VARIATIONAL SUFFICIENCY AND SOLUTION STABILITY IN OPTIMIZATION

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Abstract

Variational stability, in the sense of local good behavior of optimal values and solutions in problems of optimization under shifts in parameters, is important not only for validating model robustness in practical applications but also for confidence of outcomes in the design of solution algorithms. Fundamental results are presented here about how such stability relates to a recently developed sufficient condition for local optimality called strong variational sufficiency.

Keywords: *local optimality, variational stability, tilt stability, full stability, prox-regularity, variational convexity, variational sufficiency, second-order variational analysis.*

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Dedicated to the memory of Hédy Attouch

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1 Introduction

In classical analysis, where problems are so often posed in terms of equations, the importance of understanding how a solution might depend on parameters in the equations has long been recognized. The implicit function theorem addresses differentiable dependence, while concepts of well-posedness look to continuous dependence as an essential underpinning for a good mathematical model in applications. In variational analysis, as the extension of classical analysis for handling problems of optimization, the counterpart is understanding how optimal solutions and optimal values in a local sense might depend on parameters. Again, besides being of interest directly in many applications, this is fundamental to the development of numerical methodology and whether a problem is sufficiently well posed in solvability. Inexactness in computations can often be tied to errors in evaluating parameters, which amount to errors in problem identification, yet it's important for an algorithm still to produce solutions close to exact ones. Not only continuous dependence on parameters but also Lipschitz continuous dependence comes in then, as corresponding to solution shifts being in proportion to error sizes.

These motivations have led to decades of research on stability of optimal solutions and optimal values in which "stability" refers to Lipschitz continuous dependence. Two papers in particular loom large in the picture of what will be undertaken here: Poliquin-Rockafellar [11] for "tilt" stability (1998) and Levy-Poliquin-Rockafellar [4] for "full" stability (2000). In both, under assumptions of prox-regularity and subdifferential continuity, a condition characterizing the concept was identified that utilized coderivatives of associated subgradient mappings. Those conditions were tantamount to enhanced second-order sufficient conditions for local optimality. Subsequent contributions on those lines in [8], [5], [6], and elsewhere, elaborated the applicability of those coderivative conditions in particular settings. Our paper [1] paired optimal solutions with multiplier vectors in "primal-dual" stability. It also appealed to a newer form of second-order optimality condition, called "strong variational sufficiency," as a substitute for coderivatives in a central role, and that will be carried further here. Variational sufficiency, which originated in [14] (2019), has been shown to induce a kind of local duality [15] and to support solution methods like augmented Lagrangian algorithms in nonconvex optimization [16]. In nonlinear programming it's equivalent to the classical strong second-order sufficient condition, as established in [15].

Our main results in this paper will reveal what more needs to be combined with strong variational sufficiency to get full stability, and on the other hand, the lesser kind of stability that strong variational sufficiency can guarantee just from a constraint qualification. Prox-regularity will be in the forefront, as always, but subdifferential continuity as a separate assumption will be avoided. New facts about prox-regularity will be brought out which are based on, and add to, the connections with variational convexity in the recent papers [17], [18].[3]

As in our previous work [1], we adopt a very broad format for parameterized optimization in finite dimensions where the given problem is to

$$\bar{\mathcal{P}} \qquad \text{minimize } \ \varphi(x,0) \ \text{ with respect to } x$$

for a closed proper function φ on $I\!R^n \times I\!R^m$ and the focusis on local optimality issues around a particular $\bar{x}$. We view $\bar{P}$ as embedded in a parameterized family of problems

$$\mathcal{P}(v,u) \qquad \text{minimize } \ \varphi(x,u) - v{\cdot}[x-\bar{x}] \ \text{ with respect to } x$$

as variants, so that

$$\bar{\mathcal{P}} = \mathcal{P}(\bar{v},\bar{u}) \ \text{ for } \ (\bar{v},\bar{u}) = (0,0).$$

[3] All of these topics in second-order variational analysis were deeply influenced in their evolution by the pioneering work of H. Attouch that tied epiconvergence of functions to graphical convergence of their subgradient mappings.

We invesigate, for $\delta > 0$, the properties of

$$\begin{aligned} M_\delta(v,u) &:= \operatorname*{argmin}_{|x-\bar{x}|<\delta}\{\,\varphi(x,u) - v{\cdot}[x-\bar{x}]\},\\ m_\delta(v,u) &:= \min_{|x-\bar{x}|<\delta}\{\,\varphi(x,u) - v{\cdot}[x-\bar{x}]\}. \end{aligned} \tag{1.1}$$

In the terminology of [11] and [4], *stability* of $\bar{x}$ refers to $M_\delta(0,u)$ and $m_\delta(0,u)$ being, for small enough δ, Lipschitz continuous in u in a neighborhood of $u = 0$, with M_δ single-valued and given at 0 by $\bar{x}$, while *tilt stability* refers to those properties holding for $M_\delta(v,0)$ and $m_\delta(v,0)$ for v in a neighborhood of $v = 0$; *full stability* joins these by requiring Lipschitz continuity of $M_\delta(v,u)$ and $m_\delta(v,u)$ in (v,u) around $(v,u) = (0,0)$. It will be useful to speak of *substability* when $M_\delta(0,u)$ and $m_\delta(0,u)$ are just continuous in u, not necessarily Lipschitz continuous. A hybrid soon to come up is

$$\begin{gathered} \textit{full substability of } \bar{x}\text{: having } M_\delta(v,u) \text{ and } m_\delta(v,u) \text{ just continuous}\\ \text{in } (v,u) \text{ but Lipschitz continuous in } v\text{, locally uniformly in } u. \end{gathered} \tag{1.2}$$

In all cases, $\bar{x}$ is obviously an *isolated* locally optimal solution, and this persists parametrically.

The challenge is to tie these properties to some condition or other on the subgradients of φ in the general sense of variational analysis [19]. In this, the mappings M and Y defined by

$$M(v,u) = \{\,x \,|\, Y(x,u,v) \neq \emptyset\}, \qquad Y(x,u,v) := \{\,y \,|\, (v,y) \in \partial\varphi(x,u)\} \tag{1.3}$$

have a big role around

$$\overline{Y} = Y(\bar{x},0,0),\ \ \text{the set of } \textit{multiplier vectors } \bar{y} \text{ associated with } \bar{x} \text{ in } \bar{P}. \tag{1.4}$$

The necessary condition $0 \in \partial_x\varphi(\bar{x},0)$ for the local optimality of $\bar{x}$ in $\bar{P}$ yields a $\bar{y} \in \overline{Y}$ through [19, 10.6] under the *basic constraint qualification* on horizon subgradients, namely

$$(0,y) \in \partial^\infty\varphi(\bar{x},0) \implies y = 0. \tag{1.5}$$

Then, in fact, the multiplier set $\overline{Y}$ is not just nonempty, but also compact, and useful properties of the set-valued mapping Y for (x,u,v) in a neighborhood of $(\bar{x},0,0)$ are made available as well.

