

Uniform Quadratic Growth and Second-Order Variational Curvature in Hilbert Spaces

Samir Adly*

R. Tyrrell Rockafellar[†]

Abstract. In second-order variational analysis, a fundamental property of interest is the level of quadratic growth of a function at a point relative to sublinearization for a subgradient at that point. Uniformity in a primal-dual sense is captured by a modulus which, in finite dimensions, can be expressed by a second-subderivative formula. But what about infinite dimensions? The answer, supplied here, is that the formula can fail even for a very nice function on an infinite-dimensional Hilbert space, reducing only to an inequality. However, equality can be restored under an extra condition of directional compactness that is brought to light. This condition is automatic in finite dimensions. For an L^1 -penalized Dirichlet energy on $L^2(\Omega)$, it holds at every point of the subgradient graph. If the multiplier is globally saturated and has the same sign as the state, the modulus coincides with the lowest eigenvalue of the associated elliptic operator.

Keywords. Modulus of uniform quadratic growth; strict second subderivative; directional compactness; Hilbert spaces; L^1 penalization; Dirichlet energy; lowest eigenvalue.

AMS 2020 Subject Classification. 49J52; 49J53; 49J50; 90C26; 90C30.

Contents

1	Introduction	2
2	Basics	4
3	From the strong strict second subderivative to the basic inequality	6
3.1	Strong strict second subderivative	6
3.2	The basic inequality	7
4	A counterexample in ℓ^2: the reverse inequality fails for $d_{*,s}^2$	8
4.1	The example on ℓ^2	8
4.2	Computation of $\text{uqgf}(0 \mid 0)$	10
4.3	Computation of $d_{*,s}^2 f(0 \mid 0)(\xi)$	10
4.4	Conclusion: strict gap between uqgf and $\inf d_{*,s}^2$	11
5	A compactness condition restoring the growth-subderivative formula	11
5.1	Directional compactness of approximate minimizing directions	12
5.2	Equality under directional compactness	13
5.3	Quadratic perturbations: attainment and directional compactness	14

*Laboratoire XLIM, Université de Limoges, 123 avenue Albert Thomas, 87060 Limoges CEDEX, France. Email: samir.adly@unilim.fr, https://www.unilim.fr/pages_perso/samir.adly/.

[†]University of Washington, Department of Mathematics, Box 354350, Seattle, WA 98195-4350. Email: rtr@uw.edu, <https://sites.math.washington.edu/~rtr/mypage.html>.

6	Smooth perturbations and a nonsmooth elliptic application	15
6.1	An addition rule for the strong strict second subderivative	15
6.2	An L^1 -penalized Dirichlet energy on $L^2(\Omega)$	16
6.3	Directional compactness and representation	18
6.4	The strong strict second subderivative under saturation	18
6.5	The modulus of uniform quadratic growth and a nonsmooth example	20
7	A sequential weak-direction relaxation and its limitations	21
7.1	Definition and two opposite failures	21
8	Conclusion and perspectives	23

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and the associated norm $\| \cdot \|$. Let f be a lower-semicontinuous, extended-real-valued function on $\mathcal{H}$ that is proper: the effective domain is $\text{dom } f = \{ x \mid f(x) < \infty \}$, and f is finite on it. First-order variational analysis of f revolves around its subgradients $v \in \partial f(x)$, the definition of which will be reviewed below. Second-order variational analysis looks at, among other things, how f deviates from its “sublinearizations” given by those subgradients and local uniformities of that behavior relative to the graph of the subdifferential mapping ∂f , which is the set

$$\text{gph } \partial f = \{ (x, v) \in \mathcal{H} \times \mathcal{H} \mid v \in \partial f(x) \}. \quad (1.1)$$

This paper explores how far the theory already developed about this in finite dimensions might extend to infinite dimensions.

The key property is local *quadratic growth* of f at level s relative to $\bar{v} \in \partial f(\bar{x})$. It refers to having the difference between $f(x)$ and its sublinearization $f(\bar{x}) + \langle \bar{v}, x - \bar{x} \rangle$ be at least $\frac{s}{2} \|x - \bar{x}\|^2$ on some f -attentive neighborhood of $\bar{x}$, that is, on a set of the form

$$X_f^\alpha = \{ x \in X \mid f(x) < \alpha \} \quad (1.2)$$

for some ordinary neighborhood X of $\bar{x}$ and some $\alpha > f(\bar{x})$. Local quadratic growth is relevant to local optimality at $\bar{x}$ when $\bar{v} = 0$. For algorithms that involve nearby points and subgradients, a uniform version of the property is also useful. *Uniform* quadratic growth of f at level s locally relative to $\bar{v} \in \partial f(\bar{x})$ refers to the existence of both an f -attentive neighborhood X_f^α of $\bar{x}$ as in (1.2) and an ordinary neighborhood V of $\bar{v}$ such that

$$\begin{aligned} f(x') &\geq f(x) + \langle v, x' - x \rangle + \frac{s}{2} \|x' - x\|^2 \text{ for all } x' \in X, \\ &\text{when } v \in \partial f(x) \text{ with } x \in X_f^\alpha \text{ and } v \in V. \end{aligned} \quad (1.3)$$

The level s may be negative. For fixed neighborhoods of $\bar{x}$ and $\bar{v}$, the available s levels form an interval that is unbounded below when it is nonempty. Shrinking the neighborhoods can only increase the upper bound of that interval. This leads to the following definition of the *modulus of uniform local quadratic growth*:

$$\begin{aligned} \text{uqgf}(\bar{x} \mid \bar{v}) &:= \sup \left\{ s \in \mathbb{R} \mid \exists \alpha > f(\bar{x}) \text{ and open } X \times V \ni (\bar{x}, \bar{v}) \text{ such that} \right. \\ &\quad \left. f(x') \geq f(x) + \langle v, x' - x \rangle + \frac{s}{2} \|x' - x\|^2 \text{ for all } x' \in X, \right. \\ &\quad \left. \text{whenever } (x, v) \in (X_f^\alpha \times V) \cap \text{gph } \partial f \right\}. \end{aligned} \quad (1.4)$$

The value of the modulus can clearly be finite or ∞ (as when f is the indicator of a point, say), but also $-\infty$, that being the convention when a supremum is taken over the empty set.

A closely related property is the *prox-regularity* of f at level r relative to $\bar{v} \in \partial f(\bar{x})$, which is expressed by (1.3) in replacing s by $-r$. It has long been recognized as essential in second-order variational analysis, although its partnership with uniform quadratic growth has been obscured through an unnecessary fixation on having $r > 0$ (thus negative growth, $s < 0$). A modulus can likewise be introduced for prox-regularity, but it comes out to be $-\text{uqgf}(\bar{x} | \bar{v})$. Achieving a good understanding of the modulus in (1.4) therefore can be valuable not only for applications in optimization methodology, but also in ascertaining whether prox-regularity is present, and if so, at what level. Much about this is already known about this in finite dimensions, some which carries over to infinite dimensions, as we are about to describe.

Our spotlight will be on the relationship between uniform quadratic growth and the *strict second subderivative* function $d_*^2 f(\bar{x} | \bar{v})$. This is defined by taking lower epilimits of second-order difference quotient functions

$$\Delta_\tau^2 f(x | v)(\xi) := \frac{2}{\tau^2} [f(x + \tau\xi) - f(x) - \tau\langle v, \xi \rangle] \quad (1.5)$$

at nearby $v \in \partial f(x)$, namely¹

$$d_*^2 f(\bar{x} | \bar{v})(\bar{\xi}) := \liminf \Delta_\tau^2 f(x | v)(\xi) \text{ with respect to having} \quad (1.6)$$

$$\tau \searrow 0, \xi \rightarrow \bar{\xi}, v \rightarrow \bar{v}, x \rightarrow \bar{x} \text{ with } f(x) \rightarrow f(\bar{x}), (x, v) \in \text{gph } \partial f.$$

This contrasts with the (non-strict) second subderivative function $d^2 f(\bar{x} | \bar{v})$ defined with (x, v) fixed at $(\bar{x}, \bar{v})$ and just the limits in τ and ξ . It was proved in [20] that

$$\text{uqgf}(\bar{x} | \bar{v}) = \inf_{\|\xi\|=1} d_*^2 f(\bar{x} | \bar{v})(\xi) \text{ in finite dimensions,} \quad (1.7)$$

but this has been an open question in infinite dimensions.

The equation in question, which we'll refer to as the *growth-subderivative formula* for short, is part of a broader picture in [20] in which characterizations are presented for the *modulus of convexity* $\text{cnvf}(\bar{x} | \bar{v})$. The latter was introduced in [19], where it was shown that the modulus of uniform growth, the modulus of variational convexity and the modulus of local subdifferential monotonicity coincided and could be united under a single name - in finite dimensions. A formula for any one of them would automatically be a formula for all. An assumption that $\text{uqgf}(\bar{x} | \bar{v}) > -\infty$, meaning that prox-regularity is surely on hand, was prerequisite in [19] for the growth-subderivative formula but obviated in [20]. Other research in finite dimensions connected with characterizing variational convexity and local subdifferential monotonicity, earlier in [18] and later in [9], [6] and [10] was complementary to our concerns here, being not involved with strict second subderivatives, although looking also for ties to Moreau envelopes. The behavior of the variational modulus of convexity under Moreau envelopes has been investigated in finite dimensions in [2], where an exact transmission rule describes its rational attenuation under Moreau smoothing and its recovery as the smoothing parameter tends to zero.

For infinite-dimensional spaces, prox-regularity and related second-order stability properties have been studied in [3, 15]. More recently, relationships among variational convexity, strong variational convexity, local subdifferential monotonicity, and quadratic growth in have been developed by Khanh, Khoa, Mordukhovich, and Phat in [11]. Their results are comprehensive enough to allow the notion of the convexity modulus, as the common name for the uniform growth modulus, the variational convexity modulus and the subdifferential monotonicity modulus, to pass to infinite dimensions for Hilbert spaces, but only under the added assumptions that f is weakly sequentially lower semicontinuous and already is known to be prox-regular. Also in that vein, [5] gives graphical-derivative characterizations of strong local maximal monotonicity

¹was considered by Hang and Sarabi in [8] in taking a lower epilimit, but the function obtained by taking a full epilimit was introduced long ago in [17, Section 6] as the strict second-order epi-derivative.

and, under sequential weak lower semicontinuity, of variational convexity. But the strict second subderivatives in the formula we are targeting didn't enter these infinite-dimensional efforts.

Returning briefly to finite dimensions for more background on strict second subderivatives, we note that, prior to their deployment in [20], a calculus was developed for them by Hang and Sarabi in [7, Proposition 4.1] and Ouyang and Milzarek in [16, Theorems 3.8 and 3.20]. The strict positivity of $d_*^2 f(\bar{x} | \bar{v})(\xi)$ in every nonzero ξ as shown in [8, Proposition 3.3] as being equivalent to uniform quadratic growth localized at *some* positive level without identifying the exact modulus value, as was achieved in [20]. The proof of that nontrivial implication normalized the ξ -vectors violating the growth and used the compactness of the unit sphere to extract a norm-convergent subsequence, which capsulizes the obstacle faced in contemplating extensions to infinite-dimensional $\mathcal{H}$, where the unit sphere lacks compactness.

The direct norm-topology constructions that we will utilize in coping with this differ from previous constructions in infinite-dimensional variational analysis, such as non-strict second-order epi-derivatives along with Mosco or proto-derivatives in Hilbert spaces in [12, 13].

