Maximal monotonicity of piecewise polyhedral mappings

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Abstract

We study maximal monotonicity under graph operations for set-valued mappings whose graphs are finite unions of polyhedral convex sets. Classical preservation rules for maximal monotone mappings, including restriction of arguments, pullback by a linear map, and addition, are usually stated under relative-interior constraint qualifications. We show that, in the piecewise polyhedral setting, these relative-interior assumptions can be replaced by the corresponding nonempty-intersection assumptions.

Keywords. Maximal monotone mappings; piecewise polyhedral multifunctions; resolvents; Minty's theorem; firmly nonexpansive mappings.

1 Introduction

Piecewise linear-quadratic convex models naturally give rise to many set-valued relations with piecewise polyhedral graphs. Important examples include subdifferentials of such functions, normal cone mappings to polyhedral convex sets, and the relations arising in dual or primal-dual proximal-point schemes [4]. See also, for example, Rockafellar and Wets [5, Propositions 12.29–12.30 and Example 12.31]. These constructions go back to the classical convex analysis of subdifferentials and normal cones [3]. In such settings, monotonicity is usually inherited directly from convexity or from an underlying saddle-point structure. The subtler question is maximality, especially when new relations are constructed through graph operations.

For general maximal monotone mappings, the standard preservation theorems for linear composition, addition, and restriction of arguments are already available in the classical theory, but they are stated under relative-interior constraint qualifications [5, Theorem 12.43, Corollary 12.44, and Exercise 12.46]. The purpose of this paper is to address the following question:

> *When the underlying mappings are piecewise polyhedral, can the classical relative-interior assumptions be replaced by nonempty intersection assumptions?*

The results of this paper can therefore be viewed as sharpened finite-cell versions of the preservation rules in Section 12F of [5], rather than as replacements for the general theory.

Our approach to the above question is based on a projection theorem. Let H be a finite-dimensional real Hilbert space with an orthogonal decomposition

$$H = H_* \oplus H_{**},$$

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and let $T : H \rightrightarrows H$ be maximal monotone with piecewise polyhedral graph. Define $T_* : H_* \rightrightarrows H_*$ by

$$w_* \in T_*(z_*) \quad \Longleftrightarrow \quad \exists z_{**} \in H_{**} \text{ such that } (w_*, 0) \in T(z_*, z_{**}). \tag{1}$$

We prove that if

$$\exists (z_*, z_{**}, w_*) \in H_* \times H_{**} \times H_* \quad \text{such that} \quad (w_*, 0) \in T(z_*, z_{**}), \tag{2}$$

then T_* is maximal monotone.

The proof is resolvent-based. By Minty's parametrization [2], maximal monotonicity is equivalent to the resolvent domain being the whole ambient space. Since T is maximal monotone and $\operatorname{gph} T$ is piecewise polyhedral, its resolvent

$$R := (I + T)^{-1}$$

is a single-valued firmly nonexpansive mapping and is piecewise affine on H. We show that the resolvent of T_* can then be described through the fixed-point condition in the eliminated component. Then, the key step is a global fixed-point lemma for firmly nonexpansive piecewise affine mappings: if this fixed-point set is nonempty, then its projection onto H_* is all of H_*. This directly yields the full Minty domain and thus maximality for T_*.

The projection theorem is the basic step from which the other preservation rules follow. Restriction of arguments is obtained by applying the projection result to an inverse, while linear composition is reduced to restriction on the range of the linear map. The addition theorem then follows by applying the composition result to the product mapping along the diagonal. Therefore, once the projection theorem is established, the finite-cell versions of the standard preservation results follow systematically.

The remainder of the paper is organized as follows. Section 2 fixes the notation and collects the preliminary facts on projected graph slices and resolvents. Section 3 proves the projection theorem, which is the main technical result. Section 4 derives from it the preservation rules for restriction of arguments, linear pullbacks, and addition. In section 5, we conclude the paper.