The *variational sufficient condition* for the local optimality of $\bar{x}$ holds for a multiplier $\bar{y} \in \overline{Y}$ if there exists $e \geq 0$ ("elicitation parameter") such that the function $\varphi_e(x,u) = \varphi(x,u) + \frac{e}{2}|u|^2$ is variationally convex at $(\bar{x},0)$ for the subgradient $(0,\bar{y}) \in \partial\varphi_e(\bar{x},0) = \partial\varphi(\bar{x},0)$. The *strong* version has variational *strong* convexity of φ_e. These sufficient conditions will be reviewed in Section 2 along with the details of prox-regularity and variational convexity. They were introduced in [14] and broadly explored in [15] in the framework of *generalized nonlinear programming*, where

$$\varphi(x,u) = f_0(x) + g(F(x)+u) \ \text{ with } g \text{ closed proper convex, } f_0 \text{ and } F \text{ in } \mathcal{C}^2. \tag{1.6}$$

Classical nonlinear programming is the case of this in which g is the indicator δ_K of the cone $K = \mathbb{R}^s_- \times \mathbb{R}^{m-s}$. There, the basic constraint qualification is equivalent to the Mangasarian-Fromovitz constraint qualification, and the strong variational sufficient condition is equivalent to the standard strong second-order sufficient condition for local optimality [15, Example 1]. But many other examples of variational sufficiency were developed in [15] as well.

We can now state our main results about stability of a locally optimal solution $\bar{x}$ to problem $\bar{P}$ in relation to variational sufficiency. It will be convenient in this to adopt the terminology that

$$\begin{aligned} &\text{"variational sufficiency at } \bar{x}\text{" means having it at } \bar{x} \text{ for all } \bar{y} \in \overline{Y}\text{, i.e., } (0,\bar{y}) \in \partial\varphi(\bar{x},0),\\ &\text{"full multiplier prox-regularity at } \bar{x}\text{" means } \varphi \text{ prox-regular at } (\bar{x},0) \text{ for all such } (0,\bar{y}). \end{aligned} \tag{1.7}$$

Theorem 1.1 (variational sufficiency versus full stability and full substability)**.** *Under the basic constraint qualification (1.5),*

$$\begin{array}{l} \text{(A) full stability holds at } \bar{x} \text{ with full multiplier prox-regularity} \quad \Longrightarrow \\ \text{(B) strong variational sufficiency holds at } \bar{x} \quad \Longrightarrow \\ \text{(C) full substability holds at } \bar{x} \text{ with full multiplier prox-regularity.} \end{array}$$

Under (C), hence also (A) and (B), an open convex neighborhood $\mathcal{V}\times\mathcal{U}$ of $(0,0)$ exists such that, for small enough δ and low enough $\alpha > m_\delta(0,0) = \varphi(\bar{x},0)$,

$$M_\delta(v,u) = M(v,u) \cap \{\, x \mid |x-\bar{x}| < \delta, \varphi(x,u) < \alpha \} =: M^{\delta,\alpha}(v,u), \tag{1.8}$$

and $m_\delta(v,u)$ is $\mathcal{C}^1$ in $v \in \mathcal{V}$ for each $u \in \mathcal{U}$ with $\nabla_v m_\delta(v,u) = M_\delta(v,u)$. Under (B), furthermore $m_\delta(v,u)$ is uniformly hypo-convex in $u \in \mathcal{U}$ for $v \in \mathcal{V}$ with

$$\partial_u m_\delta(v,u) = Y(x,u,v) \;\text{ for }\; x = M_\delta(v,u), \tag{1.9}$$

where the uniform hypo-convexity refers to the existence of $e \geq 0$ such that $m_\delta(v,u) + \frac{e}{2}|u|^2$ is convex in $u \in \mathcal{U}$ for each $v \in \mathcal{V}$.

The hypo-convexity observation at the end of Theorem 1.1 is new even for the case of φ in (1.6) that corresponds to classical nonlinear programming.

A counterexample will show that (B) $\not\Rightarrow$ (A), but whether (C) $\Rightarrow$ (B), making that pair equivalent, is open to serious conjecture. Another unsettled question is whether the basic constraint qualification might automatically hold under full substability, hence also full stabilty. It holds under the latter at least for φ in the composite format (1.6); see Mordukhovich-Sarabi [7, Theorem 5.1] (note that the relevant part of [7, Theorem 5.1] doesn't rely on the special case of g considered in the paper). The basic constraint qualification (1.5) is known to be equivalent the Aubin property of the mapping $u \mapsto \operatorname{epi} \varphi(\cdot,u)$ at $\bar{u}$ for the image point $(\bar{x},\varphi(\bar{x},\bar{u}))$ [19, 10.16], and in that way connects naturally to the behavior of $M_\delta(v,u)$ with respect to u.

Since (B) and (C) definitely fall short of (A), there is the question of what needs to be combined with them to bring about an equivalence, other than an appeal back to the original coderivative conditions in [4]. We'll provide an answer which, for its formulation, requires graphical derivatives. Recall that the graphical derivative of the single-valued truncated mapping $M^{\delta,\alpha}$ in (1.8) at (v,u) is the generally set-valued mapping $DM^{\delta,\alpha}(v,u) : I\!R^n \times I\!R^m \rightrightarrows I\!R^n$ having, as its graph, the tangent cone to the graph of $M^{\delta,\alpha}$ at (v,u,x) for $x = M^{\delta,\alpha}(v,u)$. We will be looking at, more specially, the graphical derivatives $DM_v^{\delta,\alpha}(u)$ of the "partial mappings"

$$M_v^{\delta,\alpha} := M^{\delta,\alpha}(v,\cdot) : u \mapsto x = M^{\delta,\alpha}(v,u), \tag{1.10}$$

which have inner and outer norms, given in the notation $\xi \in DM_v^{\delta,\alpha}(u)(\omega)$ by

$$|DM_v^{\delta,\alpha}(u)|^- = \sup_{|\omega|\leq 1} \inf_{\xi \in M_v^{\delta,\alpha}(u)(\omega)} |\xi|, \qquad |DM_v^{\delta,\alpha}(u)|^+ = \sup_{|\omega|\leq 1} \sup_{\xi \in M_v^{\delta,\alpha}(u)(\omega)} |\xi|. \tag{1.11}$$

Theorem 1.2 (full stability from strong variational sufficiency or just full substability)**.** *Condition (A) in Theorem 1.1 is equivalent to the combination of condition (C), or (B), with*

$$\limsup_{(v,u)\to(0,0)} |DM_v^{\delta,\alpha}(u)|^- < \infty. \tag{1.12}$$

The proofs of these theorems will be presented in Section 3 after groundwork in Section 2 on understanding prox-regularity and its connections to variational convexity and tilt stability.