Our main achievements are a counterexample showing that the growth-subderivative formula can indeed fail in infinite dimensions and, on the other hand, a sufficient condition of directional compactness that restores the equation from the inequality " $\leq$ " that it would otherwise naturally revert to. To the best of our knowledge, this is the first such negative example. The workability of the sufficient condition, which is not claimed to be necessary, is demonstrated by a positive example. In a PDE application formulated on the ambient space $L^2(\Omega)$, the Dirichlet energy is extended by ∞ outside $H_0^1(\Omega)$ and is combined with an L^1 term. Directional compactness and the growth-subderivative formula hold at every point of its subgradient graph. When the multiplier is globally saturated and has the same sign as the state, we compute the strong strict second subderivative and identify the modulus of uniform quadratic growth with the lowest eigenvalue of a Dirichlet-Schrödinger operator. The functional in this application is genuinely nondifferentiable at states whose zero set contains a nonempty open subset. The explicit computation is targeted at this saturation condition.

The paper is organized as follows. Section 2 recalls the subdifferential notions and gives a localized quotient characterization of the modulus. Section 3 brings out the basic inequality for the strong strict second subderivative. Section 4 constructs in $\ell^2(\mathbb{N})$ a counterexample to equality. Section 5 introduces the directional compactness condition and proves that the growth-subderivative formula holds under it. Section 6 establishes a smooth addition rule and applies the growth-subderivative formula to the L^1 -penalized Dirichlet energy. Section 7 studies a sequential weak-direction relaxation and shows through two examples why it's a dead end that doesn't recover the unit-sphere formula.

2 Basics

We now specify the notations and variational objects used in the introduction. Throughout, we assume that $\dim \mathcal{H} \geq 1$, which would exclude the trivial case $\mathcal{H} = \{0\}$. All neighborhoods of $\mathcal{H}$ are understood for the strong topology of the norm, unless otherwise explicitly stated. We denote by $\mathbb{B}(a, r)$ the open ball centered at a and with radius $r > 0$. We write ' $\rightarrow$ ' and ' $\rightharpoonup$ ' for strong and weak convergence, respectively, in $\mathcal{H}$. The unit sphere of H is denoted by

$$\mathbb{S}_{\mathcal{H}} := \{\xi \in \mathcal{H} : \|\xi\| = 1\}.$$

In what follows, $\tau_k \downarrow 0$ means that (τ_k) is a decreasing sequence of positive numbers converging to zero. For the infimum and supremum on sets that may be empty, we will use the following standard conventions: $\inf \emptyset := \infty$ and $\sup \emptyset := -\infty$. Inequalities involving extended real-valued functions are always understood in the sense of extended real numbers.

We recall that the regular (or Fréchet) subdifferential of f at $\bar{x} \in \text{dom } f$ is defined by the

following formula

$$\widehat{\partial}f(\bar{x}) := \left\{ v \in \mathcal{H} : \liminf_{\substack{x \rightarrow \bar{x} \\ x \neq \bar{x}}} \frac{f(x) - f(\bar{x}) - \langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

For $\bar{x} \in \text{dom } f$, we write $x_k \xrightarrow{f} \bar{x}$ when $x_k \rightarrow \bar{x}$ and $f(x_k) \rightarrow f(\bar{x})$. Since $\mathcal{H}$ is a real Hilbert space, it is also a reflexive space and hence an Asplund space, so the sequential representation of the basic limiting subdifferential of the proper lower semicontinuous function f reads [14, Theorem 2.34(b)]

$$\partial f(\bar{x}) = \left\{ v \in \mathcal{H} : \exists x_k \xrightarrow{f} \bar{x}, \exists v_k \in \widehat{\partial}f(x_k) \text{ such that } v_k \rightharpoonup v \right\} s$$

Under the Riesz identification of $\mathcal{H}^*$ with $\mathcal{H}$, the dual weak-star convergence in the Banach-space formulation is precisely weak convergence in $\mathcal{H}$. Equivalently,

$$\partial f(\bar{x}) = \text{Limsup}_{x \xrightarrow{f} \bar{x}} \widehat{\partial}f(x),$$

where Limsup denotes the Painlevé-Kuratowski outer limit for the norm topology in the primal variable and the weak topology in the subgradient variable. The weak topology is used only in this limit construction, and all neighborhoods of the subgradients in the local uniform estimates below are norm-neighborhoods.

The formulation above is the precise version of the description in the introduction. We can impose equivalently in (1.3) the additional condition $\|x' - x\| \leq \beta$ for some $\beta > 0$. This follows by shrinking the neighborhood X of $\bar{x}$.

In the whole paper the symbol ∞ designates $+\infty$.

The second-order difference quotients defined in (1.5) provide a localized representation of the modulus of uniform quadratic growth.

Proposition 2.1. *Let $(\bar{x}, \bar{v}) \in \text{gph } \partial f$ with $f(\bar{x}) \in \mathbb{R}$. Then*

$$\text{uqgf}(\bar{x}|\bar{v}) = \sup_{\alpha > f(\bar{x})} \sup_{\substack{X \ni \bar{x}, V \ni \bar{v} \\ X, V \text{ open}, \beta > 0}} \inf_{\substack{(x, v) \in (X_f^\alpha \times V) \cap \text{gph } \partial f \\ 0 < \tau < \beta, \|\xi\|=1, x + \tau\xi \in X}} \Delta_\tau^2 f(x|v)(\xi). \quad (2.1)$$

Proof. Let us denote the right side of (2.1) by $\Lambda(\bar{x}, \bar{v}) \in [-\infty, \infty]$. We first prove that

$$\text{uqgf}(\bar{x}|\bar{v}) \leq \Lambda(\bar{x}, \bar{v}). \quad (2.2)$$

If no level is admissible in the definition of $\text{uqgf}(\bar{x}|\bar{v})$, then $\text{uqgf}(\bar{x}|\bar{v}) = -\infty$, and the inequality follows. Otherwise, let s be an admissible level. Choose $\alpha > f(\bar{x})$ and open neighborhoods $X \ni \bar{x}$ and $V \ni \bar{v}$ for which (1.3) holds, and fix any $\beta > 0$. Every tuple entering the corresponding localized infimum then satisfies

$$\Delta_\tau^2 f(x|v)(\xi) \geq s.$$

It follows that $\Lambda(\bar{x}, \bar{v}) \geq s$. Taking the supremum over all admissible levels gives (2.2).

We now prove the reverse inequality. If $\Lambda(\bar{x}, \bar{v}) = -\infty$, then $\text{uqgf}(\bar{x}|\bar{v}) \geq \Lambda(\bar{x}, \bar{v})$ holds automatically. Suppose now that $\Lambda(\bar{x}, \bar{v}) > -\infty$, and fix any $s \in \mathbb{R}$ such that $s < \Lambda(\bar{x}, \bar{v})$. There exist $\alpha > f(\bar{x})$, open neighborhoods $X \ni \bar{x}$ and $V \ni \bar{v}$, and $\beta > 0$ such that the corresponding localized infimum in (2.1) is strictly greater than s . Let us choose an open neighborhood $\widehat{X} \ni \bar{x}$ contained in X and having diameter less than β . Let

$$(x, v) \in (\widehat{X}_f^\alpha \times V) \cap \text{gph } \partial f \quad \text{and} \quad x' \in \widehat{X}.$$

If $x' \neq x$, set

$$\tau := \|x' - x\|, \quad \xi := \frac{x' - x}{\tau}.$$

Then $0 < \tau < \beta$, $\|\xi\| = 1$, and $x + \tau\xi = x' \in X$. Hence these variables enter the localized infimum, and

$$\Delta_\tau^2 f(x|v)(\xi) > s.$$

This implies

$$f(x') \geq f(x) + \langle v, x' - x \rangle + \frac{s}{2} \|x' - x\|^2.$$

The same inequality holds when $x' = x$. Therefore, s is admissible with α and the neighborhoods $\widehat{X}$ and V , and $\text{uqg}f(\bar{x}|\bar{v}) \geq s$.

If $\Lambda(\bar{x}, \bar{v}) \in \mathbb{R}$, letting $s \uparrow \Lambda(\bar{x}, \bar{v})$ gives

$$\text{uqg}f(\bar{x}|\bar{v}) \geq \Lambda(\bar{x}, \bar{v}).$$

If $\Lambda(\bar{x}, \bar{v}) = \infty$, the preceding argument applies to every $s \in \mathbb{R}$, and hence $\text{uqg}f(\bar{x}|\bar{v}) = \infty$. The reverse inequality follows then in all cases. $\square$

3 From the strong strict second subderivative to the basic inequality

In this section, let $f : \mathcal{H} \rightarrow (-\infty, \infty]$ be proper and lower semicontinuous, and fix a base pair $(\bar{x}, \bar{v}) \in \text{gph } \partial f$ with $f(\bar{x}) \in \mathbb{R}$.

3.1 Strong strict second subderivative

In finite dimensional settings, the corresponding moving-base, f -attentive lower-limit construction is denoted by $d_s^2 f$, with s referring to strictness, in [8, Section 3]. The notation $d_*^2 f$ is used in [20, Definition 2.1], where the corresponding growth-subderivative formula (1.7) is established in [20, Theorem 2.2]. Its extension to an infinite-dimensional Hilbert space requires an explicit choice of the topology for the convergence of the directions. We adopt the asterisk notation of [20] and write $d_{*,s}^2 f$ when the directions converge strongly in $\mathcal{H}$.

Norm convergence of the directions gives the *strong strict second subderivative* defined below.

Definition 3.1. For $(\bar{x}, \bar{v}) \in \text{gph } \partial f$ and $\xi \in \mathcal{H}$, the *strong strict second subderivative* for f at $(\bar{x}, \bar{v})$ in direction ξ is

$$d_{*,s}^2 f(\bar{x} | \bar{v})(\xi) := \liminf_{\substack{\xi_k \rightarrow \xi, \tau_k \downarrow 0 \\ (x_k, v_k) \rightarrow (\bar{x}, \bar{v}), v_k \in \partial f(x_k), f(x_k) \rightarrow f(\bar{x})}} \Delta_{\tau_k}^2 f(x_k | v_k)(\xi_k), \quad (3.1)$$

where Δ_τ^2 is defined in (1.5).

The lower limit in (3.1) is taken jointly and sequentially on all variables satisfying the displayed conditions.

The strong construction in (3.1) retains the lower semicontinuity and positive homogeneity of degree 2 as its finite-dimensional counterpart [20].

Proposition 3.2. *The function $d_{*,s}^2 f(\bar{x} | \bar{v}) : \mathcal{H} \rightarrow [-\infty, \infty]$ is lower semicontinuous for the norm topology and positively homogeneous of degree 2, i.e.,*

$$d_{*,s}^2 f(\bar{x} | \bar{v})(\lambda\xi) = \lambda^2 d_{*,s}^2 f(\bar{x} | \bar{v})(\xi) \quad \text{for every } \xi \in \mathcal{H} \text{ and } \lambda > 0. \quad (3.2)$$

Proof. The proof is standard, we give here only a sketch. Using a standard argument for joint sequential lower limits, the function $d_{*,s}^2 f(\bar{x} \mid \bar{v})$ is sequentially lower semicontinuous in its directional argument. Since the norm topology of $\mathcal{H}$ is metrizable, it is lower semicontinuous for this topology.

Let $\lambda > 0$. For every $(x, v) \in \text{gph } \partial f$, $\tau > 0$, and $\eta \in \mathcal{H}$, one has

$$\Delta_\tau^2 f(x \mid v)(\lambda\eta) = \lambda^2 \Delta_{\lambda\tau}^2 f(x \mid v)(\eta).$$

We note that the transformation $(\eta_k, \tau_k) \mapsto (\lambda\eta_k, \tau_k/\lambda)$ establishes a one-to-one correspondence between the admissible sequences in the definitions of $d_{*,s}^2 f(\bar{x} \mid \bar{v})(\xi)$ and $d_{*,s}^2 f(\bar{x} \mid \bar{v})(\lambda\xi)$. By taking their lower limits and then the infimum over all admissible sequences, we get (3.2). $\square$

3.2 The basic inequality

In the following proposition, we compare the *modulus of uniform local quadratic growth* $\text{uqgf}(\bar{x} \mid \bar{v})$ with $\inf_{\|\xi\|=1} d_{*,s}^2 f(\bar{x} \mid \bar{v})(\xi)$. We show that an inequality, called the *basic inequality* always holds without further assumptions.