2 Preliminaries

We collect here the notation and elementary facts used in the sequel. A subset of a finite-dimensional Hilbert space is called piecewise polyhedral if it is a finite union of convex polyhedra. A set-valued mapping is called piecewise polyhedral when its graph is piecewise polyhedral. A single-valued mapping is called piecewise affine if its domain can be covered by finitely many polyhedra on each of which the mapping is affine.

For any mapping M, we write $\operatorname{gph} M$ and $\operatorname{dom} M$ for its graph and its domain. For a set-valued mapping F, its range is

$$\operatorname{range} F := \bigcup_{x \in \operatorname{dom} F} F(x).$$

When F is single-valued, this is just the usual image of F. Throughout the paper, an orthogonal decomposition is written as

$$H = H_* \oplus H_{**}.$$

The associated orthogonal projections are denoted by P_* and P_{**}, and the identity mappings on H, H_*, and H_{**} are denoted by I, I_*, and I_{**}, respectively. The same convention will be used with other ambient spaces when they are introduced locally in a statement. If $V \subset H$ is a subspace, then

$$J_V : V \hookrightarrow H$$

denotes the canonical inclusion, that is, $J_V v = v$ for every $v \in V$. Its Hilbert-space adjoint is the orthogonal projection onto V:

$$J_V^* = P_V : H \to V.$$

When the subspace is clear from context, we write simply J instead of J_V. Let $S \subset H$ be a piecewise polyhedral subset. A finite collection $\mathcal{P} = \{P_1, \dots, P_N\}$ of nonempty convex polyhedra is called a polyhedral subdivision of S if

$$S = \bigcup_{i=1}^{N} P_i, \qquad \operatorname{ri} P_i \cap \operatorname{ri} P_j = \emptyset \quad \text{whenever } i \neq j.$$

We first record a simple graph-projection fact. It will be applied in the main theorem to the projected relation T_*, but the statement is written in a general form for later reference.

Proposition 2.1 (Projected graph slices)**.** *Let $Z = Z_* \oplus Z_{**}$ be a finite-dimensional Hilbert space, and let $M : Z \rightrightarrows Z$ be monotone and piecewise polyhedral. Define $M_* : Z_* \rightrightarrows Z_*$ by*

$$v_* \in M_*(z_*) \iff \exists z_{**} \in Z_{**} \text{ such that } (v_*, 0) \in M(z_*, z_{**}).$$

Then M_* is monotone and piecewise polyhedral.

Proof. Let $(z_i, v_i) \in \operatorname{gph} M_*$, $i = 1, 2$. By definition, there are $z_{i,**} \in Z_{**}$ such that

$$(v_i, 0) \in M(z_i, z_{i,**}).$$

The monotonicity of M gives

$$\big\langle (z_1, z_{1,**}) - (z_2, z_{2,**}), (v_1, 0) - (v_2, 0) \big\rangle \geq 0,$$

and hence

$$\langle z_1 - z_2, v_1 - v_2 \rangle \geq 0.$$

Thus M_* is monotone.

Moreover,

$$\operatorname{gph} M_* = \big\{ (z_*, v_*) : \exists z_{**} \in Z_{**} \text{ such that } (z_*, z_{**}, v_*, 0) \in \operatorname{gph} M \big\}.$$

This is the linear projection of the intersection of $\operatorname{gph} M$ with the subspace $Z_* \oplus Z_{**} \oplus Z_* \oplus \{0\}$. Since $\operatorname{gph} M$ is a finite union of polyhedra, $\operatorname{gph} M_*$ is again a finite union of polyhedra. Hence M_* is piecewise polyhedral. □

The next two facts are standard consequences of Minty's parametrization. Specifically, Minty's criterion detects maximality, while the polyhedral resolvent lemma supplies the piecewise affine structure of resolvents.