2 Prox-regularity background with new developments

Prox-regularity is a primal-dual-localized property of a function that has received many years of close attention since being introduced in [10], especially for its role in second-order variational analysis and its connection to behaviors tied, one way or another, to convexity. Its connection to *variational* convexity was strengthened in [17] and [18]. Consequences will be taken farther in this section in building a platform for the proofs of our main results.

For now, we consider just an extended-real-valued function f on $I\!R^n$ instead of φ on $I\!R^n \times I\!R^m$, supposing it to be closed (lower semicontinuous) and proper. We draw on the theory in [19] of general (limiting) subgradients $v \in \partial f(x)$ and horizon subgradients $v \in \partial^\infty f(x)$. Prox-regularity of f at $\bar{x}$ with respect to a subgradient $\bar{v} \in \partial f(\bar{x})$ involves open neighborhoods $\mathcal{X}$ of $\bar{x}$ and $\mathcal{V}$ of $\bar{v}$ together with an $\alpha > f(\bar{x})$ that truncates $\mathcal{X}$ to

$$\mathcal{X}_f^\alpha = \mathcal{X} \cap \{\, x \,|\, f(x) < \alpha\}, \tag{2.1}$$

a so-called *f-attentive neighborhood* of $\bar{x}$. Such truncation is important in the absence of continuity of f at $\bar{x}$ in ensuring that localization only captures geometric properties of epi f in a neighborhood of $(\bar{x}, f(\bar{x}))$. In the case of prox-regularity, the property is the existence of $r \in (-\infty, \infty)$ such that

$$f(x') \geq f(x) + v{\cdot}[x' - x] - \frac{r}{2}|x' - x|^2, \ \forall x' \in \mathcal{X}, \ (x, v) \in (\mathcal{X}_f^\alpha \times \mathcal{V}) \cap \operatorname{gph} \partial f. \tag{2.2}$$

More specifically, this is prox-regularity at level r. But r could be replaced in (2.1) by $s = -r$. in which case this would be the property of uniform quadratic growth at level s, which like r can be negative as well as positive.

For $r = s = 0$, there is a close resemblance to the global subgradient inequality that signals convexity, but in a localization that is primal-dual and f-attentive. More broadly, the global s version with $s > 0$ would signal strong convexity at that level, but even for negative s, we can think of it in connection with f being s-convex in the sense of $f(x) - \frac{s}{2}|x|^2$ being convex. These insights have motivated the following definition: f is *variationally s-convex* at $\bar{x}$ for $\bar{v} \in \partial f(\bar{x})$ if, for an f-attentive neighborhood as in prox-regularity, with $\mathcal{X}$ and $\mathcal{V}$ open convex, there is a truly s-convex closed proper function $\widehat{f}$ such that

$$\begin{array}{l} \widehat{f} \leq f \ \text{ on } \mathcal{X}, \quad (\mathcal{X} \times \mathcal{V}) \cap \operatorname{gph} \partial \widehat{f} = (\mathcal{X}_f^\alpha \times \mathcal{V}) \cap \operatorname{gph} \partial f \\ \text{and, for all } (x, v) \text{ in this truncated graph, } \widehat{f}(x) = f(x). \end{array} \tag{2.3}$$

This concept surfaced already in 1998 in [11] for $s > 0$, but didn't get a name until 2019 in [13]. Full articulation came in [17], where it was coordinated with ∂f being *maximally s-monotone* locally at $\bar{x}$ for $\bar{v} \in \partial f(\bar{x})$ in the sense that, for some f-attentive neighborhood $\mathcal{X}_f^\alpha \times \mathcal{V}$ of $(\bar{x}, \bar{v})$ with open $\mathcal{X}$ and $\mathcal{V}$,

$$(v' - v){\cdot}(x' - x) \geq s|x' - x|^2 \ \text{ when } \ (x, v), \, (x', v') \in (\mathcal{X}_f^\alpha \times \mathcal{V}) \cap \operatorname{gph} \partial f, \tag{2.4}$$

but without this holding for any strictly larger subset of $\mathcal{X} \times \mathcal{V}$.

Theorem 2.1 (equal reflections of prox-regularity [17])**.** *For any $r \in I\!R$ and $s = -r$, the following properties at $\bar{x}$ for $\bar{v} \in \partial f(\bar{x})$ are equivalent:*

(a) *prox-regularity of f at level r,*

(b) *variational convexity of f at level s,*

(c) *f-attentive locally maximal s-monotonicity of ∂f.*

It's *not* claimed that (a), (b) and (c) are always equivalent for the same $\mathcal{X}_f^\alpha \times \mathcal{V}$. Shifts may be needed in passing between the three properties, but there is nonetheless out of the equivalence a single value $\operatorname{cnv} f(\bar{x}\,|\,\bar{v}) \in [\infty,\infty]$ such that

$$\operatorname{cnv} f(\bar{x}\,|\,\bar{v}) = -\begin{cases} \text{liminf of the available } r\text{-values in (2.2) as} \\ \mathcal{X}\times\mathcal{V} \text{ shrinks down to } (\bar{x},\bar{v}) \text{ and } \alpha \searrow f(\bar{x}), \end{cases} \tag{2.5}$$

$$\operatorname{cnv} f(\bar{x}\,|\,\bar{v}) = \begin{cases} \text{limsup of the available } s\text{-values in (2.3) as} \\ \mathcal{X}\times\mathcal{V} \text{ shrinks down to } (\bar{x},\bar{v}) \text{ and } \alpha \searrow f(\bar{x}), \end{cases} \tag{2.6}$$

$$\operatorname{cnv} f(\bar{x}\,|\,\bar{v}) = \begin{cases} \text{limsup of the available } s\text{-values in (2.4) as} \\ \mathcal{X}\times\mathcal{V} \text{ shrinks down to } (\bar{x},\bar{v}) \text{ and } \alpha \searrow f(\bar{x}). \end{cases} \tag{2.7}$$

This common value $\operatorname{cnv} f(\bar{x}\,|\,\bar{v})$ is the *convexity modulus* of f at $\bar{x}$ for the subgradient $\bar{v}$. There are more equivalences in the theory in [17] beyond those in Theorem 2.1, but this is all we'll need here.