Proposition 3.3. *One always has*

$$\text{uqgf}(\bar{x} \mid \bar{v}) \leq \inf_{\|\xi\|=1} d_{*,s}^2 f(\bar{x} \mid \bar{v})(\xi). \quad (3.3)$$

Proof. Let us notice first that the conclusion is immediate when $\text{uqgf}(\bar{x} \mid \bar{v}) = -\infty$. Assume that $\text{uqgf}(\bar{x} \mid \bar{v}) > -\infty$, and fix $s < \text{uqgf}(\bar{x} \mid \bar{v})$. Proposition 2.1 gives $\alpha > f(\bar{x})$, open neighborhoods $X \ni \bar{x}$ and $V \ni \bar{v}$, and $\beta > 0$ such that

$$\inf_{\substack{(x,v) \in (X_f^\alpha \times V) \cap \text{gph } \partial f \\ 0 < \tau < \beta, \|\xi\|=1, x+\tau\xi \in X}} \Delta_\tau^2 f(x \mid v)(\xi) > s. \quad (3.4)$$

Fix $\xi \in \mathcal{H}$ with $\|\xi\| = 1$, and consider an arbitrary sequence admissible in Definition 3.1:

$$\xi_k \rightarrow \xi, \quad \tau_k \downarrow 0, \quad (x_k, v_k) \rightarrow (\bar{x}, \bar{v}), \quad v_k \in \partial f(x_k), \quad f(x_k) \rightarrow f(\bar{x}). \quad (3.5)$$

Since $\|\xi_k\| \rightarrow 1$, then there exists $k_0 \in \mathbb{N}$ such that $\xi_k \neq 0$ for every $k \geq k_0$. For such indices, set

$$r_k := \|\xi_k\|, \quad u_k := \frac{\xi_k}{r_k}, \quad \sigma_k := \tau_k r_k.$$

Then $\|u_k\| = 1$, $r_k \rightarrow 1$, and $\sigma_k \rightarrow 0$. Moreover, $x_k + \sigma_k u_k = x_k + \tau_k \xi_k \rightarrow \bar{x}$. The convergence assumptions (3.5) and the inequality $\alpha > f(\bar{x})$ therefore give, from k sufficiently large, we have

$$(x_k, v_k) \in (X_f^\alpha \times V) \cap \text{gph } \partial f, \quad 0 < \sigma_k < \beta, \quad x_k + \sigma_k u_k \in X.$$

For all sufficiently large k , the quadruple $(x_k, v_k, \sigma_k, u_k)$ is admissible in the infimum appearing in (3.4). Therefore,

$$\Delta_{\sigma_k}^2 f(x_k \mid v_k)(u_k) > s.$$

The definitions of r_k , u_k , and σ_k give the scaling identity

$$\Delta_{\tau_k}^2 f(x_k \mid v_k)(\xi_k) = r_k^2 \Delta_{\sigma_k}^2 f(x_k \mid v_k)(u_k).$$

Consequently,

$$\Delta_{\tau_k}^2 f(x_k \mid v_k)(\xi_k) > s r_k^2.$$

Since $r_k \rightarrow 1$, it follows that

$$\liminf_{k \rightarrow \infty} \Delta_{\tau_k}^2 f(x_k \mid v_k)(\xi_k) \geq s.$$

The admissible sequence was arbitrary, it follows

$$d_{*,s}^2 f(\bar{x} \mid \bar{v})(\xi) \geq s.$$

Taking the infimum over all unit vectors ξ and then letting $s \uparrow \text{uqgf}(\bar{x} \mid \bar{v})$ proves (3.3). $\square$

4 A counterexample in ℓ^2 : the reverse inequality fails for $d_{*,s}^2$

We write $\mathbb{N}^* := \{1, 2, 3, \dots\}$ for the set of positive integers. We present a concrete example on $\mathcal{H} = \ell^2(\mathbb{N}^*)$ showing that, in infinite dimension, the reverse inequality

$$\text{uqg}f(\bar{x} \mid \bar{v}) \stackrel{?}{\geq} \inf_{\|\xi\|=1} d_{*,s}^2 f(\bar{x} \mid \bar{v})(\xi)$$

may fail for the *strong* strict second subderivative $d_{*,s}^2$. By Proposition 3.3, one always has

$$\text{uqg}f(\bar{x} \mid \bar{v}) \leq \inf_{\|\xi\|=1} d_{*,s}^2 f(\bar{x} \mid \bar{v})(\xi),$$

so the point is to show that this inequality may be strict. The mechanism is the escape of unit directions producing arbitrarily small localized quotients.

4.1 The example on ℓ^2

We work on $\mathcal{H} = \ell^2(\mathbb{N}^*)$ with canonical orthonormal basis $(e_n)_{n \geq 1}$ and write $x = (x_n)_{n \geq 1}$. Set

$$r_n := 2^{-n} \quad \text{and} \quad \varepsilon_n := 2^{-n}, \quad n \geq 1.$$

For each n , choose a function $a_n \in C^\infty(\mathbb{R})$ satisfying

$$0 < \varepsilon_n \leq a_n(t) \leq 1 \text{ for all } t \in \mathbb{R}, \quad a_n(t) = 1 \text{ for } |t| \leq \frac{r_n}{2}, \quad a_n(t) = \varepsilon_n \text{ for } |t - r_n| \leq \frac{r_n}{8}. \quad (4.1)$$

Such a smooth function $t \mapsto a_n(t)$ is obtained by a standard smooth interpolation between the constant values 1 and ε_n .

Define $\varphi_n : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\varphi_n(t) := \int_0^t \int_0^s a_n(u) du ds, \quad \varphi'_n(t) = \int_0^t a_n(u) du, \quad \varphi''_n(t) = a_n(t). \quad (4.2)$$

Finally define $f : \ell^2(\mathbb{N}^*) \rightarrow \mathbb{R}$ by the separable sum

$$f(x) := \sum_{n=1}^{\infty} \varphi_n(x_n). \quad (4.3)$$

The coordinate construction is globally $C^{1,1}$, while its restrictions to affine lines retain the second-order regularity needed below.

Lemma 4.1. *The function f in (4.3) is finite-valued and convex on $\ell^2(\mathbb{N}^*)$. It belongs to $C^{1,1}(\ell^2(\mathbb{N}^*))$, with*

$$\nabla f(x) = \sum_{n \geq 1} \varphi'_n(x_n) e_n,$$

and its gradient is globally 1-Lipschitz continuous:

$$\|\nabla f(x) - \nabla f(y)\| \leq \|x - y\| \quad \text{for all } x, y \in \ell^2(\mathbb{N}^*). \quad (4.4)$$

Consequently,

$$\partial f(x) = \{\nabla f(x)\} \quad \text{for all } x \in \ell^2(\mathbb{N}^*),$$

and $(0, 0) \in \text{gph } \partial f$. Moreover, for every $x, \xi \in \ell^2(\mathbb{N}^*)$, the function $t \in \mathbb{R} \mapsto g_{x,\xi}(t) := f(x + t\xi)$, belongs to $C^2(\mathbb{R})$ and satisfies

$$g'_{x,\xi}(t) = \sum_{n \geq 1} \varphi'_n(x_n + t\xi_n) \xi_n, \quad g''_{x,\xi}(t) = \sum_{n \geq 1} a_n(x_n + t\xi_n) \xi_n^2,$$

with

$$0 \leq g''_{x,\xi}(t) \leq \|\xi\|^2, \quad \text{for all } t \in \mathbb{R}.$$

In particular, for every $\tau > 0$,

$$\Delta_\tau^2 f(x \mid \nabla f(x))(\xi) = \frac{2}{\tau^2} \int_0^\tau (\tau - t) g''_{x,\xi}(t) dt. \quad (4.5)$$

Proof. By (4.1), one has $0 < a_n \leq 1$. Hence φ_n is convex, and (4.2) gives

$$0 \leq \varphi_n(t) \leq \frac{1}{2}t^2 \quad \text{and} \quad |\varphi'_n(t)| \leq |t| \quad \text{for all } t \in \mathbb{R}.$$

It follows that

$$0 \leq f(x) \leq \frac{1}{2}\|x\|^2 < \infty, \quad \text{for every } x \in \ell^2(\mathbb{N}^*).$$

Since the partial sums in (4.3) are convex, we deduce that their pointwise limit f is also convex. For $x \in \ell^2(\mathbb{N}^*)$, we have $|\varphi'_n(x_n)| \leq |x_n|$, and consequently $(\varphi'_n(x_n))_{n \geq 1} \in \ell^2(\mathbb{N}^*)$. For $h \in \ell^2(\mathbb{N}^*)$, we have

$$0 \leq f(x+h) - f(x) - \langle \nabla f(x), h \rangle \leq \frac{1}{2} \sum_{n \geq 1} h_n^2 = \frac{1}{2} \|h\|^2.$$

Thus f is Fréchet differentiable. Furthermore, since $\varphi'_n = a_n$ and $0 < a_n \leq 1$, we have

$$|\varphi'_n(s) - \varphi'_n(t)| \leq |s - t| \quad \text{for all } s, t \in \mathbb{R}.$$

Therefore,

$$\|\nabla f(x) - \nabla f(y)\|^2 = \sum_{n \geq 1} |\varphi'_n(x_n) - \varphi'_n(y_n)|^2 \leq \sum_{n \geq 1} |x_n - y_n|^2 = \|x - y\|^2.$$

which proves (4.4). Hence $f \in C^{1,1}(\ell^2(\mathbb{N}^*))$. Since f is convex and continuously Fréchet differentiable, its convex subdifferential, and hence its limiting subdifferential, reduces to $\{\nabla f(x)\}$ for every $x \in \ell^2(\mathbb{N}^*)$. Using the fact that $\varphi'_n(0) = 0$ for every $n \in \mathbb{N}^*$, we have $\nabla f(0) = 0$. Fix $x, \xi \in \ell^2(\mathbb{N}^*)$. Since $f \in C^1(\ell^2(\mathbb{N}^*))$, the chain rule gives

$$g'_{x,\xi}(t) = \langle \nabla f(x + t\xi), \xi \rangle = \sum_{n \geq 1} \varphi'_n(x_n + t\xi_n) \xi_n.$$

Let $I \subset \mathbb{R}$ be a compact interval and fix $t_0 \in I$. By Cauchy-Schwarz,

$$\sum_{n \geq 1} |\varphi'_n(x_n + t_0 \xi_n) \xi_n| \leq \|x + t_0 \xi\| \|\xi\| < \infty.$$

Since $\xi \in \ell^2(\mathbb{N}^*)$ and $0 < a_n \leq 1$, one has

$$\sum_{n \geq 1} \sup_{t \in I} |a_n(x_n + t\xi_n) \xi_n^2| \leq \sum_{n \geq 1} |\xi_n|^2 = \|\xi\|_{\ell^2}^2 < \infty.$$

Thus, the series of derivatives

$$\sum_{n \geq 1} a_n(x_n + t\xi_n) \xi_n^2$$

converges normally, and hence uniformly, on I . Since $\sum_{n \geq 1} \varphi'_n(x_n + t_0 \xi_n) \xi_n$ converges and $\sum_{n \geq 1} a_n(x_n + t\xi_n) \xi_n^2$ converges uniformly on I , using the termwise differentiation theorem for series of C^1 functions we get the uniform convergence on I of $\sum_{n \geq 1} \varphi'_n(x_n + t\xi_n) \xi_n$ and

$$g''_{x,\xi}(t) = \sum_{n \geq 1} a_n(x_n + t\xi_n) \xi_n^2.$$

Since the compact interval I is arbitrary, $g_{x,\xi} \in C^2(\mathbb{R})$, and $0 \leq g''_{x,\xi}(t) \leq \|\xi\|^2$ for every $t \in \mathbb{R}$.