Lemma 2.2 (Minty resolvent criterion)**.** *Let $M : Z \rightrightarrows Z$ be monotone on a finite-dimensional Hilbert space Z. Then, on its domain, the resolvent $(I + M)^{-1}$ is single-valued and firmly nonexpansive. Moreover, M is maximal monotone if and only if*

$$\operatorname{dom}(I + M)^{-1} = Z.$$

Proof. This is Minty's theorem in finite dimensions. See Minty [2] and [5, Theorem 12.12]. $\square$

Lemma 2.3 (Polyhedral resolvent domain)**.** *Let $M : Z \rightrightarrows Z$ be monotone and piecewise polyhedral on a finite-dimensional Hilbert space Z. Then $D_M := \operatorname{dom}(I+M)^{-1}$ is a closed piecewise polyhedral subset of Z. The resolvent $(I+M)^{-1}$ is affine on each member of a finite polyhedral subdivision of D_M.*

Proof. Write $\operatorname{gph} M = \bigcup_{i=1}^{N} C_i$, where each C_i is a convex polyhedron in $Z \times Z$. For each i, define

$$D_i := \{x + u \mid (x, u) \in C_i\}.$$

Then each D_i is a polyhedron, and

$$D_M = \bigcup_{i=1}^{N} D_i.$$

Hence D_M is a finite union of polyhedra, and in particular is closed.

For the affine structure, set

$$G_i := \{(a, x) \in Z \times Z \mid (x, a - x) \in C_i\}.$$

Then G_i is a convex polyhedron and its projection onto the first component is D_i. By Lemma 2.2, the resolvent is single-valued on D_M. Hence G_i is the graph of the restriction of $(I + M)^{-1}$ to D_i. Since G_i is convex, this restriction is affine on D_i. Taking a common finite polyhedral refinement of the sets D_i gives the claimed subdivision. $\square$

3 The Projection Theorem

We now return to the projection operation introduced in (1). Throughout this section we fix the orthogonal decomposition

$$H = H_* \oplus H_{**},$$

and use the notation introduced in Section 2. The mapping $T : H \rightrightarrows H$ is assumed to be maximal monotone and piecewise polyhedral, and $T_* : H_* \rightrightarrows H_*$ is defined by (1). We also assume the nonempty slice condition (2).

By Proposition 2.1, the projected relation T_* is monotone and piecewise polyhedral. Thus maximality is the only remaining issue. We prove it by passing to Minty coordinates. The maximality of T gives the firmly nonexpansive resolvent

$$R := (I + T)^{-1} : H \to H.$$

Then Lemma 2.3 implies that R is piecewise affine on H. The next lemma identifies the resolvent domain of T_* as a projected fixed-point set for R.

Lemma 3.1 (Projected resolvent representation)**.** *Let $R_* := (I_* + T_*)^{-1}$. For $a_*, x_* \in H_*$, the following are equivalent:*

(i) $x_* = R_* a_*$;

(ii) *there exists $b_{**} \in H_{**}$ such that*

$$R(a_*, b_{**}) = (x_*, b_{**}).$$

Consequently,

$$\operatorname{dom} R_* = P_*\Big\{(a_*, b_{**}) \in H_* \oplus H_{**} \;\Big|\; P_{**}R(a_*, b_{**}) = b_{**}\Big\}. \tag{3}$$

Proof. The condition $x_* = R_* a_*$ means

$$a_* - x_* \in T_*(x_*).$$

By the definition of T_*, this is equivalent to the existence of $b_{**} \in H_{**}$ such that

$$(a_* - x_*, 0) \in T(x_*, b_{**}).$$

Adding (x_*, b_{**}) gives

$$(a_*, b_{**}) \in (I + T)(x_*, b_{**}),$$

which is equivalent to

$$R(a_*, b_{**}) = (x_*, b_{**}).$$

The converse follows by reversing these implications. Eliminating x_* gives (3). □

From Lemmas 2.2 and 3.1, we see that to show the maximality of T_*, it suffices to show that for every $a_* \in H_*$, there exists $b_{**} \in H_{**}$ such that

$$P_{**}R(a_*, b_{**}) = b_{**}.$$

Thus, the maximality of T_* is reduced to a fixed-point statement for firmly nonexpansive piecewise affine mappings. This is precisely where the piecewise polyhedral structure enters the argument. Piecewise affineness implies that the displacement range of the sliced map is a finite union of polyhedra and hence is closed. On the other hand, maximal monotonicity yields the near convexity of this range, which combined with closedness implies the convexity. These two facts make it possible to apply a separation argument. The following lemma proves the required fixed-point claim.