A distinction in terminology requires care, however. Saying that f is prox-regular at $\bar{x}$ for a subgradient $\bar{v} \in \partial f(\bar{x})$ means prox-regularity holds there at *some* level $r \in I\!R$, but saying that f is variationally convex at $\bar{x}$ for $\bar{v}$ specifically means variational convexity at level $s = 0$. Level $s > 0$ is variational *strong* convexity. Level $s < 0$ is variational *hypo*-convexity. By itself without a designated function and location, "variational convexity" refers instead to the general topic at all levels, which through Theorem 2.1 is identical to the general topic of prox-regularity.

Variational strong convexity of f at $\bar{x}$ for $\bar{v} = 0$ as subgradient is important for understanding tilt stability of f at $\bar{x}$ in the sense of the mappings

$$\begin{aligned} m_\delta(v) \quad &:= \inf_{|x-\bar{x}|<\delta} \{\, f(x) - v\cdot[x-\bar{x}]\,\}, \\ M_\delta(v) \quad &:= \operatorname*{argmin}_{|x-\bar{x}|<\delta} \{\, f(x) - v\cdot[x-\bar{x}]\,\}, \end{aligned} \tag{2.8}$$

being, for small enough $\delta > 0$, Lipschitz continuous on a neighborhood of 0 with M_δ single-valued and assigning $\bar{x}$ to 0. Originally in [11, Theorem 1.3], variational strong convexity, without it yet having that name, was shown to be necessary and sufficient for tilt stability under the assumption not only of prox-regularity of f at $\bar{x}$ for $\bar{v} = 0$ but also *subdifferential continuity* there, which refers to having

$$f(x) \to f(\bar{x}) \;\text{ when }\; (x,v) \to (\bar{x},\bar{v}) \;\text{ within }\; \operatorname{gph} \partial f. \tag{2.9}$$

The additional assumption has turned out to be superfluous, however.

Theorem 2.2 (tilt stability without assuming subdifferential continuity, Gfrerer [3, Theorem 5.1])**.** *The variational strong convexity of f at $\bar{x}$ for subgradient $\bar{v} = 0$ is equivalent to f being prox-regular at $\bar{x}$ for $\bar{v} = 0$ and tilt stable there. In more detail, this means having an interval $(0,\sigma)$ such that variational s-convexity of f for all $s \in (0,\sigma)$ corresponds to M_δ being, for all $s \in (0,\sigma)$, Lipschitz continuous with constant s^{-1} in some neighborhood of 0.*

This result relies on facts about prox-regularity that concern a modified version of the subdifferential continuity in (2.9), namely

$$\begin{aligned} &\text{under the prox-regularity condition (2.2),} \\ &\quad \text{(a)}\;\; f(x) \text{ depends continously on } \;(x,v) \in (\mathcal{X}^\alpha_f \times \mathcal{V}) \cap \operatorname{gph} \partial f, \\ &\quad \text{(b)}\;\; (\mathcal{X}^\alpha_f \times \mathcal{V}) \cap \operatorname{gph} \partial f \text{ is closed relative to } \mathcal{X}\times\mathcal{V}, \end{aligned} \tag{2.10}$$

where (a) comes from [10, Proposition 2.3] and (b) has been added by Gfrerer in [3, Lemma 3.4]. (In the cited works, $\mathcal{X}$ and $\mathcal{V}$ are open ε-balls around $\bar{x}$ and $\bar{v}$, and $\alpha = f(\bar{x})+\varepsilon$, but the broader formulation in (2.10) is supported by the simple observation in Proposition 2.5(a) below.) In particular, (2.10) covers the version of (2.9) restricting to $f(x) < \alpha$. Another consequence is part (b) of the theorem coming next, which records additional facts needed later.

Theorem 2.3 (properties joined to tilt stability).

(a) *The function m_δ in (2.8) is always finite, concave and globally Lipschitz continuous on $I\!R^n$. In tilt stability, it is moreover differentiable on a neighborhood of 0 with $\nabla m_\delta(v) = -M_\delta(v)$.*

(b) *Under prox-regularity of f at $\bar{x}$ for subgradient $\bar{v} = 0$, there exists $\alpha > m_\delta(0) = f(\bar{x})$ such that, in tilt stability with v close enough to 0,*

$$M_\delta(v) = (\partial f)^{-1}(v) \cap \{\, x \,|\, |x - \bar{x}| < \delta,\ f(x) < \alpha\}. \tag{2.11}$$

When f is subdifferentially continuous at $\bar{x}$, this holds with the α replaceable by ∞.

(c) *Variational strong convexity at level $s > 0$ corresponds in the equivalence in Theorem 2.2 to having s^{-1} be a Lipschitz constant for M_δ when δ is sufficiently small.*

Proof. (a): The concavity is evident because the formula presents m_δ as the pointwise infimum of a collection of affine functions. A lower semicontinuous proper function is bounded from below on every compact subset, hence f is bounded from below by some β on the ball in (2.8). In that case,

$$\infty > f(\bar{x}) \geq m_\delta(v) \geq \beta - \delta|v|,$$

which tells us that $-m_\delta$ is a finite convex function on $I\!R^n$ with recession function $\leq |\cdot|$. Hence

$$-m_\delta(v') \leq -m_\delta(v) + \delta|v' - v| \ \text{ for all } \ v,\, v',$$

signaling Lipschitz continuity with constant δ. That property of $-m_\delta$ carries over to m_δ.

For the claim in (a) about $M_\delta(v)$, we note that if x is a minimizer for v in (2.8), $x \in M_\delta(v)$, then $m_\delta(v) = f(x) - v\cdot[x - \bar{x}]$, while for other v' merely $m_\delta(v') \leq f(x) - v'\cdot[x - \bar{x}]$. That combination implies $m_\delta(v') - m_\delta(v) \leq [v - v']\cdot x$, according to which x is a subgradient of the convex function $-m_\delta$ at v. The set of subgradients is convex, so that if $M_\delta(v)$ consists of a unique x, as in tilt stability, that x is the unique subgradient of $-m_\delta$ at v. In convex analysis this corresponds to $-m_\delta$ being differentiable at v with x as its gradient there. Then the vector $-x = -M_\delta(v)$ is the gradient of m_δ at v.