Finally, the one-dimensional integral remainder formula gives

$$g_{x,\xi}(\tau) - g_{x,\xi}(0) - \tau g'_{x,\xi}(0) = \int_0^\tau (\tau - t) g''_{x,\xi}(t) dt,$$

which proves (4.5). $\square$

The proof of Lemma 4.1 does not use any uniform bounds on a'_n . It shows that $f \in C^{1,1}(\ell^2(\mathbb{N}^*))$ and $t \mapsto f(x + t\xi) \in C^2(\mathbb{R})$ for all $x, \xi \in \ell^2(\mathbb{N}^*)$, but it does not imply that $f \in C^2(\ell^2(\mathbb{N}^*))$.

4.2 Computation of $\text{uqgf}(0 \mid 0)$

Convexity induces a non-negative modulus of uniform quadratic growth, while regions where $a_n = \varepsilon_n$ produce second-order quotients converging to zero in the vicinity of the origin. The following proposition gives the value of the Uniform Quadratic Growth of f at $(0, 0)$.

Proposition 4.2. *For the function f in (4.3), one has $\text{uqgf}(0 \mid 0) = 0$.*

Proof. Since f is convex, the subgradient inequality gives (1.3) with $s = 0$ for every pair of the graph. Therefore, $\text{uqgf}(0 \mid 0) \geq 0$.

Assume that some $s > 0$ are admissible. Then there exist $\alpha > 0$ and open neighborhoods $X \ni 0$ and $V \ni 0$ such that (1.3) holds. Choose $\rho > 0$ such that $\mathbb{B}(0, \rho) \subset X$. For each n , define $x^{(n)} := r_n e_n$ and $v^{(n)} := \nabla f(x^{(n)}) = \varphi'_n(r_n) e_n$. Since $r_n \rightarrow 0$, $|\varphi'_n(r_n)| \leq r_n$, and $f(x^{(n)}) \leq r_n^2/2 \rightarrow 0$, the pairs of the graph $(x^{(n)}, v^{(n)})$ converge attentively to $(0, 0)$. Thus, for some sufficiently large n , $\varepsilon_n < s$, $r_n < \rho/2$, and $(x^{(n)}, v^{(n)}) \in (X_f^\alpha \times V) \cap \text{gph } \partial f$.

Fix such an index and define $t := r_n/16$ and $x' := x^{(n)} + t e_n$. Then $\|x'\| = r_n + t = 17r_n/16 < \rho$, therefore $x' \in X$. For every $u \in [r_n, r_n + t]$, we have $|u - r_n| \leq r_n/16 < r_n/8$. It follows from (4.1) that $a_n(u) = \varepsilon_n$ on this interval. We have $\varphi''_n = \varepsilon_n$ on $[r_n, r_n + t]$. Using Taylor's formula, we obtain

$$\begin{aligned} f(x') - f(x^{(n)}) - \langle v^{(n)}, x' - x^{(n)} \rangle &= \varphi_n(r_n + t) - \varphi_n(r_n) - \varphi'_n(r_n)t \\ &= \int_0^t (t - s) \varphi''_n(r_n + s) ds \\ &= \varepsilon_n \int_0^t (t - s) ds = \frac{\varepsilon_n}{2} t^2. \end{aligned}$$

On the other hand, (1.3) gives $\varepsilon_n t^2/2 \geq s t^2/2$. Since $t > 0$, this implies $\varepsilon_n \geq s$, which contradicts the choice $\varepsilon_n < s$. Therefore, no positive value is admissible, and $\text{uqgf}(0 \mid 0) \leq 0$. With the first inequality, this proves that $\text{uqgf}(0 \mid 0) = 0$. $\square$

4.3 Computation of $d_{*,s}^2 f(0 \mid 0)(\xi)$

We now show that every fixed unit direction has a strong strict second subderivative equal to 1n and by taking the infimum on the unit sphere we get also 1, whereas $\text{uqgf}(0 \mid 0) = 0$. The explanation comes from the fact that the strong convergence of the directions prevents their mass from escaping to higher coordinates. The next proposition is in this sense.

Proposition 4.3. *For the function f defined by (4.3),*

$$d_{*,s}^2 f(0 \mid 0)(\xi) = 1 \quad \text{for every } \|\xi\| = 1, \quad \inf_{\|\xi\|=1} d_{*,s}^2 f(0 \mid 0)(\xi) = 1.$$

Proof. Fix $\xi \in \mathcal{H}$ such that $\|\xi\| = 1$. To prove that $d_{*,s}^2 f(0 \mid 0)(\xi) \leq 1$, take $x_k = v_k = 0$, $\xi_k = \xi$, and $\tau_k = 1/k$. This family is admissible, and Lemma 4.1 together with (4.5) gives $0 \leq \Delta_{\tau_k}^2 f(0 \mid 0)(\xi) \leq \|\xi\|^2 = 1$. Therefore $d_{*,s}^2 f(0 \mid 0)(\xi) \leq 1$.

For the reverse inequality, consider an arbitrary family admissible in the sense of Definition 3.1. Thus $\xi_k \rightarrow \xi$, $\tau_k \downarrow 0$, $(x_k, v_k) \rightarrow (0, 0)$, and $v_k \in \partial f(x_k)$. By Lemma 4.1, $v_k = \nabla f(x_k)$. Fix $\eta > 0$ and choose N such that $\sum_{n>N} \xi_n^2 < \eta$. Strong convergence gives $\sum_{n>N} \xi_{k,n}^2 < 2\eta$ for all sufficiently large k , while $\|\xi_k\|^2 \rightarrow 1$.

Set $r_* := r_N$ and $M := 1 + \sup_k \max_{1 \leq n \leq N} |\xi_{k,n}|$. The constant M is finite because the sequence (ξ_k) is bounded. There exists k_0 such that, for every $k \geq k_0$, $|x_{k,n}| \leq r_*/4$ for all $n \leq N$ and $\tau_k M \leq r_*/4$. Hence, for $t \in [0, \tau_k]$ and $n \leq N$, $|x_{k,n} + t\xi_{k,n}| \leq r_*/2 \leq r_n/2$. Formula (4.1) then gives $a_n(x_{k,n} + t\xi_{k,n}) = 1$.

By setting $g_k(t) := f(x_k + t\xi_k)$ and using Lemma 4.1, we obtain

$$g_k''(t) = \sum_{n \geq 1} a_n(x_{k,n} + t\xi_{k,n}) \xi_{k,n}^2 \geq \sum_{n \leq N} \xi_{k,n}^2 \geq \|\xi_k\|^2 - 2\eta \quad \text{for } 0 \leq t \leq \tau_k.$$

Taking into account the fact that $g_k'(0) = \langle v_k, \xi_k \rangle$, formula (4.5) gives $\Delta_{\tau_k}^2 f(x_k \mid v_k)(\xi_k) \geq \|\xi_k\|^2 - 2\eta$. Since $\|\xi_k\|^2 \rightarrow 1$, it follows that $\liminf_{k \rightarrow \infty} \Delta_{\tau_k}^2 f(x_k \mid v_k)(\xi_k) \geq 1 - 2\eta$. The admissible family is arbitrary, $d_{*,s}^2 f(0 \mid 0)(\xi) \geq 1 - 2\eta$. Letting $\eta \downarrow 0$ gives $d_{*,s}^2 f(0 \mid 0)(\xi) \geq 1$. The two inequalities prove $d_{*,s}^2 f(0 \mid 0)(\xi) = 1$. $\square$

4.4 Conclusion: strict gap between $\text{uqg}f$ and $\inf d_{*,s}^2$

The combination of the two previous calculations gives a strict gap between the uniform quadratic growth of f at $(0, 0)$ and the strong strict second-order curvature at the same point.

Theorem 4.4. *The function f defined by (4.3) is finite-valued and convex on $\ell^2(\mathbb{N}^*)$. It is of class $C^{1,1}$, and its gradient is globally Lipschitz continuous with constant 1. At the graph point $(0, 0)$, one has*

$$\text{uqg}f(0 \mid 0) = 0 < 1 = \inf_{\|\xi\|=1} d_{*,s}^2 f(0 \mid 0)(\xi).$$

Proof. The regularity assertions follow from Lemma 4.1, while the two identities follow directly from Proposition 4.2 and Proposition 4.3. $\square$

Remark 4.5. (i) The strict gap observed in Theorem 4.4 results from the lack of compactness in the coordinate directions. In fact, with $x^{(n)} = r_n e_n$, $v^{(n)} = \nabla f(x^{(n)})$ and $\tau_n = r_n/16$, the construction gives:

$$\Delta_{\tau_n}^2 f(x^{(n)} \mid v^{(n)})(e_n) = \varepsilon_n \longrightarrow 0.$$

However, (e_n) does not have a strongly convergent subsequence. These quotients therefore cannot appear along an admissible sequence in the definition of $d_{*,s}^2 f(0 \mid 0)(\xi)$, because the corresponding directions must converge strongly towards a fixed vector ξ .

(ii) The strict inequality in Theorem 4.4, together with Corollary 6.2, shows that f is not of class C^2 on any neighborhood of the origin, although $f \in C^{1,1}(\ell^2(\mathbb{N}^*))$.

5 A compactness condition restoring the growth-subderivative formula

In this section we identify a structural compactness condition under which the modulus of uniform local quadratic growth can be recovered from the strong strict second subderivative by minimization over unit directions in a Hilbert space. The condition requires a relatively compact selection of approximate minimizing directions whenever the directional infima remain bounded above along an attentive graph sequence and a sequence of steps tending to zero.

5.1 Directional compactness of approximate minimizing directions

For $(x, v) \in \text{gph } \partial f$ and $\tau > 0$, we define

$$\phi(x, v, \tau) := \inf_{\|\xi\|=1} \Delta_\tau^2 f(x \mid v)(\xi).$$

Definition 5.1. We say that f satisfies the *directional compactness condition* at $(\bar{x}, \bar{v})$ if the following holds: for every sequence $(x_k, v_k) \rightarrow (\bar{x}, \bar{v})$ with $v_k \in \partial f(x_k)$ and $f(x_k) \rightarrow f(\bar{x})$, and every sequence $\tau_k \downarrow 0$ such that

$$\sup_k \phi_k < \infty, \quad \text{with} \quad \phi_k := \phi(x_k, v_k, \tau_k),$$

there exist positive numbers $\varepsilon_k \rightarrow 0$ and unit vectors $\xi_k \in \mathcal{H}$ such that the sequence (ξ_k) is relatively compact in $\mathcal{H}$ and

$$\Delta_{\tau_k}^2 f(x_k \mid v_k)(\xi_k) \leq \begin{cases} \phi_k + \varepsilon_k, & \text{if } \phi_k \in \mathbb{R}, \\ -\varepsilon_k^{-1}, & \text{if } \phi_k = -\infty. \end{cases}$$

A related compactness condition concerning the graphical limits of maximally monotone operators was used in [1].

Remark 5.2. The directional compactness condition in Definition 5.1 is motivated by the compactness assumption used in [1, Theorem 3.1] to preserve maximal monotonicity under Painlevé-Kuratowski graph convergence. The assumption in [1] requires relative compactness of every bounded sequence (z_k) for which $A_k^0(z_k)$ remains bounded, where $A_k^0(z_k)$ is the element of minimal norm in the set $A_k(z_k)$. Here compactness is required only for a selection of approximate minimizing unit directions along attentive graph sequences satisfying $\sup_k \phi_k < \infty$. Note that in both proofs, the compactness condition provides the subsequence with convergence in norm necessary to pass to the limit.

The directional compactness condition is automatic in finite dimensions.

Remark 5.3. Suppose that $\mathcal{H}$ is finite dimensional. For fixed $(x, v) \in \text{gph } \partial f$ and $\tau > 0$, the mapping $\xi \mapsto \Delta_\tau^2 f(x \mid v)(\xi)$ is lower semicontinuous and therefore attains its infimum on the compact unit sphere. For the sequences considered in Definition 5.1, this infimum is finite. For each k , choose a unit vector ξ_k attaining ϕ_k . Since the unit sphere is compact, (ξ_k) is relatively compact. These choices satisfy Definition 5.1 with $\varepsilon_k = 1/k$.

