MATH 1A FINAL (7:00 PM VERSION) SOLUTION

(Last edited December 25, 2013 at 9:14pm.)

Problem 1. (i) *Give the precise definition of the definite integral using Riemann sums.*

(ii) *Write an expression for the definite integral* $\int_1^5 \frac{e^x}{e^x+3}\,dx$*, giving your answer as a limit and using right hand endpoints as sample points.*

(iii) *Calculate the integral* $\int_1^5 \frac{e^x}{e^x+3}\,dx$ *using any methods you like.*

Solution. (i) Let $f : [a,b] \to \mathbb{R}$ be a function. For each $n \in \mathbb{N}$, pick a collection of sample points $x_1^*, \ldots, x_n^*$ so that x_i^* lies in $[a + (i-1)\frac{b-a}{n}, a + i\frac{b-a}{n}]$. The definite integral of f from a to b is (defined as)

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\left(\frac{b-a}{n}\right)$$

provided the limit exists and gives the same value for all possible choices of sample point. (It is not assumed that f is continuous.)

(ii) The required expression is

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{e^{1+i(\frac{4}{n})}}{e^{1+i(\frac{4}{n})}+3} \cdot \frac{4}{n}.$$

(A common mistake was to write the Riemann sum for the case $n = 4$.)

(iii) Let $u = e^x + 3$. Then $\frac{du}{dx} = e^x$. Thus

$$\begin{aligned}\int_1^5 \frac{e^x}{e^x+3}\,dx &= \int_1^5 \frac{1}{u}\frac{du}{dx}\,dx \\ &= \int_{u(1)}^{u(5)} \frac{1}{u}\,du \qquad \text{by the Substitution Rule} \\ &= \ln(u(5)) - \ln(u(1)) \\ &= \ln\left(\frac{e^5+3}{e^1+3}\right).\end{aligned}$$

□

Problem 2. (i) *State carefully and in full the Fundamental Theorem of Calculus.*

(ii) *Suppose that a particle moves along the x-axis with velocity* $\sin(\frac{\pi t}{2})$*, where t is time in seconds. What is the net change in position between time $t = 0$ seconds and time $t = 1$ seconds?*

Solution. (i) Let f be a continuous real-valued function on $[a,b]$. Let $g : [a,b] \to \mathbb{R}$ be defined by $g(x) = \int_a^x f(t)\,dt$. Then g is continuous on $[a,b]$, differentiable on (a,b), and $g'(x) = f(x)$. Furthermore, if F is any antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

(ii) We have

$$\begin{aligned}\text{Net change in position} &= \int_{t=0}^{t=1} \sin\left(\frac{\pi t}{2}\right)\, dt \\ &= \left(-\frac{2}{\pi}\cos\left(\frac{\pi t}{2}\right)\right)_{t=0}^{t=1} \\ &= (0) - \left(-\frac{2}{\pi}\right) \\ &= \frac{2}{\pi}\ .\end{aligned}$$

□

Problem 3. (i) *Let f be a real-valued function, and let a and L be real numbers. What does it mean to say that $\lim_{x\to a^+} f(x) = L$?*

(ii) *State the domain of the function $f(x) = \sqrt[8]{x-15}$.*

(iii) *Prove, using the definition you gave in part (i), that*

$$\lim_{x\to 15^+} \sqrt[8]{x-15} = 0\ .$$

Solution. (i) For all $\epsilon > 0$, there exists $\delta > 0$ such that if $a < x < a + \delta$ then $|f(x) - L| < \epsilon$.

(ii) The domain is $[15, \infty)$.

(iii) Let $\epsilon > 0$. Put $\delta = \epsilon^8$. Suppose $15 < x < 15 + \delta$. Then $0 < x - 15 < \epsilon^8$, so $0 < \sqrt[8]{x-15} < \epsilon$. Thus $|\sqrt[8]{x-15} - 0| < \epsilon$. Thus $\lim_{x\to 15^+} \sqrt[8]{x-15} = 0$.