(b): This takes advantage of the equivalence in Theorem 2.2, which provides a strongly convex function $\widehat{f}$ satisfying (2.3) for $\bar{v} = 0$ and some $\mathcal{X}$, $\mathcal{V}$ and $\alpha > f(\bar{x})$ as described. Then $f(\bar{x}) \geq m_\delta(0)$, with equality holding under tilt stability. From part (a), we get $m_\delta(v) < \alpha$ for v in a neighborhood of 0 inside $\mathcal{V}$, so that having $x \in M_\delta(v)$ for δ such that $\{\, x \,|\, |x - \bar{x}| < \delta\} \subset \mathcal{X}$ will entail having $v \in \partial f(x)$ with $(x, v) \in \mathcal{X}^\alpha_f \times \mathcal{V}$. That's the same in (2.3) as having $v \in \partial \widehat{f}(x)$ with $(x, v) \in \mathcal{X} \times \mathcal{V}$. The convexity of $\partial \widehat{f}$ associates that in turn with x giving the minimum of $\widehat{f}(x) - v\cdot[x - \bar{x}]$ over $\mathcal{X}$ and hence, by the relationship in (2.3), also the minimum of $f(x) - v\cdot[x - \bar{x}]$ over $\mathcal{X}$. That leads to the equation in (2.11). In the presence of the subdifferential continuity in (2.9), just having $(x, v) \in \operatorname{gph} \partial f$ near enough to $(\bar{x}, 0)$ ensures having $f(x) < \alpha$. The α condition in (2.11) is then satisfied and doesn't need to be entered explicitly.

(c): Again this utilizes the equivalence in Theorem 2.2, with the function $\widehat{f}$ in (2.3) being s-convex. The s-convexity with $s > 0$ implies

$$(v' - v)\cdot(x' - x) \geq s|x' - x|^2 \ \text{ when } \ (x, v),\, (x', v') \in (\mathcal{X} \times \mathcal{V}) \cap \operatorname{gph} \partial \widehat{f},$$

with the consequence that $|x' - x| \leq s^{-1}|v' - v|$. Through (2.3) and part (b), this carries over in (2.3) to $x = M_\delta(v)$ and $x' = M_\delta(v')$. $\square$

In connection with Theorem 2.3(c), we note that this property doesn't quite follow from Gfrerer's equivalence in Theorem 2.2, because that applies at a level s only when variational strong convexity of f also holds at some level $s' > s$.

The original characterization of tilt stability in [11, Theorem 1.3], which utilized subdifferential continuity, included also equivalence with the positive-definiteness of the Mordukhovich second-order subdifferential $\partial^2 f(\bar{x}\,|\,0)$. A substitute for that subdifferential is needed in the absence of subdifferential continuity, because the graph of $\partial^2 f(\bar{x}\,|\,0)$, being defined in terms of limits of normal cones to the graph of ∂f that aren't f-attentive, can be much larger than the graph of $\partial^2 f^\alpha(\bar{x}\,|\,0)$ for $f^\alpha(x) = f(x)$ when $f(x) \leq \alpha$, otherwise $f^\alpha(x) = \infty$. The remedy is replacing $\partial^2 f(\bar{x}\,|\,0)$ by the Gfrerer second-order subdifferential, defined in [3], which we'll denote here by $\partial^2_G f(\bar{x}\,|\,0)$. It simply modifies the Mordukhovich definition by requiring f-attentivity, so that $\partial^2_G f(\bar{x}\,|\,0) = \partial^2_G f^\alpha(\bar{x}\,|\,0) = \partial^2 f^\alpha(\bar{x}\,|\,0)$. That way, in terms of $\partial^2_G f(\bar{x}\,|\,0)$, the second-order subdifferential characterization in [11] carries over without the assumption of subdifferential continuity; cf. [3, Theorem 5.1].

Here, however, instead such of coderivative-based subdifferentials we'll employ for our purposes the *strict* second-order subdifferential $\partial^2_* f(\bar{x}\,|\,0)$ that was introduced in [18] by

$$\begin{array}{l} (\xi,\mu) \in \operatorname{gph} \partial^2_* f(\bar{x}\,|\,\bar{v}) \iff \exists\,(x^\nu, v^\nu) \to (\bar{x}, \bar{v}) \text{ in } \operatorname{gph}\partial f \text{ and} \\ (\xi^\nu, \mu^\nu) \to (\xi,\mu),\ \tau^\nu \searrow 0, \text{ with } (x^\nu + \tau^\nu \xi^\nu, v^\nu + \tau^\nu \mu^\nu) \in \operatorname{gph}\partial f, \\ \text{and furthermore both } f(x^\nu) \to f(\bar{x}) \text{ and } f(x^\nu + \tau^\nu \xi^\nu) \to f(\bar{x}). \end{array} \tag{2.12}$$

This graphical prescription means that the mapping $\partial^2_* f(\bar{x}\,|\,\bar{v})$ is the outer graphical limit of the first-order difference quotient mappings $\Delta_\tau[\partial f](x\,|\,v)$ as $\tau \searrow 0$ while also $(x, v) \to (\bar{x}, \bar{v})$ f-attentively. The *definiteness* modulus of $\partial^2_* f(\bar{x}\,|\,\bar{v})$ is

$$\operatorname{def}[\partial^2_* f(\bar{x}\,|\,\bar{v})] = \sup\{\, s \in I\!R \mid \mu{\cdot}\xi \geq s|\xi|^2 \text{ when } \mu \in \partial^2_* f(\bar{x}\,|\,\bar{v})(\xi)\}. \tag{2.13}$$

Theorem 2.4 (alternative criterion for tilt stability)**.** *The following are equivalent when f is prox-regular at $\bar{x}$ with respect to having 0 as a subgradient there:*

(a) *f is tilt-stable at $\bar{x}$.*

(b) *f has a local minimum at $\bar{x}$, and $0 \in \partial^2_* f(\bar{x}\,|\,0)(\xi)$ only for $\xi = 0$.*

(c) *$\partial^2_* f(\bar{x}\,|\,0)$ is positive-definite: $\operatorname{def}[\partial^2_* f(\bar{x}\,|\,0)] > 0$.*

In these circumstances, for any positive $s < \operatorname{def}[\partial^2_* f(\bar{x}\,|\,0)]$ in (c), s^{-1} serves as a local Lipschitz constant for the tilt stability in (a).