The condition will be tested rather than used only as an abstract assumption. Proposition 5.4 gives a practical compact-embedding criterion, Subsection 5.3 characterizes the condition for quadratic energies, and Proposition 6.5 verifies it for the nonsmooth elliptic functional of Section 6.

A coercive estimate in a space compactly embedded into $\mathcal{H}$ ensures the relative compactness of approximate minimizing directions.

Proposition 5.4. *Let V be a Banach space continuously and compactly embedded into $\mathcal{H}$, and set $\|\xi\|_V := \infty$ for $\xi \notin V$. Suppose that, for every attentive graph sequence $(x_k, v_k) \rightarrow (\bar{x}, \bar{v})$ and every $\tau_k \downarrow 0$, there exist constants $c, C > 0$ such that, for all sufficiently large k ,*

$$\Delta_{\tau_k}^2 f(x_k \mid v_k)(\xi) \geq c\|\xi\|_V^2 - C \quad \text{for every } \|\xi\| = 1.$$

Then f satisfies the directional compactness condition at $(\bar{x}, \bar{v})$.

Proof. Fix an attentive graph sequence $(x_k, v_k) \rightarrow (\bar{x}, \bar{v})$ and a sequence $\tau_k \downarrow 0$ for which $\phi_k := \phi(x_k, v_k, \tau_k)$ is bounded above. Choose $M > 0$ such that $\phi_k \leq M$ for every k , and set $\varepsilon_k := 1/k$. Let $c, C > 0$ be given by the assumption, and choose k_0 such that the coercive estimate holds for every $k \geq k_0$. For these indices, $-C \leq \phi_k \leq M$, so $\phi_k \in \mathbb{R}$. Choose a unit vector ξ_k such that $\Delta_{\tau_k}^2 f(x_k | v_k)(\xi_k) \leq \phi_k + \varepsilon_k$. Then

$$c\|\xi_k\|_V^2 - C \leq \Delta_{\tau_k}^2 f(x_k | v_k)(\xi_k) \leq \phi_k + \varepsilon_k \leq M + 1.$$

Thus $\xi_k \in V$ and $\|\xi_k\|_V^2 \leq (M + C + 1)/c$ for every $k \geq k_0$. The compact embedding makes $(\xi_k)_{k \geq k_0}$ relatively compact in $\mathcal{H}$.

For $k < k_0$, choose a unit vector ξ_k with quotient at most $\phi_k + \varepsilon_k$ if $\phi_k \in \mathbb{R}$, and at most $-\varepsilon_k^{-1}$ if $\phi_k = -\infty$. These choices exist by the definition of ϕ_k . Adding finitely many vectors preserves relative compactness, so Definition 5.1 holds with $\varepsilon_k = 1/k$. $\square$

5.2 Equality under directional compactness

Directional compactness prevents approximate minimizing directions from escaping and restores equality in (3.3).

Theorem 5.5. *Let $f : \mathcal{H} \rightarrow (-\infty, \infty]$ be proper and lower semicontinuous, and let $(\bar{x}, \bar{v}) \in \text{gph } \partial f$ with $f(\bar{x}) \in \mathbb{R}$. If f satisfies the directional compactness condition, in Definition 5.1, at $(\bar{x}, \bar{v})$, then the growth-subderivative formula holds:*

$$\text{uqg}f(\bar{x} | \bar{v}) = \inf_{\|\xi\|=1} d_{*,s}^2 f(\bar{x} | \bar{v})(\xi).$$

Proof. We set $m := \inf_{\|\xi\|=1} d_{*,s}^2 f(\bar{x} | \bar{v})(\xi)$. By Proposition 3.3, we have $\text{uqg}f(\bar{x} | \bar{v}) \leq m$. If $m = -\infty$, the reverse inequality is automatic. Assume that $m > -\infty$, fix $s \in \mathbb{R}$ with $s < m$, and suppose that $s \geq \text{uqg}f(\bar{x} | \bar{v})$. By Proposition 2.1, every localized infimum is bounded above by $\text{uqg}f(\bar{x} | \bar{v}) \leq s$.

For each $k \in \mathbb{N}^*$, set $\alpha_k := f(\bar{x}) + 1/k$, $X_k := \mathbb{B}(\bar{x}, 1/k)$, $V_k := \mathbb{B}(\bar{v}, 1/k)$, and $\beta_1 := 1$. For $k \geq 2$, once τ_{k-1} has been chosen, we select β_k such that $0 < \beta_k < \min\{1/k, \tau_{k-1}\}$. The localized quotient characterization gives $(x_k, v_k) \in (X_k)^{\alpha_k} \times V_k \cap \text{gph } \partial f$, $0 < \tau_k < \beta_k$, and a unit vector u_k such that $x_k + \tau_k u_k \in X_k$ and $\Delta_{\tau_k}^2 f(x_k | v_k)(u_k) \leq s + 1/k$.

Since $0 < \tau_k < \beta_k < \min\{1/k, \tau_{k-1}\}$ for $k \geq 2$, we have $\tau_k \downarrow 0$. Moreover, $x_k \rightarrow \bar{x}$, $v_k \rightarrow \bar{v}$, and $f(x_k) < f(\bar{x}) + 1/k$. Lower semicontinuity gives $f(\bar{x}) \leq \liminf_k f(x_k)$, and therefore $f(x_k) \rightarrow f(\bar{x})$. Thus (x_k, v_k) is an attentive graph sequence. With $\phi_k := \phi(x_k, v_k, \tau_k)$, we have $\phi_k \leq s + 1/k$ and hence $\sup_k \phi_k \leq s + 1 < \infty$.

Directional compactness provides positive numbers $\varepsilon_k \rightarrow 0$ and unit vectors ξ_k forming a relatively compact sequence such that

$$\Delta_{\tau_k}^2 f(x_k | v_k)(\xi_k) \leq \begin{cases} \phi_k + \varepsilon_k, & \phi_k > -\infty, \\ -\varepsilon_k^{-1}, & \phi_k = -\infty. \end{cases}$$

For every k such that $\phi_k > -\infty$, one has

$$\Delta_{\tau_k}^2 f(x_k | v_k)(\xi_k) \leq \phi_k + \varepsilon_k \leq s + \frac{1}{k} + \varepsilon_k.$$

If $\phi_k = -\infty$, then $\Delta_{\tau_k}^2 f(x_k | v_k)(\xi_k) \leq -\varepsilon_k^{-1}$. This upper bound tends to $-\infty$ along any infinite subsequence of such indices. Since $\varepsilon_k \rightarrow 0$, it follows that

$$\limsup_{k \rightarrow \infty} \Delta_{\tau_k}^2 f(x_k | v_k)(\xi_k) \leq s.$$

After passing to a subsequence, $\xi_k \rightarrow \xi$ for some unit vector ξ . Definition 3.1 gives

$$d_{*,s}^2 f(\bar{x} \mid \bar{v})(\xi) \leq \liminf_k \Delta_{\tau_k}^2 f(x_k \mid v_k)(\xi_k) \leq \limsup_k \Delta_{\tau_k}^2 f(x_k \mid v_k)(\xi_k) \leq s.$$

Since $\|\xi\| = 1$, it follows that $m \leq s$, which contradicts the choice $s < m$. Hence $\text{uqg}f(\bar{x} \mid \bar{v}) > s$ for every $s < m$. Taking the supremum over such s gives $\text{uqg}f(\bar{x} \mid \bar{v}) \geq m$. If $m = \infty$, the first inequality holds for every $s \in \mathbb{R}$ and therefore $\text{uqg}f(\bar{x} \mid \bar{v}) = \infty$. $\square$

Remark 5.6. The directional compactness condition fails in the ℓ^2 example of Section 4. Otherwise, Theorem 5.5 would contradict Theorem 4.4. For the graph pairs $(x^{(n)}, v^{(n)})$ used in the proof of Proposition 4.2 and $\tau_n := r_n/16$, one has $0 \leq \phi(x^{(n)}, v^{(n)}, \tau_n) \leq \varepsilon_n$. Thus these directional infima are bounded above, but no relatively compact selection of approximate minimizing directions is possible.

5.3 Quadratic perturbations: attainment and directional compactness

For quadratic functions, directional compactness holds if and only if the Rayleigh quotient attains its infimum on the unit sphere. This characterization also shows that compactness of the perturbation alone is not sufficient.

Proposition 5.7. *Let $\mathcal{H}$ be a real Hilbert space, let $K : \mathcal{H} \rightarrow \mathcal{H}$ be bounded and self-adjoint, and define*

$$f(x) := \frac{1}{2} \langle (\text{Id} + K)x, x \rangle, \quad \mu := \inf_{\|\xi\|=1} \langle K\xi, \xi \rangle.$$

The following assertions are equivalent:

- (i) *f satisfies the directional compactness condition at some point of $\text{gph } \partial f$;*
- (ii) *f satisfies this condition at every point of $\text{gph } \partial f$;*
- (iii) *μ is attained on the unit sphere.*

Proof. Since $\partial f(x) = \{(\text{Id} + K)x\}$, the expansion of the quadratic form gives, at every point of the graph and for all $\tau > 0$,

$$\Delta_\tau^2 f(x \mid v)(\xi) = \langle (\text{Id} + K)\xi, \xi \rangle, \quad \text{and} \quad \phi(x, v, \tau) = 1 + \mu.$$

If a unit vector ξ_* attains μ , the constant choices $\xi_k = \xi_*$ and $\varepsilon_k = 1/k$ satisfy Definition 5.1 at every point of the graph. Thus, (iii) implies (ii), while (ii) implies (i).

Suppose that (i) holds at $(\bar{x}, \bar{x} + K\bar{x})$. Since K is bounded, $\mu \in \mathbb{R}$. For the constant graph sequence and $\tau_k = 1/k$, one has $\phi_k = 1 + \mu$, so the directional infima are bounded above. Definition 5.1 therefore gives positive numbers $\varepsilon_k \rightarrow 0$ and a relatively compact sequence of unit vectors (ξ_k) such that

$$\mu \leq \langle K\xi_k, \xi_k \rangle \leq \mu + \varepsilon_k.$$

Along a norm-convergent subsequence, $\xi_k \rightarrow \xi_*$ for some unit vector ξ_* . The continuity of the quadratic form gives: $\langle K\xi_*, \xi_* \rangle = \mu$. Therefore, (i) implies (iii). $\square$

For a compact perturbation, the sign and kernel of K determine whether the minimum Rayleigh value is reached.

Remark 5.8. Suppose that $\mathcal{H}$ is infinite dimension and that K is a compact operator. If $\langle K\xi, \xi \rangle < 0$ for some unit vector ξ , then $\mu < 0$. The spectral theorem for compact self-adjoint operators implies that μ is an eigenvalue and is therefore attained.

Now suppose K is non-negative. Let (e_n) be an orthonormal sequence, we have $e_n \rightarrow 0$, and the compactness of K implies $Ke_n \rightarrow 0$. Therefore,

$$0 \leq \mu \leq \langle Ke_n, e_n \rangle \leq \|Ke_n\| \rightarrow 0,$$

and therefore $\mu = 0$. Moreover, $\langle K\xi, \xi \rangle = \|K^{1/2}\xi\|^2$, which vanishes if and only if $\xi \in \ker K$. Thus, μ is attained on the unit sphere if and only if $\ker K \neq \{0\}$.

In particular, if K is non-negative and injective, the function f does not satisfy the condition of directional compactness. Its second-order development is nevertheless exact and uniform on the unit sphere, so that the Rayleigh identity remains valid. For general C^2 functions, Corollary 6.2 deduces (6.3) from operator-norm continuity of the Hessian without assuming directional compactness.

