(\sqrt{1+x} + \sqrt{1-x})} \\
&= \lim_{x \to 0} \frac{2}{\sqrt{1+x} + \sqrt{1-x}} \\
&= \frac{2}{\sqrt{1+0} + \sqrt{1-0}} \\
&= \frac{2}{1+1} \\
&= 1 \, .
\end{aligned}$$

$\square$

Problem 3. (i) *Let f be a real valued function, and let a and L be real numbers. What does it mean to say that $\lim_{x \to a} f(x) = L$?*

(ii) *Prove carefully, using the definition you gave in part (i), that*

$$\lim_{x \to 1} (x^3 + x^2 + x - 5) = -2 \, .$$

Solution. (i) The statement "$\lim_{x \to a} f(x) = L$" means "for every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$."

(ii) Let $\epsilon > 0$. Put $\delta = \min\{\frac{\epsilon}{1000}, \frac{1}{1000}\}$. Suppose $0 < |x - 1| < \delta$. Then $0 < |x - 1| < \frac{\epsilon}{1000}$ and $0 < |x - 1| < \frac{1}{1000}$. Thus we have

$$\begin{aligned}
|x^2 + 2x + 3| &= |(x-1)^2 + 4(x-1) + 6| \\
&= |x-1|^2 + 4|x-1| + 6 \quad \text{(by the triangle inequality)} \\
&= (\tfrac{1}{1000})^2 + 4(\tfrac{1}{1000}) + 6 \quad (\text{since } |x-1| < \tfrac{1}{1000}) \\
&< 7 \, .
\end{aligned}$$

So

$$\begin{aligned} |(x^3+x^2+x-5)-(-2)| &= |x^3+x^2+x-3| \\ &= |x-1|\cdot|x^2+2x+3| \\ &< \frac{\epsilon}{1000}\cdot 7 \\ &< \epsilon\,. \end{aligned}$$

Thus $\lim_{x\to 1}(x^3+x^2+x-5) = -2$.

□

Problem 4. (i) *State carefully the Intermediate Value Theorem.*

(ii) *Prove that there is a root of the equation*

$$2x^3 = 3^x$$

in the interval $(1,2)$.

Solution. (i) Let f be a continuous function on the interval $[a,b]$. Suppose $f(a) \neq f(b)$ and let N be a value strictly between $f(a)$ and $f(b)$. Then there exists some $c \in (a,b)$ such that $f(c) = N$.

(Remark: f needs to be continuous on the closed interval $[a,b]$, not just on the open interval (a,b), and we can guarantee that such c exists in the open interval (a,b), not just the closed interval $[a,b]$.)

(ii) Let $f(x) = 2x^3 - 3^x$. Since f is the sum of a polynomial and an exponential function, it is continuous on the interval $[1,2]$ (it is even continuous on the entire real line, but we only need it to be continuous on $[1,2]$). Observe that

$$\begin{aligned} f(1) &= 2(1)^3 - 3^1 = -1 \\ f(2) &= 2(2)^3 - 3^2 = 7\,. \end{aligned}$$

Set $N = 0$; it is strictly between $f(1)$ and $f(2)$. By the Intermediate Value Theorem, there exists $c \in (1,2)$ such that $f(c) = 0$. This implies that c is a solution to $2x^3 = 3^x$.

□

Problem 5. *The figure below shows the graph of* $y = f(x)$ *when*

$$f(x) = \begin{cases} \frac{5}{x+1} & \textit{if } x < -\pi \\ \tan x & \textit{if } -\pi \le x < 0 \textit{ but } x \neq -\frac{\pi}{2} \\ 0 & \textit{if } x = -\frac{\pi}{2} \\ x^2 & \textit{if } 0 \le x < 1 \\ 3 & \textit{if } 1 \le x < 4 \\ 1 & \textit{if } x = 4 \\ \frac{1}{(x-5)^2} + 2 & \textit{if } 4 < x < 7 \textit{ but } x \neq 5 \\ -2 & \textit{if } x = 5 \\ \ln x & \textit{if } x \ge 7 \end{cases}$$

For each of the following statements, indicate if it is true or false.

(i) $\lim_{x\to 4} f(x) = 3$

(ii) $\lim_{x\to 5^+} f(x) = \infty$
(iii) $\lim_{x\to 5} f(x) = \infty$
(iv) $\lim_{x\to -3} f(x)$ *exists*
(v) $\lim_{x\to -\frac{\pi}{2}^+} f(x) = \infty$
(vi) $\lim_{x\to \infty} f(x) = \infty$
(vii) *The graph* $y = f(x)$ *has a horizontal asymptote at* $y = 0$.
(viii) *The graph* $y = f(x)$ *has two horizontal asymptotes.*
(ix) *The graph* $y = f(x)$ *has two vertical asymptotes.*
(x) $f(x)$ *is continuous at* $x = 0$.
(xi) $f(x)$ *is continuous at* $x = 1$.
(xii) $f(x)$ *is continuous on the interval* $[1, 2]$.