□

Problem 4. (i) *Let $y = y(t)$ be the number of bacterial cells growing in a culture, where t is time measured in days. Suppose the rate of growth of $y(t)$ is proportional to $y(t)$. That is, suppose that $y'(t) = Cy(t)$ where C is constant. Prove carefully that*

$$y(t) = y(0)e^{Ct}\ .$$

(ii) *You are given that at time $t = 0$ there are 3 cells in the culture. After 6 days there are 27 cells in the culture. How many days will it take for there to be 81 cells?*

Solution. (i) See Alexander's handouts.

(ii) Let $y(t)$ be the number of cells in the culture at time t. We have $y(t) = y(0)e^{Ct}$ for some constant C. We have $y(0) = 3$ and $y(6) = 27$. Thus $27 = 3e^{6C}$, which implies $C = \frac{1}{6}\ln 9$. Let t_0 be the time at which there are 81 cells in culture, i.e. $y(t_0) = 81$. Then $81 = 3e^{Ct_0}$ implies $t_0 = \frac{\ln 27}{C} = \frac{6\ln 27}{\ln 9} = 9$.

□

Problem 5. (i) *A bowl is made by rotating the region bounded by the curve $y = x^3$, the x-axis, and the line $x = 2$ about the y-axis. Find the volume of the bowl.*

(ii) *A football is made in the shape of $y = \sin x$, rotated about the x-axis between $x = 0$ and $x = \pi$. Find the volume of the football. (You may use the fact that $\cos(2\theta) = 1 - 2\sin^2\theta$ without proof.)*

Solution. (i) The volume of the bowl is

$$\int_0^2 (2\pi x)(x^3)\, dx = \int_0^2 2\pi x^4\, dx = \frac{2\pi}{5}(2^5 - 0^5) = \frac{64\pi}{5}\ .$$

(ii) The volume of the football is

$$\int_0^\pi \pi(\sin x)^2\,dx = \int_0^\pi \pi\left(\frac{1-\cos(2x)}{2}\right)\,dx$$
$$= \frac{\pi}{2}\left(x - \frac{1}{2}\sin(2x)\right)\Big|_0^\pi$$
$$= \frac{\pi^2}{2}\,.$$

□

Problem 6. *A ladder that is* 13 *meters long is leaning against a vertical wall and standing on horizontal ground. The bottom of the ladder slips. Assume that the top of the ladder stays in contact with the wall, and the bottom of the ladder stays in contact with the ground. Calculate the speed that the bottom of the ladder is moving away from the wall when the top of the ladder is* 5 *meters above the ground and moving downwards at* 3 *meters per second.*

Solution. Let $x(t)$ be the distance between the bottom of the ladder and the wall, and let $y(t)$ be the distance between the top of the ladder and the ground. We have

$$(x(t))^2 + (y(t))^2 = 13^2$$

for all t. Let t_0 be the time of interest. Differentiating with respect to time gives

$$2x(t)x'(t) + 2y(t)y'(t) = 0\,.$$

We have $y(t_0) = 5$ (which implies $x(t_0) = 12$) and $y'(t_0) = -3$. Thus $x'(t_0) = \frac{y(t_0)y'(t_0)}{x(t_0)} = \frac{5(-3)}{12} = -\frac{5}{4}$. □

Problem 7. (i) *Suppose that* $yx - \sin(x^4) = 5^x$. *Find* $\frac{dy}{dx}$. *You may leave your answer in terms of both* x *and* y.

(ii) *Suppose* $y = x^{5x}$. *Find* $\frac{dy}{dx}$.

Solution. (i) Differentiating implicitly gives

$$y'x + y - \cos(x^4)(4x^3) = 5^x \ln 5$$

and solving for y' gives

$$y' = \frac{5^x \ln 5 + 4x^3\cos(x^4) - y}{x}\,.$$

(ii) Taking natural logarithm of both sides gives $\ln(y) = 5x\ln x$. Differentiating with respect to x gives $\frac{1}{y}y' = 5\ln x + 5x(\frac{1}{x})$. Thus $y' = x^{5x}(5\ln x + 5)$.