6 Smooth perturbations and a nonsmooth elliptic application

This section establishes an addition rule for the strong strict second subderivative and applies the representation theorem to a Dirichlet energy with an L^1 term. The energy is defined on $L^2(\Omega)$ and extended by ∞ outside $H_0^1(\Omega)$. We first check the directional compactness condition at every point of the subgradient graph and obtain the growth-subderivative formula. After that, we compute the strong strict second subderivative when the multiplier is globally saturated (in a sense that will be precise later) and has the same sign as the state. Under this condition, the modulus of uniform quadratic growth coincides with the lowest eigenvalue of the associated elliptic operator.

6.1 An addition rule for the strong strict second subderivative

The following result extends the smooth addition rule in [20, Proposition 2.3] to the strong strict second subderivative on a Hilbert space.

Proposition 6.1. *Let $\mathcal{H}$ be a real Hilbert space. Let $f_1 : \mathcal{H} \rightarrow (-\infty, \infty]$ be proper and lower semicontinuous, and let $f_2 : \mathcal{H} \rightarrow \mathbb{R}$ be twice continuously Fréchet differentiable on a neighborhood of $\bar{x} \in \mathcal{H}$. Set $f := f_1 + f_2$ and assume that $(\bar{x}, \bar{v}) \in \text{gph } \partial f$. Set $\bar{v}_1 := \bar{v} - \nabla f_2(\bar{x})$. Then $\bar{v}_1 \in \partial f_1(\bar{x})$ and, for every $\xi \in \mathcal{H}$,*

$$d_{*,s}^2 f(\bar{x} \mid \bar{v})(\xi) = d_{*,s}^2 f_1(\bar{x} \mid \bar{v}_1)(\xi) + \langle \nabla^2 f_2(\bar{x})\xi, \xi \rangle. \quad (6.1)$$

Proof. Since f_2 is continuously Fréchet differentiable near $\bar{x}$, it is strictly differentiable at every point x sufficiently close to $\bar{x}$. By [14, Proposition 1.107(ii)],

$$\partial f(x) = \partial f_1(x) + \nabla f_2(x) \quad (6.2)$$

for such x . In particular, $\bar{v}_1 \in \partial f_1(\bar{x})$.

Fix $\xi \in \mathcal{H}$. Let $x_k \rightarrow \bar{x}$, $\xi_k \rightarrow \xi$, and $\tau_k \downarrow 0$, and set $v_{1,k} := v_k - \nabla f_2(x_k)$. For all sufficiently large k , (6.2) gives $v_k \in \partial f(x_k)$ if and only if $v_{1,k} \in \partial f_1(x_k)$. On the other hand, the continuity of f_2 and ∇f_2 gives

$$f(x_k) \rightarrow f(\bar{x}) \iff f_1(x_k) \rightarrow f_1(\bar{x}), \quad v_k \rightarrow \bar{v} \iff v_{1,k} \rightarrow \bar{v}_1.$$

Thus the admissible sequences in the two lower-limit definitions correspond. For these sequences, Taylor's formula with integral remainder gives, for all sufficiently large k ,

$$\Delta_{\tau_k}^2 f(x_k \mid v_k)(\xi_k) = \Delta_{\tau_k}^2 f_1(x_k \mid v_{1,k})(\xi_k) + R_k,$$

where

$$R_k := 2 \int_0^1 (1-t) \langle \nabla^2 f_2(x_k + t\tau_k \xi_k) \xi_k, \xi_k \rangle dt.$$

Since $x_k + t\tau_k\xi_k \rightarrow \bar{x}$ uniformly for $t \in [0, 1]$, the operator norm continuity of $\nabla^2 f_2$ and $\xi_k \rightarrow \xi$ imply $R_k \rightarrow \langle \nabla^2 f_2(\bar{x})\xi, \xi \rangle$. Taking the lower limit along each admissible sequence and then the infimum over these sequences proves (6.1). $\square$

For a twice continuously differentiable function, the strict curvature and the modulus of uniform quadratic growth are both determined by the Hessian.

Corollary 6.2. *Let $\mathcal{H}$ be a real Hilbert space, and let $f : \mathcal{H} \rightarrow \mathbb{R}$ be twice continuously Fréchet differentiable on a neighborhood of $\bar{x}$. Then, for every $\xi \in \mathcal{H}$,*

$$d_{*,s}^2 f(\bar{x} \mid \nabla f(\bar{x}))(\xi) = \langle \nabla^2 f(\bar{x})\xi, \xi \rangle.$$

Moreover,

$$\text{uqg}f(\bar{x} \mid \nabla f(\bar{x})) = \inf_{\|\xi\|=1} \langle \nabla^2 f(\bar{x})\xi, \xi \rangle. \quad (6.3)$$

Proof. The first formula follows from Proposition 6.1, applied with $f_1 \equiv 0$ and $f_2 = f$. Set $m := \inf_{\|\xi\|=1} \langle \nabla^2 f(\bar{x})\xi, \xi \rangle$. Proposition 3.3 and the first formula give $\text{uqg}f(\bar{x} \mid \nabla f(\bar{x})) \leq m$.

Let $s < m$. The continuity of $\nabla^2 f$ in the operator norm gives an open convex neighborhood X of $\bar{x}$ such that

$$\langle \nabla^2 f(x)h, h \rangle \geq s\|h\|^2 \quad \text{for every } x \in X \text{ and } h \in \mathcal{H}.$$

For $x, x+h \in X$, Taylor's formula with integral remainder gives

$$f(x+h) - f(x) - \langle \nabla f(x), h \rangle \geq \frac{s}{2}\|h\|^2.$$

Since $\partial f(x) = \{\nabla f(x)\}$ on X , the level s is admissible in the definition of the modulus of uniform quadratic growth. Taking the supremum over $s < m$ gives $\text{uqg}f(\bar{x} \mid \nabla f(\bar{x})) \geq m$. Together with the other inequality, we obtain (6.3). $\square$

The corollary also gives the following conclusions about directional compactness and the regularity of the counterexample.

Remark 6.3. The proof of Corollary 6.2 does not use the directional compactness assumption. Theorem 5.5 gives a sufficient condition for general lower semicontinuous functions, but that condition is not necessary in the C^2 case. Corollary 6.2 also shows that the $C^{1,1}$ function in Theorem 4.4 cannot be of class C^2 on any neighborhood of the origin, since its modulus of uniform quadratic growth is different from the infimum of its strong strict second subderivative on the unit sphere.

6.2 An L^1 -penalized Dirichlet energy on $L^2(\Omega)$

Let $\Omega \subset \mathbb{R}^d$ be a nonempty bounded Lipschitz domain. For the remainder of this section, we take $\mathcal{H} = L^2(\Omega)$ and denote its norm by $\|\cdot\|_2$. The Dirichlet Laplacian A is the self-adjoint operator associated with

$$a(u, z) := \int_{\Omega} \nabla u \cdot \nabla z \, dx, \quad u, z \in H_0^1(\Omega).$$

Thus $u \in D(A)$ and $Au = h \in L^2(\Omega)$ precisely when

$$\int_{\Omega} \nabla u \cdot \nabla z \, dx = \int_{\Omega} h z \, dx \quad \text{for every } z \in H_0^1(\Omega).$$

Let $q \in L^\infty(\Omega; \mathbb{R})$, $g \in L^2(\Omega)$, and $\beta > 0$. We use $q_-(x) := \max\{-q(x), 0\}$.

Let $M_q : L^2(\Omega) \rightarrow L^2(\Omega)$ be the bounded self-adjoint multiplication operator defined by

$$M_q \xi := q\xi. \quad (6.4)$$

Define $F_1 : L^2(\Omega) \rightarrow (-\infty, \infty]$ by

$$F_1(u) := \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} qu^2 dx + \beta \int_{\Omega} |u| dx - \int_{\Omega} gu dx, & u \in H_0^1(\Omega), \\ \infty, & u \notin H_0^1(\Omega). \end{cases} \quad (6.5)$$

Set

$$\Psi(u) := \frac{1}{2} \int_{\Omega} qu^2 dx - \int_{\Omega} gu dx, \quad G := F_1 - \Psi.$$

Then $F_1 = G + \Psi$, where G is the sum of the Dirichlet energy and the L^1 term and $\Psi \in C^2(L^2(\Omega))$, with $\nabla \Psi(u) = qu - g$ and $\nabla^2 \Psi(u) = M_q$.

For $r \in \mathbb{R}$, set

$$\text{Sign}(r) = \partial|\cdot|(r) := \begin{cases} \{1\}, & r > 0, \\ [-1, 1], & r = 0, \\ \{-1\}, & r < 0. \end{cases}$$

We use the convention that f is s -convex if $f - \frac{s}{2} \|\cdot\|_2^2$ is convex. The case $s < 0$ is called hypoconvexity, also known as weak convexity or semiconvexity [20, Section 1, equation (1.8) and footnote 4].

The following proposition gives the subgradient graph and the quotient formula used in the compactness and curvature arguments.

Proposition 6.4. *Set $\rho := \|q_-\|_{\infty}$. The functional F_1 is proper, lower semicontinuous, and s -convex for $s = -\rho$. Moreover,*

$$v \in \partial F_1(u) \iff \begin{cases} u \in D(A), \\ \text{there exists } \zeta \in L^{\infty}(\Omega) \text{ such that} \\ v = Au + qu + \beta\zeta - g, \quad \zeta(x) \in \text{Sign}(u(x)) \text{ for a.e. } x \in \Omega. \end{cases} \quad (6.6)$$

For each graph pair (u, v) , the multiplier ζ is unique. For this multiplier, every $\xi \in H_0^1(\Omega)$ and every $\tau > 0$ satisfy

$$\Delta_{\tau}^2 F_1(u | v)(\xi) = \int_{\Omega} |\nabla \xi|^2 dx + \int_{\Omega} q\xi^2 dx + \frac{2\beta}{\tau^2} \int_{\Omega} (|u + \tau\xi| - |u| - \tau\zeta\xi) dx. \quad (6.7)$$

The last integral is nonnegative. If $\xi \notin H_0^1(\Omega)$, then $\Delta_{\tau}^2 F_1(u | v)(\xi) = \infty$.

Proof. Define

$$Q(u) := \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx, & u \in H_0^1(\Omega), \\ \infty, & u \notin H_0^1(\Omega), \end{cases} \quad J(u) := \int_{\Omega} |u| dx.$$

Then $G = Q + \beta J$. The closedness of the Dirichlet form and the definition of its associated operator give

$$\partial Q(u) = \begin{cases} \{Au\}, & u \in D(A), \\ \emptyset, & u \notin D(A). \end{cases}$$

In particular, Q is proper, convex, and lower semicontinuous on $L^2(\Omega)$. Since Ω has finite measure, J is continuous and convex on $L^2(\Omega)$, and

$$\partial J(u) = \{\zeta \in L^{\infty}(\Omega) : \zeta(x) \in \text{Sign}(u(x)) \text{ for a.e. } x \in \Omega\}.$$

The convex sum rule for G , followed by the smooth perturbation rule (6.2) for $F_1 = G + \Psi$, gives (6.6). Since $\beta > 0$, that identity also determines ζ uniquely.

The functional F_1 is proper and lower semicontinuous because Q has these properties and J and Ψ are continuous on $L^2(\Omega)$. Since $q + \rho \geq 0$ almost everywhere, $F_1 + \frac{\rho}{2}\|\cdot\|_2^2$ is convex. Thus F_1 is s -convex for $s = -\rho$.

For $\xi \in H_0^1(\Omega)$, substitution of (6.6) into the second-order quotient and expansion of the quadratic terms give (6.7). Its last integrand is nonnegative by the scalar subgradient inequality for $|\cdot|$. Finally, if $\xi \notin H_0^1(\Omega)$, then $u + \tau\xi \notin H_0^1(\Omega)$ because $u \in D(A) \subset H_0^1(\Omega)$ and the quotient equals ∞ . $\square$

6.3 Directional compactness and representation

The lower bound in (6.7) and the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ give directional compactness at every point of the subgradient graph.