Proof. (i) True. We have $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} 3 = 3$ and $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}(\frac{1}{(x-5)^2} + 2) = \frac{1}{(4-5)^2} + 2 = 3$. This implies $\lim_{x\to 4} f(x) = 3$. (Notice that the limit is still 3 even if $f(4) \neq 3$.)

(ii) True. We prove that "for every $N > 0$ there exists $\delta > 0$ such that if $0 < x - 5 < \delta$ then $f(x) > N$", which is what the statement "$\lim_{x\to 5^+} \frac{1}{(x-5)^2} + 2 = \infty$" means.

Let $N > 0$. Set $\delta = \min\{\frac{1}{\sqrt{N-2}}, 1\}$ if $N > 2$ and $\delta = 1$ otherwise. (Having $\delta \leq 1$ is nice because it ensures that $f(x) = \frac{1}{(x-5)^2} + 2$ for all x satisfying $0 < x - 5 < \delta$). Suppose $0 < x - 5 < \delta$. If $N > 2$, then $0 < x - 5 < \frac{1}{\sqrt{N-2}}$, so $N - 2 < \frac{1}{(x-5)^2}$, thus $N < \frac{1}{(x-5)^2} + 2 = f(x)$. If $N \leq 2$, then $0 < x - 5 < 1$, so $1 \leq \frac{1}{(x-5)^2}$; thus $N \leq 2 < 3 \leq \frac{1}{(x-5)^2} + 2 = f(x)$.

(iii) True. We prove that "for every $N > 0$ there exists $\delta > 0$ such that if $0 < |x-5| < \delta$ then $f(x) > N$", which is what the statement "$\lim_{x\to 5} \frac{1}{(x-5)^2} + 2 = \infty$" means.

Let $N > 0$. Set $\delta = \min\{\frac{1}{\sqrt{N-2}}, 1\}$ if $N > 2$ and $\delta = 1$ otherwise. Suppose $0 < |x - 5| < \delta$. If $N > 2$, then $0 < |x - 5| < \frac{1}{\sqrt{N-2}}$, so $N - 2 < \frac{1}{(x-5)^2}$, thus $N < \frac{1}{(x-5)^2} + 2 = f(x)$. If $N \leq 2$, then $0 < |x - 5| < 1$, so $1 \leq \frac{1}{(x-5)^2}$; thus $N \leq 2 < 3 \leq \frac{1}{(x-5)^2} + 2 = f(x)$.

(iv) True. Since $f(x) = \tan x$ on the interval $(-\pi, -\frac{\pi}{2})$, which contains -3, f is continuous at $x = -3$ and $f(-3) = \lim_{x\to -3} f(x)$ (by the definition of continuity).

(v) False. We actually have $\lim_{x\to -\frac{\pi}{2}^+} f(x) = -\infty$ and $\lim_{x\to -\frac{\pi}{2}^-} f(x) = \infty$.

(vi) True. Note that $f(x) = \ln x$ on $[7, \infty)$; for any positive $N > 0$, we have $\ln x > N$ for all $x > e^N$.

(vii) True. We interpret the statement "the graph $y = f(x)$ has a horizontal asymptote at $y = 0$" to mean "at least one of $\lim_{x\to -\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 0$ holds". In this case we have $\lim_{x\to -\infty} f(x) = 0$.

(viii) False. The graph $y = f(x)$ has exactly one horizontal asymptote, namely at $y = 0$. Although it might seem like there is a horizontal asymptote at $y \approx 3.4$, it is not the case since $\lim_{x\to\infty} f(x) = \infty$. The graph shows only a finite section of $\ln x$, which grows increasingly slowly with respect to x.

(ix) True. The two vertical asymptotes are at $x = -\frac{\pi}{2}$ and $x = 5$.

(x) True. We have $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \tan x = 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$, so $f(x)$ is continuous at $x = 0$.

(xi) False. We have $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 3 = 3$, and these two one-sided limits do not agree, so $f(x)$ is not continuous at $x = 1$.

(xii) True. Professor Coward has addressed this issue; his definition of continuity of f on an interval $[a, b]$ is: "f is continuous in the usual sense for each point $x \in (a, b)$, and $f(a) = \lim_{x\to a^+} f(x)$ and $f(b) = \lim_{x\to b^-} f(x)$". This is a weaker condition than the alternative definition, which requires additionally that $\lim_{x\to a^-} f(x)$ exists and is equal to $f(a)$ and $\lim_{x\to b^-} f(x)$ exists and is equal to

$f(b)$. Importantly, the Intermediate Value Theorem is still true for functions which satisfy the weaker condition.

□

Problem 6. *Using the limit definition of derivative, show that if* $f(x) = x^2$*, then* $f'(x) = 2x$*.*

Solution. Set $f(x) = x^2$. We have

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} 2x + h \\ &= 2x \ . \end{aligned}$$

Thus $f'(x) = 2x$. □

Problem 7. *What is* $\log_4 8$*?*

Solution. Let $x = \log_4 8$. Recall that the expression "$\log_4 8$" is defined to be the unique number such that $4^{\log_4 8} = 8$. Thus $4^x = 8$. Write $4 = 2^2$ and $8 = 2^3$ so that $2^{2x} = 2^3$. Thus $2x = 3$, so $x = \frac{3}{2}$. □