Math 126 Homework hints

Note: As I’m sure you’ve all seen in section, I make mistakes too! I can’t guarantee that everything here will be right 100% of the time. If you think there’s an error in something, please e-mail me and I’ll try to get a correction up. Same goes for any clarification or further questions; if I’ve made something even more confusing, please e-mail or come to office hours to clear it up!

Taylor Notes Section 5

Problem 8: Define a function $g(x)$ by

$$g(x) = \begin{cases} \sin(x) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Find the Taylor series, write out the first three non-zero terms, and state where it converges. Do the same for $f(x) = \int_0^x g(t) \, dt$.

We already know the Taylor series for $\sin(x)$:

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

So, to get the Taylor series for $g(x)$ away from the point $x = 0$, we just divide by $x$:

$$g(x) = \frac{\sin(x)}{x} = \frac{1}{x} \left( \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \right) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!}$$

A quick check tells us that this is 1 at $x = 0$, so this is indeed the Taylor series for $g(x)$. Writing out the first three non-zero terms amounts to plugging in values of $k$ until you get three non-zero terms. As for the interval of convergence, what was the interval of convergence for the Taylor series for $\sin(x)$? We should expect this one to converge on the same interval, right?

To find the Taylor series for $f(x)$, use the standard trick of interchanging the order of summation and integration:

$$f(x) = \int_0^x g(t) \, dt = \int_0^x \left( \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k+1)!} \right) \, dt = \sum_{k=0}^{\infty} \left( \int_0^x \frac{(-1)^k t^{2k}}{(2k+1)!} \, dt \right)$$

Once you integrate these terms, you’ve got your series. Finding the non-zero terms and the interval of convergence is the same as in the last problem.