Proposition 6.5. *Let $(\bar{u}, \bar{v}) \in \text{gph } \partial F_1$. Then F_1 satisfies the directional compactness condition of Definition 5.1 at $(\bar{u}, \bar{v})$.*

Proof. We apply Proposition 5.4 with $V = H_0^1(\Omega)$, equipped with the equivalent norm $\|\xi\|_V := (\int_\Omega |\nabla \xi|^2 dx)^{1/2}$, and $\mathcal{H} = L^2(\Omega)$. For every $(u, v) \in \text{gph } \partial F_1$, $\tau > 0$, and $\xi \in V$ with $\|\xi\|_2 = 1$, formula (6.7) gives

$$\Delta_\tau^2 F_1(u | v)(\xi) \geq \|\xi\|_V^2 - \|q_-\|_\infty,$$

since the L^1 remainder is nonnegative. For $\xi \notin V$, the quotient is ∞ , so the same estimate holds with $\|\xi\|_V := \infty$. The compact embedding and this estimate verify the hypotheses of Proposition 5.4, with $c = 1$ and $C = 1 + \|q_-\|_\infty$. $\square$

Consequently, Theorem 5.5 gives, for every $(\bar{u}, \bar{v}) \in \text{gph } \partial F_1$,

$$\text{uqg} F_1(\bar{u} | \bar{v}) = \inf_{\|\xi\|_2=1} d_{*,s}^2 F_1(\bar{u} | \bar{v})(\xi). \quad (6.8)$$

6.4 The strong strict second subderivative under saturation

At a fixed base, critical directions for the L^1 term are constrained by the multiplier on the zero set. In finite dimensions, the strict second subderivative of a convex polyhedral function is computed in [7, Proposition 4.1]. This formula is used in the proof of the chain rule in [8, Theorem 3.5]. The moving-base construction below removes the sign restriction when the multiplier is saturated on one side.

For every $(u, v) \in \text{gph } \partial F_1$, set

$$v_0 := v - \nabla \Psi(u) = v - qu + g.$$

Since $F_1 = G + \Psi$, Proposition 6.1 gives $v_0 \in \partial G(u)$ and, for every $\xi \in L^2(\Omega)$,

$$d_{*,s}^2 F_1(u | v)(\xi) = d_{*,s}^2 G(u | v_0)(\xi) + \int_\Omega q \xi^2 dx. \quad (6.9)$$

Thus the potential enters the strict curvature through the smooth addition rule, whereas the affine load has no second-order contribution.

The saturated graph condition gives an explicit strong strict second subderivative.

Theorem 6.6. *Let $(\bar{u}, \bar{v}) \in \text{gph } \partial F_1$, and let $\bar{\zeta}$ be the multiplier in (6.6). Suppose that, for some $\sigma \in \{-1, 1\}$,*

$$\sigma \bar{u} \geq 0 \quad \text{a.e. in } \Omega, \quad \bar{\zeta} = \sigma \quad \text{a.e. in } \Omega. \quad (6.10)$$

Then

$$d_{*,s}^2 F_1(\bar{u} \mid \bar{v})(\xi) = \begin{cases} \int_{\Omega} |\nabla \xi|^2 dx + \int_{\Omega} q \xi^2 dx, & \xi \in H_0^1(\Omega), \\ \infty, & \xi \notin H_0^1(\Omega). \end{cases} \quad (6.11)$$

Proof. Set $\bar{v}_0 := \bar{v} - q\bar{u} + g = A\bar{u} + \beta\bar{\zeta}$. By (6.9), it suffices to prove that

$$d_{*,s}^2 G(\bar{u} \mid \bar{v}_0)(\xi) = \begin{cases} \int_{\Omega} |\nabla \xi|^2 dx, & \xi \in H_0^1(\Omega), \\ \infty, & \xi \notin H_0^1(\Omega). \end{cases}$$

Since $G = Q + \beta J$, the calculation leading to (6.7) gives, for every $(u, v_0) \in \text{gph } \partial G$ and $\tau > 0$,

$$\Delta_{\tau}^2 G(u \mid v_0)(\xi) \geq \int_{\Omega} |\nabla \xi|^2 dx \quad \text{for } \xi \in H_0^1(\Omega).$$

If $\xi \notin H_0^1(\Omega)$, the quotient is infinite.

Let $(u_k, v_{0,k}, \tau_k, \xi_k)$ be admissible for $d_{*,s}^2 G(\bar{u} \mid \bar{v}_0)(\xi)$, and suppose that the lower limit of its quotients is finite. Passing to a subsequence, we may assume that the quotients converge to this lower limit. The preceding bound then shows that (ξ_k) is bounded in $H_0^1(\Omega)$. A further subsequence converges weakly in $H_0^1(\Omega)$. Its limit is ξ , since $\xi_k \rightarrow \xi$ strongly in $L^2(\Omega)$.

Weak lower semicontinuity therefore gives

$$\liminf_{k \rightarrow \infty} \Delta_{\tau_k}^2 G(u_k \mid v_{0,k})(\xi_k) \geq \int_{\Omega} |\nabla \xi|^2 dx.$$

If $\xi \notin H_0^1(\Omega)$, the same argument shows that a finite lower limit is impossible. This proves the lower bound.

For the recovery family, fix $\xi \in H_0^1(\Omega)$ and choose $\xi_n \in C_c^\infty(\Omega)$ such that $\xi_n \rightarrow \xi$ in $H_0^1(\Omega)$. Choose $\rho_n \in C_c^\infty(\Omega)$ satisfying

$$\rho_n \geq 0, \quad \rho_n \geq 1 \quad \text{on } \text{supp } \xi_n,$$

and then choose $\delta_n \downarrow 0$ so that

$$\delta_n (\|\rho_n\|_{H_0^1} + \|A\rho_n\|_2) \rightarrow 0.$$

Define

$$u_n := \bar{u} + \sigma \delta_n \rho_n, \quad \zeta_n := \sigma, \quad v_{0,n} := Au_n + \beta \zeta_n.$$

Since $\bar{u}, \rho_n \in D(A)$, one has $u_n \in D(A)$. Moreover, (6.10) gives $\zeta_n(x) \in \text{Sign}(u_n(x))$ almost everywhere. Hence $(u_n, v_{0,n}) \in \text{gph } \partial G$, and the choice of δ_n ensures

$$u_n \rightarrow \bar{u}, \quad v_{0,n} \rightarrow \bar{v}_0, \quad G(u_n) \rightarrow G(\bar{u}).$$

Choose $\tau_n \downarrow 0$ such that $\tau_n \|\xi_n\|_\infty \leq \delta_n/2$. On $\text{supp } \xi_n$,

$$\sigma(u_n + \tau_n \xi_n) \geq \delta_n - \tau_n |\xi_n| \geq \frac{\delta_n}{2}.$$

Outside this support, $\xi_n = 0$. Consequently,

$$|u_n + \tau_n \xi_n| - |u_n| - \tau_n \zeta_n \xi_n = 0 \quad \text{a.e. in } \Omega.$$

Thus

$$\Delta_{\tau_n}^2 G(u_n \mid v_{0,n})(\xi_n) = \int_{\Omega} |\nabla \xi_n|^2 dx \rightarrow \int_{\Omega} |\nabla \xi|^2 dx.$$

This proves the reverse inequality for G . Formula (6.9) then gives (6.11). $\square$

6.5 The modulus of uniform quadratic growth and a nonsmooth example

Set

$$A_q := A + M_q, \quad D(A_q) := D(A),$$

where M_q is defined in (6.4). The operator A_q is self-adjoint. Moreover, the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ implies that A_q has compact resolvent; see [4, Chapitre IX].

Corollary 6.7. *Under the assumptions of Theorem 6.6, we have*

$$\begin{aligned} \text{uqg}F_1(\bar{u} \mid \bar{v}) &= \min_{\substack{\xi \in H_0^1(\Omega) \\ \|\xi\|_2=1}} \left\{ \int_{\Omega} |\nabla \xi|^2 dx + \int_{\Omega} q \xi^2 dx \right\} \\ &= \lambda_1(A_q), \end{aligned} \tag{6.12}$$

where $\lambda_1(A_q)$ denotes the lowest eigenvalue of A_q .

Proof. Let m denote the infimum of the quadratic form in (6.12) over $\{\xi \in H_0^1(\Omega) : \|\xi\|_2 = 1\}$. Since $q \in L^\infty(\Omega)$, one has $m \in \mathbb{R}$. Formula (6.7) gives

$$F_1(u') \geq F_1(u) + \langle v, u' - u \rangle + \frac{m}{2} \|u' - u\|_2^2$$

for every $(u, v) \in \text{gph } \partial F_1$ and $u' \in L^2(\Omega)$. Thus m is admissible in the definition of the modulus, and $\text{uqg}F_1(\bar{u} \mid \bar{v}) \geq m$. Proposition 3.3 and Theorem 6.6 give the reverse inequality. Hence $\text{uqg}F_1(\bar{u} \mid \bar{v}) = m$.

A minimizing sequence for m is bounded in $H_0^1(\Omega)$ because q is bounded below. After extraction, it converges weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. The limit has L^2 norm one and minimizes the quadratic form, by weak lower semicontinuity of the Dirichlet energy and continuity of the potential term for strong L^2 convergence. Denote a minimizer by ξ . The first-order optimality condition for this constrained minimum, with the multiplier identified by taking $z = \xi$, gives

$$\int_{\Omega} \nabla \xi \cdot \nabla z dx + \int_{\Omega} q \xi z dx = m \int_{\Omega} \xi z dx \quad \text{for every } z \in H_0^1(\Omega).$$

Since $(m - q)\xi \in L^2(\Omega)$, the definition of $D(A)$ gives $\xi \in D(A)$ and $A_q \xi = m\xi$. The operator A_q is self-adjoint with compact resolvent, so this Rayleigh minimum is its lowest eigenvalue. This proves (6.12). $\square$

A compactly supported state gives a graph point at which the integral term is nondifferentiable and the preceding formulas apply.

Example 6.8. Choose $0 \neq \bar{u} \in C_c^\infty(\Omega)$ with $\bar{u} \geq 0$, and set

$$g := -\Delta \bar{u} + q\bar{u} + \beta.$$

Then $g \in L^2(\Omega)$ and

$$0 \in \partial F_1(\bar{u})$$

with the saturated multiplier $\bar{\zeta} = 1$ almost everywhere. The set $\{\bar{u} = 0\}$ contains a nonempty open subset. This subset has positive measure, and the subdifferential of

$$J(u) := \int_{\Omega} |u| dx$$

at $\bar{u}$ contains distinct elements obtained by choosing different values in $[-1, 1]$ on $\{\bar{u} = 0\}$. Hence $\partial J(\bar{u})$ is not a singleton, so J is not Fréchet differentiable at $\bar{u}$ as a functional on $L^2(\Omega)$. Nevertheless, Theorem 6.6 and Corollary 6.7 give

$$d_{*,s}^2 F_1(\bar{u} \mid 0)(\xi) = \int_{\Omega} |\nabla \xi|^2 dx + \int_{\Omega} q \xi^2 dx$$

for $\xi \in H_0^1(\Omega)$ and

$$\text{uqg} F_1(\bar{u} \mid 0) = \lambda_1(A_q).$$

The representation (6.8) remains valid for interior multipliers, but the present results do not give an explicit formula for the strict second subderivative or the modulus.

7 A sequential weak-direction relaxation and its limitations

We examine a sequential relaxation in which only the directional convergence is weakened by replacing the strong convergence by weak convergence. The base points and subgradients still converge in norm, and the base points remain f -attentive. The construction is used only to check whether weak convergence of directions can restore the growth-subderivative formula (1.7).

7.1 Definition and two opposite failures

For $(x, v) \in \text{gph } \partial f$, $\tau > 0$, and $\xi \in \mathcal{H}$, recall that

$$\Delta_{\tau}^2 f(x \mid v)(\xi) = \frac{2}{\tau^2} \left(f(x + \tau \xi) - f(x) - \tau \langle v, \xi \rangle \right).$$

The sequential relaxation replaces norm convergence of the directions by weak convergence.