Proof. The equivalence of (c) with the variational strong convexity of f at $\bar{x}$ for subgradient $\bar{v} = 0$ is known from [18, Theorem 3.6] and the fact that prox-regularity makes subgradients be proximal subgradients. In that equivalence, the variational strong convexity holds at level s for any positive $s < \operatorname{def}[\partial^2_* f(\bar{x}\,|\,\bar{v})]$. Then from Theorem 2.2 we have (c) being equivalent to (a) with s^{-1} serving as a Lipschitz constant locally for M_δ in the tilt stability. We also have (b) implied by the minimization in (a) and the positive-definiteness in (c). All that remains is demonstrating that (b) implies (a).

The assumption that $\bar{x}$ is a local minimizer of f guarantees that $\bar{x} \in M_\delta(0)$ when δ is small enough. The nonsingularity condition in (b) is equivalent by [18, Proposition 3.3] to the existence of a neighborhood $\mathcal{V}$ of 0 and an f-attentive neighborhood $\mathcal{X}$ of $\bar{x}$ such that the mapping $M^* : v \in \mathcal{V} \mapsto \mathcal{X}^\alpha_f \cap (\partial f)^{-1}(v)$ is single-valued and Lipschitz continuous. Since $m_\delta(v) \leq f(\bar{x}) < \alpha$ for all v, we will therefore have, when δ is small enough, that $x \in M_\delta(v)$ and $v \in \mathcal{V}$ entail $x \in \mathcal{X}^\alpha_f$ along with $v \in \partial f(x)$, hence $x = M^*(v)$. This confirms that *to the extent that M_δ is nonempty-valued* it is single-valued and Lipschitz continuous, assigning $\bar{x}$ to 0. This leaves us with demonstrating that the nonemptiness can be arranged by taking δ still smaller, if necessary.

From $\bar{x}$ being the unique minimizer of f over the open δ-ball around $\bar{x}$, we know that $f(x) > f(\bar{x})$ when $|x - \bar{x}| < \delta$. Taking δ to be slightly smaller, we can arrange that $\bar{x}$ is the unique element of $\operatorname{argmin} f_\delta$, with f_δ being the sum of f and the indicator of the closed δ-ball around $\bar{x}$. For any

v, the function $f_\delta^v(x) = f_\delta(x) - v{\cdot}[x-\bar{x}]$ is lower semicontinuous with bounded domain, so that $\operatorname{argmin} f_\delta^v \neq \emptyset$. Moreover, $\operatorname{argmin} f_\delta^v = M_\delta(v)$ when $\operatorname{argmin} f_\delta^v \subset \{\, x \mid |x-\bar{x}| < \delta\}$. Our goal can be achieved by verifying that the latter must hold for sufficiently small δ. The key to that is the fact that f_δ^v converges epigraphically to f_δ as $v \to 0$ by [19, 7.8a]. This ensures by [19, 7.33] that the outer limit of the nonempty compact sets $\operatorname{argmin} f_\delta^v$ as $v \to 0$ lies within $\operatorname{argmin} f_\delta$, which is just $\{\bar{x}\}$. □

The remainder of this section turns from tilt stability to more basic consequences of prox-regularity and their impact on subgradient calculus. We note first that, although the definitions of levels of prox-regularity and variational convexity center on a particular pair $(\bar{x},\bar{v})$ in $\operatorname{gph}\partial f$, with $\mathcal{X}_f^\alpha \times \mathcal{V}$ being a sufficiently small neighborhood, the properties in (2.2) and (2.3) have a life beyond just that, because they can operate for multiple pairs simultaneously.

Proposition 2.5 (simultaneous prox-regularity)**.** *Let the prox-regularity condition (2.2) hold for some open set $\mathcal{X} \times \mathcal{V}$ and $\alpha \in I\!R$. Then for every $\bar{x} \in \mathcal{X}_f^\alpha$,*

(a) *f is prox-regular at $\bar{x}$ for every $\bar{v} \in \mathcal{V} \cap \partial f(\bar{x})$,*

(b) *$\partial f(\bar{x})$ is convex if $\mathcal{V} \supset \partial f(\bar{x})$, while $\mathcal{V} \cap \partial f(\bar{x})$ is convex if $\mathcal{V}$ is convex.*

Proof. (a): This is obvious from the fact that $\mathcal{X}_f^\alpha$ is an f-attentive neighborhood of each of its elements. (b): Any vector in the convex hull of $\mathcal{V} \cap \partial f(\bar{x})$ is a convex combination $\bar{v} = \sum_{i=0}^n \lambda_i \bar{v}_i$ of vectors $\bar{v}_i \in \mathcal{V} \cap \partial f(\bar{x})$. By taking the same convex combination of the inequalities (2.2) holding for each i, we see the inequality holding for $\bar{v}$. That resultant inequality says in particular that $\bar{v}$ is a proximal subgradient, so in particular $\bar{v} \in \partial f(\bar{x})$. When $\mathcal{V}$ is convex, $\bar{v}$ belongs also to $\mathcal{V}$, but even when $\mathcal{V}$ isn't convex, if $\mathcal{V} \supset \partial f(\bar{x})$ we anyway have $\mathcal{V} \cap \partial f(\bar{x}) = \partial f(\bar{x})$ and can conclude that the convex hull of $\partial f(\bar{x})$ is a subset of $\partial f(\bar{x})$. That means $\partial f(\bar{x})$ is convex. □

Analogous statements hold also for the variational convexity condition (2.3) and for the local maximality in the monotonicity condition (2.4). The two counterparts to (a) are just as obvious. The counterparts for (b) are clear from the convexity of $\partial \widehat{f}(x)$ in (2.3) and the maximality imposed in (2.4), since that inequality would persist in adding any $v \in \mathcal{V}$ that was missing from $\mathcal{V} \cap \partial f(\bar{x})$ but belonged to its convex hull.

The statement and proof of the next result make use of the convexity modulus $\operatorname{cnv} f(\bar{x} \,|\, \bar{v})$ given simultaneously by (2.5), (2.6) and (2.7).

