SOLVING THE DIFFERENTIAL EQUATION $y' = Cy$

MINSEON SHIN

(*Last edited December 13, 2013 at 2:13pm.*)

Proposition 1. *Let C be a constant. Suppose there exists a differentiable function $y : \mathbb{R} \to \mathbb{R}$ such that*

$$\frac{dy}{dt} = Cy \ . \tag{1}$$

Then $y(t) = y(0)e^{Ct}$.[1]

Proof. We'll prove that $y(t) = y(0)e^{Ct}$. We have

$$\frac{dy}{dt} - Cy = 0 \ . \tag{2}$$

Multiplying (2) by e^{-Ct} gives[2]

$$e^{-Ct} \cdot \frac{dy}{dt} - Ce^{-Ct}y = 0 \ . \tag{3}$$

Note that (applying the product rule backwards)

$$e^{-Ct} \cdot \frac{dy}{dt} - Ce^{-Ct}y = \frac{d}{dt}(ye^{-Ct}) \ . \tag{4}$$

Combining (3) and (4), we have that ye^{-Ct} is a function whose derivative is 0 everywhere. Then Lemma 2 implies that ye^{-Ct} is a constant.[3] Thus there exists a constant k such that $y(t) \cdot e^{-Ct} = k$ for all t. Multiplying both sides by e^{Ct} gives

$$y(t) = ke^{Ct} \ .$$

Now we solve for k: substituting $t = 0$ into (5) gives $y(0) = ke^{C \cdot 0}$, which means $k = y(0)$. Thus

$$y(t) = y(0)e^{Ct} \ . \tag{5}$$

□

Date: Fri, Dec 13, 2013.

[1] Notice that (1) is an equality of functions: the LHS is the derivative of y (namely, the value of the function $\frac{dy}{dt}$ at t is the number $\lim_{h \to 0} \frac{y(t+h)-y(h)}{h}$), and the RHS is a constant multiple of y.

[2] This trick may seem unmotivated. Why do we do this? This works because multiplying by e^{-Ct} puts the LHS of (3) into a form to which we can apply the product rule backwards. This is called the "integrating factor" trick. You might ask, "can we solve all differential equations using this method?" The answer is "in general, no." For Math 1A, you'll only have to know how to solve (1), which has a very special, simple form. Take a course on differential equations if you're interested in learning how to solve other kinds of differential equations.

[3] This makes sense intuitively (the slope of the function is zero everywhere) but we should prove it.

Another argument. You may have seen a trick called "separation of variables" before. We rearrange (1) to get

$$\frac{1}{y} \cdot \frac{dy}{dt} = C \ .$$

Fix two real numbers $a < b$; then we have[4]

$$\int_a^b \frac{1}{y} \cdot \frac{dy}{dt} \ dt = \int_a^b C \ dt \ . \tag{6}$$

By the Substitution Rule (substituting $u = y$), we have

$$\int_{y(a)}^{y(b)} \frac{1}{u} \ du = \int_a^b \frac{1}{y} \cdot \frac{dy}{dt} \ dt \ . \tag{7}$$

Combining (6) and (7) gives

$$\ln(y(b)) - \ln(y(a)) = C(b - a)$$

so

$$y(b) = y(a)e^{C(b-a)} \ .$$

In particular, this is true when $b = t$, a variable, and $a = 0$, so we have

$$y(t) = y(0)e^{Ct} \ .$$

□

Lemma 2. *Let $f : [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f'(x) = 0$ for all $x \in (a, b)$. Then f is constant, i.e. $f(x) = f(y)$ for any $x, y \in [a, b]$.*

Proof. Let $x, y \in [a, b]$ be two real numbers such that $x < y$. Note that f is continuous on $[x, y]$ and differentiable on (x, y). Thus, by the Mean Value Theorem, there exists $c \in (x, y)$ such that $f'(c) = \frac{f(y)-f(x)}{y-x}$. Since $f'(c) = 0$, we must have $f(y) - f(x) = 0$, or $f(x) = f(y)$. □

[4]This says that the area under the graph of the function $\frac{1}{y} \cdot \frac{dy}{dt}$ between $t = a$ and $t = b$ is the same as the area under the graph of the constant function C between $t = a$ and $t = b$.