Lemma 3.2 (Global projected fixed-point lemma)**.** *Let X and Y be finite-dimensional Hilbert spaces, and let $R : X \oplus Y \to X \oplus Y$ be firmly nonexpansive and piecewise affine. Define*

$$K := \{(a, b) \in X \oplus Y \mid P_Y R(a, b) = b\}.$$

If $K \neq \emptyset$, then

$$P_X K = X.$$

Proof. Choose $(\bar{a}, \bar{b}) \in K$. Thus

$$P_Y R(\bar{a}, \bar{b}) = \bar{b}.$$

Fix an arbitrary $a \in X$. We prove that there exists $b \in Y$ such that $P_Y R(a, b) = b$. Define

$$Q_a : Y \to Y, \qquad Q_a(b) := P_Y R(a, b),$$

and set

$$G_a := I_Y - Q_a.$$

We first verify directly that G_a is firmly nonexpansive. Take $b_1, b_2 \in Y$, and write

$$R(a, b_i) = (x_i, y_i), \qquad i = 1, 2.$$

Set
$$d := b_1 - b_2, \qquad q := y_1 - y_2.$$
It holds from the firmly nonexpansiveness of R that
$$\|x_1 - x_2\|^2 + \|q\|^2 \leq \langle q, d\rangle,$$
i.e., $\|q\|^2 \leq \langle q, d\rangle$. On the other hand,
$$\begin{aligned}
&\langle G_a(b_1) - G_a(b_2),\, b_1 - b_2\rangle - \|G_a(b_1) - G_a(b_2)\|^2 \\
&\quad = \langle d - q, d\rangle - \|d - q\|^2 \\
&\quad = \langle q, d\rangle - \|q\|^2 \geq 0.
\end{aligned}$$
Thus
$$\|G_a(b_1) - G_a(b_2)\|^2 \leq \langle G_a(b_1) - G_a(b_2),\, b_1 - b_2\rangle,$$
so G_a is firmly nonexpansive. In particular, G_a is continuous and monotone with $\operatorname{dom} G_a = Y$. By the standard full-domain maximality criterion for continuous monotone mappings [1, Corollary 20.28], G_a is maximal monotone on Y. Moreover, since R is piecewise affine, the definition of G_a immediately implies that G_a is piecewise affine. Thus $\operatorname{range} G_a$ is a finite union of polyhedra and is therefore closed, which together with the nearly convexity of the range of G_a [5, Theorem 12.41], implies that
$$C := \operatorname{range} G_a$$
is a closed convex subset of Y.

Suppose, for contradiction, that Q_a has no fixed point. Then
$$0 \notin C.$$
By the strong separation theorem [3, Section 11], there exist $v \in Y \setminus \{0\}$ and $\gamma > 0$ such that
$$\langle c, v\rangle \geq \gamma \qquad \forall c \in C. \tag{4}$$
Since $G_a(b) \in C$ for every $b \in Y$, this yields
$$\langle G_a(b), v\rangle \geq \gamma \qquad \forall b \in Y. \tag{5}$$
For $t > 0$, set
$$e := \frac{v}{\|v\|}, \qquad b_t := \bar{b} - te.$$
Using $G_a = I_Y - Q_a$, we have
$$Q_a(b_t) - \bar{b} = b_t - \bar{b} - G_a(b_t) = -te - G_a(b_t).$$
Therefore, by (5),
$$\begin{aligned}
\|Q_a(b_t) - \bar{b}\| &\geq \left\langle Q_a(b_t) - \bar{b}, -e\right\rangle \\
&= t + \frac{\langle G_a(b_t), v\rangle}{\|v\|} \\
&\geq t + \frac{\gamma}{\|v\|}.
\end{aligned}$$

