INTEGRATING RATIONAL FUNCTIONS

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This document gives a summary of how to find antiderivatives of rational functions. (You can also look at Section 7.4 in the textbook.) Recall that a *rational function* is a function of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials. We can always assume that the leading coefficient of $q(x)$ is 1, by dividing the numerator and denominator of $f(x)$ by the leading coefficient of $q(x)$.

If $q(x)$ is constant. If $q(x)$ is a (nonzero) constant, then $f(x)$ is a polynomial, and you can use the power rule (backwards).

Example 1. Find an antiderivative of $f(x) = \frac{x^2+3x-2}{3}$.

Solution. Write $f(x) = \frac{1}{3}x^2 + x - \frac{2}{3}$. So $\int f(x)\,dx = \frac{1}{9}x^3 + \frac{1}{2}x^2 - \frac{2}{3}x + C$, where C is a constant. □

If $q(x)$ is linear. If $q(x) = x - a$ for some a, then you can use polynomial long division to write $f(x)$ as the sum of a polynomial and a fraction $\frac{b}{x-a}$ where b is a constant. Then integrate the polynomial as above and integrate the fraction $\frac{b}{x-a}$ using the substitution $u = x - a$.

Example 2. Find an antiderivative of $f(x) = \frac{x^2-4x+1}{x-3}$.

Solution. Polynomial long division gives $x^2 - 4x + 1 = (x-3)(x-1) - 2$, hence $f(x) = (x-1) + \frac{-2}{x-3}$. We have $\int x - 1\,dx = \frac{1}{2}x^2 - x + C_1$ and $\int \frac{-2}{x-3}\,dx = -2\ln(x-a) + C_2$, where C_1, C_2 are constants. Thus $\int f(x)\,dx = \frac{1}{2}x^2 - x - 2\ln(x-a) + C$ where C is a constant. □

If $q(x)$ is quadratic. If $q(x) = x^2 + ax + b$ for some constants a, b, then you can use polynomial long division to write $f(x)$ as the sum of a polynomial and a fraction $\frac{cx+d}{x^2+ax+b}$ where c, d are constants. Integrate the polynomial as usual. We're left with the task of integrating the term $\frac{cx+d}{x^2+ax+b}$. The method depends on whether $x^2 + ax + b$ factors into linear terms or not. Recall that $x^2 + ax + b$ factors into linear terms (i.e. has a solution) if and only if its discriminant is nonnegative (i.e. $a^2 - 4b \geq 0$). If the discriminant is positive, then the two solutions of $x^2 + ax + b$ are distinct; if the discriminant is 0, then the two solutions of $x^2 + ax + b$ are the same and $x^2 + ax + b$ is a perfect square.

Suppose $a^2 - 4b > 0$. Then $x^2 + ax + b = (x - r_1)(x - r_2)$ where $r_1 \neq r_2$. You can use *partial fraction decomposition* to write

$$\frac{cx+d}{(x-r_1)(x-r_2)} = \frac{a_1}{x-r_1} + \frac{a_2}{x-r_2} \tag{1}$$

(i.e. there exist constants a_1, a_2 such that (1) is true for all x). How do we find a_1, a_2? Multiply both sides of (1) by $(x - r_1)(x - r_2)$ and rearrange terms to get

$$cx + d = (a_1 + a_2)x + (-a_1 r_2 - a_2 r_1)\ . \tag{2}$$

Equating coefficients of x and 1 in (2), we get

$$c = a_1 + a_2\ ;\ d = -a_1 r_2 - a_2 r_1 \tag{3}$$

which you can solve for a_1, a_2 (recall that c, d, r_1, r_2 are known).[1] Then you can integrate each term on the RHS of (1) using the substitutions $u = x - r_1$ and $u = x - r_2$, respectively.

Example 3. Find an antiderivative of $f(x) = \frac{2x-1}{x^2-3x+2}$.

Solution. The denominator factors as $x^2 - 3x + 2 = (x-1)(x-2)$. Write $\frac{2x-1}{(x-1)(x-2)} = \frac{a_1}{x-1} + \frac{a_2}{x-2}$ where a_1, a_2 are constants. Let's find a_1 and a_2. Multiplying both sides by $(x-1)(x-2)$ and rearranging terms gives $2x - 1 = (a_1 + a_2)x + (-2a_1 - a_2)$. This implies $a_1 + a_2 = 2$ and $-2a_1 - a_2 = -1$, which implies $a_1 = -1$ and $a_2 = 3$. Thus

$$\int \frac{2x-1}{x^2-3x+2}\, dx = \int \left(\frac{-1}{x-1} + \frac{3}{x-2} \right) dx = -\ln(x-1) + 3\ln(x-2) + C\,.$$

□

Suppose $a^2 - 4b = 0$. Then $x^2 + ax + b = (x-r)^2$ for some r. Then you can use polynomial long division to write

$$\frac{cx+d}{(x-r)^2} = \frac{a_1}{x-r} + \frac{a_2}{(x-r)^2}$$

and you can integrate $\frac{a_1}{x-r}$ by the substitution $u = x - r$ and $\frac{a_2}{(x-r)^2}$ by the power rule.

Example 4. Find an antiderivative of $f(x) = \frac{3x+1}{x^2-4x+4}$.