Definition 7.1. For $(\bar{x}, \bar{v}) \in \text{gph } \partial f$ and $\xi \in \mathcal{H}$, define

$$d_{*,w}^2 f(\bar{x} \mid \bar{v})(\xi) := \liminf_{\substack{\xi_k \rightharpoonup \xi, \tau_k \downarrow 0 \\ (x_k, v_k) \rightarrow (\bar{x}, \bar{v}), v_k \in \partial f(x_k), f(x_k) \rightarrow f(\bar{x})}} \Delta_{\tau_k}^2 f(x_k \mid v_k)(\xi_k). \quad (7.1)$$

The lower limit is understood sequentially.

Every strongly admissible family is also weakly admissible. Hence

$$d_{*,w}^2 f(\bar{x} \mid \bar{v})(\xi) \leq d_{*,s}^2 f(\bar{x} \mid \bar{v})(\xi). \quad (7.2)$$

If $\xi_k \rightharpoonup \xi$ and $\|\xi_k\| \rightarrow \|\xi\|$, then $\xi_k \rightarrow \xi$ in norm. Thus, at a unit target direction, adding the condition $\|\xi_k\| \rightarrow 1$ reduces (7.1) to the strong construction.

The two examples below isolate the loss and the uncontrolled growth of norms.

Proposition 7.2. *The following statements hold.*

(i) *For the function on $\ell^2(\mathbb{N}^*)$ constructed in Section 4,*

$$d_{*,w}^2 f(0 \mid 0)(0) = 0, \quad d_{*,w}^2 f(0 \mid 0)(\xi) = 1 \quad \text{for every } \|\xi\| = 1.$$

Consequently,

$$\text{uqg} f(0 \mid 0) = 0 < \inf_{\|\xi\|=1} d_{*,w}^2 f(0 \mid 0)(\xi) = 1.$$

(ii) *Let $\mathcal{H}$ be infinite-dimensional, let $\mu > 0$, and set $h(x) := -\mu \|x\|^2/2$. Then*

$$\text{uqg} h(0 \mid 0) = \inf_{\|\xi\|=1} d_{*,s}^2 h(0 \mid 0)(\xi) = -\mu,$$

whereas

$$d_{*,w}^2 h(0 \mid 0)(\xi) = -\infty \quad \text{for every } \|\xi\| = 1.$$

Proof. For part (i), let (e_n) be the canonical basis of $\ell^2(\mathbb{N}^*)$ and set

$$x^{(n)} := r_n e_n, \quad v^{(n)} := \nabla f(x^{(n)}), \quad \tau_n := \frac{r_n}{16}.$$

Since $r_n = 2^{-n}$, one has $\tau_n \downarrow 0$. Moreover, $e_n \rightharpoonup 0$, the graph pairs $(x^{(n)}, v^{(n)})$ converge attentively to $(0, 0)$, and the computation in Proposition 4.2 gives

$$\Delta_{\tau_n}^2 f(x^{(n)} | v^{(n)})(e_n) = \varepsilon_n \longrightarrow 0.$$

Hence $d_{*,w}^2 f(0 | 0)(0) \leq 0$. Convexity makes every second-order quotient nonnegative, so equality holds.

Now fix $\|\xi\| = 1$. Inequality (7.2) and Proposition 4.3 give $d_{*,w}^2 f(0 | 0)(\xi) \leq 1$. Consider any family admissible in (7.1). Thus $\xi_k \rightharpoonup \xi$, and consequently $M := 1 + \sup_k \|\xi_k\| < \infty$. Moreover, $v_k = \nabla f(x_k)$ by Lemma 4.1.

Fix $\eta > 0$ and choose N such that $\sum_{n>N} \xi_n^2 < \eta$. Set $r_* := r_N$. Since $x_k \rightarrow 0$ and $\tau_k \downarrow 0$, for all sufficiently large k ,

$$\max_{1 \leq n \leq N} |x_{k,n}| \leq \frac{r_*}{4}, \quad \tau_k M \leq \frac{r_*}{4}.$$

Therefore, for $0 \leq t \leq \tau_k$ and $n \leq N$,

$$|x_{k,n} + t\xi_{k,n}| \leq \frac{r_*}{2} \leq \frac{r_n}{2}.$$

Formula (4.1) gives $a_n(x_{k,n} + t\xi_{k,n}) = 1$ on these coordinates. Hence (4.5) yields

$$\Delta_{\tau_k}^2 f(x_k | v_k)(\xi_k) \geq \sum_{n \leq N} \xi_{k,n}^2.$$

Weak convergence gives coordinatewise convergence, and therefore

$$\liminf_{k \rightarrow \infty} \Delta_{\tau_k}^2 f(x_k | v_k)(\xi_k) \geq \sum_{n \leq N} \xi_n^2 > 1 - \eta.$$

The admissible family was arbitrary. Letting $\eta \downarrow 0$ proves $d_{*,w}^2 f(0 | 0)(\xi) \geq 1$, and hence part (i).

For part (ii), direct calculation gives

$$\Delta_{\tau}^2 h(x | -\mu x)(\xi) = -\mu \|\xi\|^2.$$

It follows that

$$\text{uqgh}(0 | 0) = -\mu, \quad d_{*,s}^2 h(0 | 0)(\xi) = -\mu \quad \text{for every } \|\xi\| = 1.$$

Fix such a vector ξ and choose an orthonormal sequence $(e_k) \subset \{\xi\}^\perp$. For each integer $m \geq 1$, set

$$\xi_k^{(m)} := \xi + m e_k.$$

Then $\xi_k^{(m)} \rightharpoonup \xi$ and $\|\xi_k^{(m)}\|^2 = 1 + m^2$. Together with $x_k = v_k = 0$ and $\tau_k = 1/k$, these sequences form an admissible family in (7.1). Consequently,

$$d_{*,w}^2 h(0 | 0)(\xi) \leq -\mu(1 + m^2) \quad \text{for every } m \geq 1.$$

Letting $m \rightarrow \infty$ proves part (ii). $\square$

The preceding proposition isolates two distinct failures of the sequential weak relaxation. In part (i), the escaping unit directions are detected only at the weak limit 0, after their norm has been lost. In part (ii), each weakly convergent directional sequence is bounded, but no uniform bound is imposed across the families entering the lower envelope. Consequently, on the unit sphere, the weak relaxation may remain strictly above the growth modulus or collapse to $-\infty$.

8 Conclusion and perspectives

The extent to which the modulus of uniform quadratic growth can be captured in infinite dimensions by the a strict second-subderivative, as in finite dimensions, has been explored from several angles through counterexamples and analysis of sufficient compactness conditions. A characterization by the localized quotients gives

$$\text{uqg}f(\bar{x} \mid \bar{v}) \leq \inf_{\|\xi\|=1} d_{*,s}^2 f(\bar{x} \mid \bar{v})(\xi),$$

but an ℓ^2 example is found that proves the inequality can be strict. In fact, the unit directions that produce small quotients in this example do not possess any convergent subsequence in the norm topology, which leads to identify a directional compactness condition that restores the equality. A quadratic example with compact perturbation shows that this condition, always satisfied in finite dimensions, is not necessary, but just sufficient for the equality to hold in infinite dimensions.

In a further example involving L^1 -penalized Dirichlet energy on $L^2(\Omega)$, the directional compactness and the representation formula are shown to be on hand at every point of the graph of the subdifferential. Under a same-sign global saturation of the multiplier, the strong strict second subderivative is given by the quadratic form associated with A_q , and we have

$$\text{uqg}F_1(\bar{u} \mid \bar{v}) = \lambda_1(A_q).$$

Example 6.8 gives this identity at a state where the L^1 -term is not differentiable. The examples of Section 7 demonstrate that replacing strong convergence of directions by weak convergence would not restore the growth-subderivative formula, in general.

Many open questions need further investigations. The one is how to calculate the strong strict second subderivative and the corresponding modulus when the multiplier is not saturated on the set where $\bar{u}$ vanishes. The second is how to find a verifiable condition weaker than the directional compactness that still ensures the growth-subderivative formula for nonsmooth functions. These questions are out of the scope of the current manuscript and will be the subject of a future project of research.

References

- [1] S. Adly, H. Attouch, and R. T. Rockafellar. *Preservation or not of the maximally monotone property by graph-convergence*. J. Convex Anal. **30** (2023), 413–440.
- [2] S. Adly and R. T. Rockafellar. *Variational modulus of convexity under Moreau envelopes: attenuation, recovery, and oracle tuning*. Accepted for publication in SIAM J. Optim., 2026. Preprint available at HAL: [hal-05563788v2](https://hal.archives-ouvertes.fr/hal-05563788v2).
- [3] F. Bernard and L. Thibault. *Prox-regular functions in Hilbert spaces*. J. Math. Anal. Appl. **303** (2005), 1–14.
- [4] H. Brézis. *Analyse fonctionnelle : théorie et applications*. Masson, Paris, 1983.
- [5] J. G. Garrido. *Characterization of strong local maximal monotonicity through graphical derivatives*. Set-Valued Var. Anal. **33** (2025), article 35.
- [6] H. Gfrerer. *On second-order variational analysis of variational convexity of prox-regular functions*. Set-Valued Var. Anal. **33** (2025), article 8.
- [7] N. T. V. Hang, W. Jung, and M. E. Sarabi. *Role of subgradients in variational analysis of polyhedral functions*. J. Optim. Theory Appl. **200** (2024), 1160–1192.

- [8] N. T. V. Hang and M. E. Sarabi. *A chain rule for strict twice epi-differentiability and its applications*. SIAM J. Optim. **34** (2024), 918-945.
- [9] P. D. Khanh, B. S. Mordukhovich, V. T. Phat. *Variational convexity of functions and variational sufficiency in optimization*. SIAM J. Optim. **33** (2023), 1121-1158.
- [10] P. D. Khanh, B. S. Mordukhovich, V. T. Phat, and L. D. Viet. *Characterizations of variational convexity and tilt stability via quadratic bundles*. J. Convex Anal. **33** (2026), 903-924.
- [11] P. D. Khanh, V. V. H. Khoa, B. S. Mordukhovich, and V. T. Phat. *Variational and strong variational convexity in infinite-dimensional variational analysis*. SIAM J. Optim. **34** (2024), 2756-2787.
- [12] N. Kato. *On the second derivatives of convex functions on Hilbert spaces*. Proc. Amer. Math. Soc. **106** (1989), 697-705.
- [13] A. B. Levy, R. A. Poliquin, and L. Thibault. *Partial extensions of Attouch's theorem with applications to proto-derivatives of subgradient mappings*. Trans. Amer. Math. Soc. **347** (1995), 1269-1294.
- [14] B. S. Mordukhovich. *Variational Analysis and Generalized Differentiation, I: Basic Theory*. Springer, Berlin, 2006.
- [15] B. S. Mordukhovich and T. T. A. Nghia. *Second-order variational analysis and characterizations of tilt-stable optimal solutions in infinite-dimensional spaces*. Nonlinear Anal. **86** (2013), 159-180.
- [16] W. Ouyang and A. Milzarek. *Variational properties of decomposable functions. Part I: Strict epi-calculus and applications*. (2024). [arXiv:2311.07267v2](https://arxiv.org/abs/2311.07267v2).
- [17] R. A. Poliquin, R. T. Rockafellar. *Prox-regular functions in variational analysis*. Trans. Amer. Math. Soc. **348** (1996), 1805-1838.
- [18] R. A. Poliquin, R. T. Rockafellar. *Tilt stability of a local minimum*. SIAM J. Optim. **8** (1998), 287-299.
- [19] R. T. Rockafellar. *Variational convexity and prox-regularity*. J. Convex Anal. **33** (2026), 1049-1060.
- [20] R. T. Rockafellar. *Derivative tests for prox-regularity and the modulus of convexity*. J. Convex Anal. **33** (2026), 243-266.

Acknowledgements

The first author gratefully acknowledges support from the ECOS Project C24E06.