On the other hand, because R is nonexpansive and $P_Y R(\bar{a}, \bar{b}) = \bar{b}$,

$$\begin{aligned}
\|Q_a(b_t) - \bar{b}\| &= \|P_Y R(a, b_t) - P_Y R(\bar{a}, \bar{b})\| \\
&\le \|R(a, b_t) - R(\bar{a}, \bar{b})\| \\
&\le \|(a, b_t) - (\bar{a}, \bar{b})\| \\
&= \sqrt{\|a - \bar{a}\|^2 + t^2}.
\end{aligned}$$

However, for all sufficiently large t, it holds that

$$\sqrt{\|a - \bar{a}\|^2 + t^2} < t + \frac{\gamma}{\|v\|},$$

which contradicts the two preceding estimates. Hence Q_a has a fixed point. Thus there exists $b \in Y$ such that $P_Y R(a, b) = b$. Since $a \in X$ was arbitrary, $P_X K = X$. □

We can now complete the proof of the projection theorem.

Theorem 3.3 (Projected maximal monotonicity)**.** *Let $T : H \rightrightarrows H$ be maximal monotone and piecewise polyhedral, and suppose that the slice condition (2) holds. Then the projected relation T_* defined in (1) is maximal monotone on H_*.*

Proof. By Proposition 2.1, T_* is monotone and piecewise polyhedral. Let

$$R_* := (I_* + T_*)^{-1}, \qquad D_* := \operatorname{dom} R_*.$$

By Lemma 3.1,

$$D_* = P_* K, \qquad K = \{(a_*, b_{**}) \mid P_{**} R(a_*, b_{**}) = b_{**}\},$$

where $R = (I + T)^{-1}$.

The slice condition (2) implies $K \neq \emptyset$. Indeed, if $(w_*, 0) \in T(z_*, z_{**})$, then

$$R(z_* + w_*, z_{**}) = (z_*, z_{**}),$$

and hence $(z_* + w_*, z_{**}) \in K$.

Since T is maximal monotone, R is firmly nonexpansive on all of H. Since $\operatorname{gph} T$ is piecewise polyhedral, Lemma 2.3 shows that R is piecewise affine. Applying Lemma 3.2 with

$$X = H_*, \qquad Y = H_{**},$$

we obtain

$$D_* = P_* K = H_*.$$

By Lemma 2.2, T_* is maximal monotone. □

4 Preservation Operations

The projection theorem leads to the following preservation rules. They are piecewise polyhedral counterparts of the classical results in [5, Theorem 12.43, Corollary 12.44, and Exercise 12.46]. Indeed, the relative-interior constraint qualifications appearing there are replaced below by nonempty intersection conditions.

Theorem 4.1. *The following assertions hold.*

(a) ***Restriction of arguments.*** *Let $M : H \rightrightarrows H$ be maximal monotone and piecewise polyhedral, let V be a subspace of H, and write $J := J_V$ for the canonical inclusion defined in Section 2. If*

$$V \cap \operatorname{dom} M \neq \emptyset,$$

then

$$J^*MJ : V \rightrightarrows V$$

is maximal monotone and piecewise polyhedral.

(b) ***Composition with a linear map.*** *Let $M : H \rightrightarrows H$ be maximal monotone and piecewise polyhedral, and let $L : K \to H$ be linear. If*

$$L(K) \cap \operatorname{dom} M \neq \emptyset,$$

then

$$L^*ML : K \rightrightarrows K$$

is maximal monotone and piecewise polyhedral.

(c) ***Addition.*** *Let $M_1, M_2 : H \rightrightarrows H$ be maximal monotone and piecewise polyhedral. If*

$$\operatorname{dom} M_1 \cap \operatorname{dom} M_2 \neq \emptyset,$$

then

$$M_1 + M_2$$

is maximal monotone and piecewise polyhedral.