Solution. The denominator factors as $x^2 - 4x + 4 = (x-2)^2$. We have $3x + 1 = (x-2)(3) + 7$ so

$$\int \frac{3x+1}{x^2-4x+4}\, dx = \int \frac{3(x-2)+7}{(x-2)^2}\, dx = \int \left(\frac{3}{x-2} + \frac{7}{(x-2)^2} \right) dx = 3\ln(x-2) - \frac{7}{x-2} + C\,.$$

□

Suppose $a^2 - 4b < 0$. This is the most complicated case. Write

$$\frac{cx+d}{x^2+ax+b} = \frac{c}{2}\frac{2x+a}{x^2+ax+b} + \frac{d - \frac{ac}{2}}{x^2+ax+b}\,.$$

Integrate the term $\frac{c}{2}\frac{2x+a}{x^2+ax+b}$ using the substitution $u = x^2 + ax + b$. The second term $\frac{d-\frac{ac}{2}}{x^2+ax+b}$ needs more work. We can complete the square in $x^2 + ax + b$ to get

$$x^2 + ax + b = \left(x + \frac{a}{2}\right)^2 + b - \frac{a^2}{4}$$

where $b - \frac{a^2}{4} > 0$. Let's define three new constants

$$a_1 := -\frac{a}{2} \qquad a_2 := \sqrt{b - \frac{a^2}{4}} \qquad a_3 := d - \frac{ac}{2}$$

to simplify computations. Then

$$\frac{d - \frac{ac}{2}}{x^2+ax+b} = \frac{a_3}{(x-a_1)^2 + a_2^2} = \frac{\frac{a_3}{a_2^2}}{(\frac{x-a_1}{a_2})^2 + 1} = \frac{a_3}{a_2}\frac{\frac{1}{a_2}}{(\frac{x-a_1}{a_2})^2 + 1}$$

where you can integrate the last expression using the substitution $u = \frac{x-a_1}{a_2}$:

$$\int \frac{a_3}{a_2}\frac{\frac{1}{a_2}}{(\frac{x-a_1}{a_2})^2+1}\, dx = \int \frac{a_3}{a_2}\frac{u'}{u^2+1}\, dx \overset{(*)}{=} \int \frac{a_3}{a_2}\frac{1}{u^2+1}\, du = \frac{a_3}{a_2}\arctan(u) + C = \frac{a_3}{a_2}\arctan\left(\frac{x-a_1}{a_2}\right) + C$$

where the step marked $(*)$ uses the Substitution Rule.

[1]Notice that the solution for a_1, a_2 is not unique (or even may not exist) if $r_1 = r_2$.

If $q(x)$ has degree greater than 2. (Here it gets really nasty. I'm going to give you less detail than you might hope for. You shouldn't need to know this for the final.) If $q(x)$ has degree greater than 2, then it will factor as a product of linear and quadratic polynomials:

$$q(x) = (a_1(x))^{e_1} \cdots (a_m(x))^{e_m} (b_1(x))^{f_1} \cdots (b_n(x))^{f_n}$$

where the $a_i(x)$ are linear, the $b_i(x)$ are quadratic, and the e_i, f_i are positive integers. As above, you can use polynomial long division to assume that $p(x)$ has degree strictly less than that of $q(x)$. Then there exist polynomials $c_1(x), \ldots, c_m(x), d_1(x), \ldots, d_n(x)$ such that

$$\frac{p(x)}{q(x)} = \frac{c_1(x)}{(a_1(x))^{e_1}} + \ldots + \frac{c_m(x)}{(a_m(x))^{e_m}} + \frac{d_1(x)}{(b_1(x))^{f_1}} + \ldots + \frac{d_n(x)}{(b_n(x))^{f_n}}$$

and $c_i(x)$ has degree strictly less than that of $(a_i(x))^{e_i}$ for all $i = 1, \ldots, m$ and $d_i(x)$ has degree strictly less than that of $(b_i(x))^{f_i}$ for all $i = 1, \ldots, n$. Then you'll get a system of equations in the coefficients of the $c_i(x)$ and $d_i(x)$. Solve for the $c_i(x)$ and $d_i(x)$. Using polynomial long division, write each $\frac{c_i(x)}{(a_i(x))^{e_i}}$ in the form

$$\frac{c_i(x)}{(a_i(x))^{e_i}} = \frac{\text{(constant)}}{a_i(x)} + \ldots + \frac{\text{(constant)}}{(a_i(x))^{e_i}}$$

and integrate using the substitution $u = a_1(x)$ for the first term and the power rule for the rest. Use (a modified version of) polynomial long division to write each $\frac{d_n(x)}{(b_n(x))^{f_n}}$ as

$$\frac{d_i(x)}{(b_i(x))^{f_i}} = \frac{\text{(linear)}}{b_i(x)} + \ldots + \frac{\text{(linear)}}{(b_i(x))^{f_i}}$$

so it suffices to integrate terms like $\frac{cx+d}{(x^2+ax+b)^n}$. By a substitution

$$u = \frac{x - \frac{a}{2}}{\sqrt{b - \frac{a^2}{4}}} \ ,$$

we can reduce to the case $\frac{cx+d}{(x^2+1)^n}$. From here you can use the fact that

$$\left(\frac{1}{(x^2+1)^n}\right)' = \frac{-2nx}{(x^2+1)^n} \qquad \text{and} \qquad \left(\frac{x}{(x^2+1)^n}\right)' = \frac{-2n+1}{(x^2+1)^n} + \frac{2n}{(x^2+1)^{n+1}} \ .$$

Your final answer will involve linear combinations of terms of the form $\frac{1}{(x^2+1)^n}$, $\frac{x}{(x^2+1)^n}$ (for varying n), and $\arctan(x)$.