Proposition 2.6 (uniformity in prox-regularity)**.** *If f is prox-regular at $\bar{x}$ for all $\bar{v} \in C$ for some compact set $C \subset \partial f(\bar{x})$, or equivalently $\operatorname{cnv} f(\bar{x} \,|\, \bar{v}) > -\infty$ for such $\bar{v}$, then $\operatorname{conv} C \subset \partial f(\bar{x})$. If C is convex, we further get*

(a) *$\exists\, \mathcal{X}$, $\mathcal{V}$, α and r for which the prox-regularity condition (2.2) holds with $\mathcal{V} \supset C$.*

(b) *Here $\mathcal{X}$ and $\mathcal{V}$ can be taken to be convex and, together with α and r, adjusted so that the variational convexity condition (2.3) for $s = -r$ holds along with (2.2) for all $\bar{v} \in C$, moreover with $s > \bar{s}$ if $\operatorname{cnv} f(\bar{x} \,|\, \bar{v}) > \bar{s}$ for all $\bar{v} \in C$.*

Proof. Extending in the argument for Proposition 2.5(b), we represent an arbitrary $\bar{v} \in \operatorname{conv} C$ as a convex combination $\sum_{i=0}^n \lambda_i \bar{v}_i$ of vectors $\bar{v}_i \in C$. By assumption, we have the prox-regularity condition (2.2) holding for each i for neighborhoods $\mathcal{X}_i$ of $\bar{x}$, $\mathcal{V}_i$ of $\bar{v}_i$, and parameter values r_i, α_i. Let O denote an open convex neighborhood of the origin small enough that $\bar{v}_i + O \subset \mathcal{V}_i$ for all i, and take $\mathcal{X} = \cap_i \mathcal{X}_i$, $\alpha = \min_i \alpha_i$, $r = \max_i r_i$. Then $\mathcal{X}$ is an open neighborhood of $\bar{x}$, and (2.2) holds for all i with respect to $\mathcal{X}$, r, α, and the open neighborhood $\bar{v}_i + O$ of $\bar{v}_i$:

$$f(x') \geq f(x) + v_i{\cdot}[x'-x] - \frac{r}{2}|x'-x|^2 \;\forall x' \in \mathcal{X},\; (x, v_i) \in (\mathcal{X}_f^\alpha \times [\bar{v}_i + O]\,) \cap \operatorname{gph}\partial f.$$

In particular, v is thus a proximal subgradient of f at $\bar{x}$, so $v \in \partial f(\bar{x})$. This confirms that $\operatorname{conv} C \subset \partial f(\bar{x})$ and that (2.2) holds for $\mathcal{X}$, r, α and the neighborhood $\mathcal{V} = \bar{v} + O$ of $\bar{v}$, which gives the prox-regularity of f at $\bar{x}$ for $\bar{v}$.

A quantitative refinement of this argument, in relation to the level r, will be important in (a) and (b), for which we now assume the convexity of C. Refinement comes from the fact, seen in (2.5), that the r_i-levels available for the prox-regularity at $\bar{x}$ for $\bar{v}_i$ include all the values in the interval $(-\operatorname{cnv} f(\bar{x}\,|\,\bar{v}_i), \infty)$. It follows that, in taking $r = \max_i r_{\bar{v}_i}$ we can arrange for r to be any value $> -\min_i \operatorname{cnv} f(\bar{x}\,|\,\bar{v}_i)$. Put another way, the argument shows that, for any $\bar{s} \in I\!R$,

$$\operatorname{cnv} f(\bar{x}\,|\,\bar{v}) > \bar{s},\ \forall \bar{v} \in C \quad \Longrightarrow \quad \operatorname{cnv} f(\bar{x}\,|\,\bar{v}) > \bar{s},\ \forall \bar{v} \in \operatorname{conv} C. \tag{2.14}$$

(a): On the basis of (a) and the fact that the convex hull of a compact set is compact, plus the observation in (2.14), we can henceforth assume for simplicity that C itself is convex: $C = \operatorname{conv} C$. For each $\bar{v} \in C$ there are open neighborhoods $\mathcal{X}_{\bar{v}}$ and $\mathcal{V}_{\bar{v}}$ along with $\alpha_{\bar{v}} > f(\bar{x})$ and $r_{\bar{v}}$ in the pattern of (2.2). The union of the open sets $\mathcal{V}_{\bar{v}}$ covers C, so the compactness of C allows us to extract a finite covering, say for $\bar{v}_k$, $k = 1, \ldots, m$. Take $\mathcal{V} = \cup_k \mathcal{V}_{\bar{v}_k}$, $\mathcal{X} = \cap_k \mathcal{X}_{\bar{v}_k}$, $\alpha = \min_k \alpha_{\bar{v}_k}$ and $r = \max_k r_{\bar{v}_k}$. Then $\mathcal{X}$ is an open neighborhood of $\bar{x}$, $\mathcal{V}$ is an open set containing C, and (2.2) holds.

Here again we have a refinement in terms of the convexity modulus, because in arranging the covering we can insist, for any $\varepsilon > 0$, on having $r_{\bar{v}} < -\operatorname{cnv} f(\bar{x}\,|\,\bar{v}) + \varepsilon$. That way, if $\bar{s}$ is a value such that $\operatorname{cnv} f(\bar{x}\,|\,\bar{v}) > \bar{s}$ for every $\bar{v} \in C$, we can get $r < -\bar{s}$.

(b): By choosing $\mathcal{X}_{\bar{v}}$ and $\mathcal{V}_{\bar{v}}$ in this construction to be convex, we get $\mathcal{X}$ convex, but not necessarily $\mathcal{V}$. To fix that, replace $\mathcal{V}$ by the open convex set $\{\, v \mid \operatorname{dist}_C(v) < \varepsilon\}$ for a positive $\varepsilon > 0$ small enough that it lies within $\mathcal{V}$. Such ε must exist, since otherwise it would be possible to generate a v-sequence in the closed exterior of $\mathcal{V}$ that gets closer and closer to C and is bounded because C is bounded. It would have a cluster point in C that still belongs to the exterior, in contradiction to $C \subset \mathcal{V}$.

The argument in (b) can now be sharpened by taking advantage of Theorem 2.1 through its consequence in the three descriptions of the convexity modulus in (2.5), (2.6) and (2.7). By choosing smaller neighborhoods $\mathcal{X}_{\bar{v}}$ and $\mathcal{V}_{\bar{v}}$ and lower values for $\alpha_{\bar{v}}$ and $s_{\bar{v}} = -r_{\bar{v}}$, we can arrange to have not only the prox-regularity condition (2.2), but also the variational convexity condition (2.3) for a function $\widehat{f}_{\bar{v}}$ and the associated maximal r-monotonicity in (2.4). Proceeding with those elements to a covering that corresponds to a finite collection of vectors $\bar{v}_k$, $k = 1, \ldots, m$, we can pass again to $\mathcal{X}$, $\mathcal{V}$, α and s as $-r$, know from the refinement at the end of the proof of (b) that if $\operatorname{cnv} f(\bar{x}, \bar{v}) > \bar{s}$ for all $\bar{v} \in C$, we can end up with $s > \bar{s}$.