Proof. For (a), split $H = V \oplus V^\perp$. By the notation introduced in Section 2, $J = J_V$ and $J^* = P_V$. Hence the pullback J^*MJ is the restriction of the argument of M to V, followed by projection of the values back to V:

$$\operatorname{gph}(J^*MJ) = \{(x, J^*u) : x \in V,\ u \in Mx\}.$$

Apply Theorem 3.3 to $T = M^{-1}$, with the decomposition $H = V \oplus V^\perp$. The projected relation associated with T is the inverse of J^*MJ: indeed, for $x, w \in V$,

$$w \in T_*(x) \iff \exists u_\perp \in V^\perp \text{ such that } (w, 0) \in M^{-1}(x, u_\perp) \iff w \in (J^*MJ)^{-1}(x).$$

The slice condition in Theorem 3.3 is precisely $V \cap \operatorname{dom} M \neq \emptyset$. Thus T_* is maximal monotone and piecewise polyhedral on V, and the same is true of its inverse J^*MJ.

For (b), let $N := \ker L$, $V := L(K)$, and choose the orthogonal decomposition $K = N \oplus N^\perp$. The restriction

$$L_0 := L|_{N^\perp} : N^\perp \to V$$

is a linear isomorphism. Since $V = L(K)$ and $L(K) \cap \operatorname{dom}(M) \neq \emptyset$, we know from part (a) that $R := J^*MJ$ is maximal monotone and piecewise polyhedral on V. Pulling back R by the isomorphism L_0 gives the maximal monotone piecewise polyhedral mapping

$$L_0^* R L_0 : N^\perp \rightrightarrows N^\perp.$$

Under the decomposition $K = N \oplus N^\perp$, every $k \in K$ can be written as $k = n + \xi$, with $n \in N$ and $\xi \in N^\perp$. Since $Ln = 0$, the operator L^*ML depends only on ξ, and its values lie in $N^\perp$. More precisely,

$$\operatorname{gph}(L^*ML) = \big\{((n, \xi), (0, s)) \mid n \in N,\ (\xi, s) \in \operatorname{gph}(L_0^* R L_0)\big\}.$$

Hence L^*ML is maximal monotone and piecewise polyhedral on K.

For (c), let

$$\mathcal{M} := M_1 \times M_2 : H \times H \rightrightarrows H \times H, \qquad \mathcal{M}(x_1, x_2) := M_1(x_1) \times M_2(x_2).$$

Then $\mathcal{M}$ is maximal monotone and piecewise polyhedral, since

$$\operatorname{gph} \mathcal{M} = \operatorname{gph} M_1 \times \operatorname{gph} M_2.$$

Now define the diagonal embedding

$$L : H \to H \times H, \qquad Lx = (x, x).$$

Its adjoint is

$$L^*(u_1, u_2) = u_1 + u_2.$$

Since $\operatorname{dom} M_1 \cap \operatorname{dom} M_2 \neq \emptyset$, we have from the definition of L and $\mathcal{M}$ that

$$L(H) \cap \operatorname{dom} \mathcal{M} \neq \emptyset.$$

Thus, part (b) asserts that $L^*\mathcal{M}L$ is maximal monotone and piecewise polyhedral.

Finally, for every $x \in H$,

$$\begin{aligned} L^*\mathcal{M}L(x) &= \{u_1 + u_2 \mid u_1 \in M_1(x),\ u_2 \in M_2(x)\} \\ &= (M_1 + M_2)(x). \end{aligned}$$

Hence, $M_1 + M_2$ is maximal monotone and piecewise polyhedral. □

5 Concluding Remarks

In this paper, we show that the piecewise polyhedral structure allows one to replace the relative-interior constraint qualifications in standard maximal-monotonicity preservation rules by the corresponding nonempty-intersection assumptions. This provides a simple calculus for constructing new maximal monotone piecewise polyhedral mappings from known ones. After a monotone piecewise polyhedral relation has been identified, maximal monotonicity is preserved under projection, restriction, pullback, and addition as soon as the corresponding nonempty feasibility conditions hold. This provides a useful tool for analyzing polyhedral multifunctions arising in variational inequalities, normal-cone systems, complementarity models, and proximal or primal-dual algorithms.

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