□

Problem 8. *Calculate the following limits:*

(i) $\lim_{x\to\infty} \frac{x}{e^{2x}}$

(ii) $\lim_{x\to 0} \frac{1-\cos(3x)}{x^2}$

(iii) $\lim_{x\to\infty} \frac{3x^2-7x+3}{5x^2-14}$

If you use any special rules to calculate your answers, you should state the rule each time you use it.

Solution. (i) We have

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{e^{2x}} &= \lim_{x\to\infty}\frac{1}{2e^{2x}} \qquad \text{by L'Hospital's Rule}\\ &= 0\,.\end{aligned}$$

(ii) We have

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos(3x)}{x^2} &= \lim_{x\to 0}\frac{3\sin(3x)}{2x} \qquad \text{by L'Hospital's Rule}\\ &= \lim_{x\to 0}\frac{9\cos(3x)}{2} \qquad \text{by L'Hospital's Rule}\\ &= \frac{9\cos(3\cdot 0)}{2}\\ &= \frac{9}{2}\,.\end{aligned}$$

(iii) We have

$$\begin{aligned}\lim_{x\to\infty}\frac{3x^2-7x+3}{5x^2-14} &= \lim_{x\to\infty}\frac{6x-7}{10x} \qquad \text{by L'Hospital's Rule}\\ &= \lim_{x\to\infty}\frac{6}{10} \qquad \text{by L'Hospital's Rule}\\ &= \frac{3}{5}\,.\end{aligned}$$

□

Problem 9. *Consider the graph of* $y = x^3 - 3x^2$.

(i) *Give the coordinates of the* x*-intercepts.*
(ii) *Give the coordinates of all critical points, and indicate for each if it is a local maximum or local minimum.*
(iii) *Give the coordinates of any points of inflection.*
(iv) *The graph passes through the point* $(-1, -4)$. *Describe the concavity of the graph at this point.*

Solution. (i) The x-intercepts are the points on the graph whose y-coordinates are 0. If $(x, 0)$ is an x-intercept, then $0 = x^3 - 3x^2$ implies that $x = 0$ or $x = 3$. Also, $(3, 0)$ and $(0, 0)$ are points on the graph. Thus $(3, 0)$ and $(0, 0)$ are x-intercepts.
(ii) The critical points of the graph are where $y' = 0$; if (x, y) is a critical point, then $3x^2 - 6x = 0$. This can only happen when $x = 2$ or $x = 0$. Thus the critical points are $(2, -4)$ and $(0, 0)$.
(iii) The point (x, y) is a point of inflection if $y'' = 0$. We have $y'' = 6x$, so $(0, 0)$ is the only point of inflection.
(iv) We have $y''(-1) = -6$, so the graph is concave down.

□

Problem 10. *Let* f *and* g *be real-valued functions. Suppose that* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = K$. *Prove that* $\lim_{x\to a}(f(x) + g(x)) = L + K$.

Solution. Let f and g be functions, and suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$. Let $\epsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \frac{\epsilon}{2}$. Also, since $\lim_{x\to a} g(x) = K$, there exists $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then $|g(x) - K| < \frac{\epsilon}{2}$. Put $\delta = \min\{\delta_1, \delta_2\}$.

Suppose $0 < |x - a| < \delta$. Then $|f(x) - L| < \frac{\epsilon}{2}$ and $|g(x) - K| < \frac{\epsilon}{2}$. Hence

$$\begin{aligned} |(f(x) + g(x)) - (L + K)| &= |(f(x) - L) + (g(x) - K)| \\ &\leq |f(x) - L| + |g(x) - K| \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \ . \end{aligned}$$

Hence $\lim_{x \to a}(f(x) + g(x)) = L + K$. $\square$