Each of the s-convex functions $\widehat{f}_{\bar{v}_k}$ is defined on $\mathcal{X}$ and $\leq f$ there, so by taking $\widehat{f}$ to be their pointwise max, we obtain an s-convex function that is $\leq f$ on $\mathcal{X}$. We must still confirm, though, that $\widehat{f}$ fits the rest of (2.3) as well. For this, note that for any k we have

$$\begin{aligned} &\widehat{f}_{\bar{v}_k} \leq f \ \text{ on } \mathcal{X}, \quad (\mathcal{X} \times [\mathcal{V} \cap \mathcal{V}_{\bar{v}_i}]) \cap \operatorname{gph} \partial \widehat{f}_{\bar{v}_k} = (\mathcal{X}^{\alpha}_{f} \times [\mathcal{V} \cap \mathcal{V}_{\bar{v}_i}]) \cap \operatorname{gph} \partial f \\ &\text{and, for all } (x, v) \text{ in this truncated graph, } \widehat{f}_{\bar{v}_k}(x) = f(x). \end{aligned} \tag{2.15}$$

The truncation is furthermore maximal s-monotone relative to $\mathcal{X} \times [\mathcal{V} \cap \mathcal{V}_{\bar{v}_k}]$. Note next that, because of common s-convexity and $\widehat{f}_{\bar{v}_k} \leq \widehat{f}$, we have

$$\partial \widehat{f}_{\bar{v}_k}(x) \subset \partial \widehat{f}(x) \ \text{ when } \ \widehat{f}_{\bar{v}_k}(x) = \widehat{f}(x).$$

From this and having $\widehat{f} \leq f$ on $\mathcal{X}$, we see in (2.3) that

$$(\mathcal{X} \times [\mathcal{V} \cap \mathcal{V}_{\bar{v}_k}]) \cap \operatorname{gph} \partial \widehat{f}_{\bar{v}_k} \subset (\mathcal{X} \times [\mathcal{V} \cap \mathcal{V}_{\bar{v}_k}]) \cap \operatorname{gph} \partial \widehat{f}.$$

Then equality must hold, however, because of the right side being s-monotone and the left side being maximal s-monotone. It follows that (2.15) holds with $\widehat{f}$ in place of $\widehat{f}_{\bar{v}_k}$. This being true for every k, we conclude that (2.3) does hold for $\widehat{f}$. $\square$

A parametric property of partial subgradients will complete this section. Several other special consequences of prox-regularity for subgradient calculus, such as a chain rule and a sum rule, were uncovered in [12], but the parameterization puts this new result in a different category.

Theorem 2.7 (partial subgradients under prox-regularity). *For closed proper φ on $I\!R^n \times I\!R^m$ and the mapping $Y : (x,u,v) \mapsto \{\, y \,|\, (v,y) \in \partial\varphi(x,u)\}$, suppose for given $(\bar{x},\bar{u},\bar{v})$ that*

(a) *for every $\bar{y} \in Y(\bar{x},\bar{u},\bar{v})$, prox-regularity of φ holds at $(\bar{x},\bar{u})$ for the subgradient $(\bar{v},\bar{y})$, and*

(b) *the constraint qualification holds that*

$$(0,y) \in \partial^\infty \varphi(\bar{x},\bar{u}) \;\; \text{only for} \;\; y = 0. \tag{2.16}$$

Then there is a φ-attentive neighborhood $(\mathcal{X}\times\mathcal{U})^\alpha_\varphi$ of $(\bar{x},\bar{u})$ along with open convex sets $\mathcal{V} \ni \bar{v}$ and $\mathcal{Y} \supset Y(\bar{x},\bar{u},\bar{v})$ such that the mapping Y is compact-convex-valued, outer semicontinuous and locally bounded from $(\mathcal{X}\times\mathcal{U})^\alpha_\varphi \times \mathcal{V}$ into $\mathcal{Y}$, and

$$\begin{aligned} &\forall u \in \mathcal{U},\ \exists\, x \in \mathcal{X} \text{ with } \varphi(x,u) < \alpha \text{ and, for every such } x,\\ &\partial_x\varphi(x,u) \cap \mathcal{V} = \{\, v \in \mathcal{V} \,|\, \exists\, y,\ (v,y) \in \partial\varphi(x,u)\}, \quad \text{a convex set,} \end{aligned} \tag{2.17}$$

and moreover such that uniform prox-regularity holds in the existence of $r > 0$ for which

$$\begin{aligned} &\varphi(x',u') \geq \varphi(x,u) + (v,y)\cdot[(x',u') - (x,u)] - \tfrac{r}{2}|(x',u') - (x,u)|^2\\ &\forall\, (x',u') \in \mathcal{X}\times\mathcal{U}, \quad \text{when} \;\; (x,u,v,y) \in [(\mathcal{X}\times\mathcal{U})^\alpha_\varphi \times (\mathcal{V} \times I\!R^n)] \cap \operatorname{gph}\partial\varphi, \end{aligned} \tag{2.18}$$

with $r < \bar{r}$ if for every $\bar{y}$ in assumption (a) the prox-regularity holds at a level $< \bar{r}$.

Proof. The constraint qualification in (2.16) corresponds to the epigraphical mapping $u \mapsto \operatorname{epi}\varphi(\cdot,u)$ having the Aubin property at $\bar{u}$ for the image point $(\bar{x},\varphi(\bar{x},\bar{u}))$ [19, 10.16]. That property includes inner semicontinuity in the sense that for every neighborhood $\mathcal{W}$ of $(\bar{x},\varphi(\bar{x},\bar{u}))$ the intersection of $\mathcal{W}$ with $\operatorname{epi}\varphi(\cdot,u)$ is nonempty for all u near enough to $\bar{u}$. This will later serve to justify the existence claim in (2.17). But the Aubin property persists by its definition to holding also for all u near enough to $\bar{u}$ with respect to $(x,\alpha) \in \operatorname{epi}\varphi(\cdot,u)$ near enough to $(\bar{x},\varphi(\bar{x},\bar{u}))$. Thus (2.16) persists for every (x,u) in some φ-attentive neighborhood $(\mathcal{X}_0\times\mathcal{U}_0)^{\alpha_0}_\varphi$ of $(\bar{x},\bar{u})$. We have then from [19, 10.11] that

$$\begin{aligned} &\text{for } \;(x,u) \in (\mathcal{X}_0\times\mathcal{U}_0)^{\alpha_0}_\varphi,\\ &\quad \partial_x\varphi(x,u) \subset \{\, v \,|\, \exists\, y,\ (v,y) \in \partial\varphi(x,u)\} = \{\, v \,|\, Y(x,u,v) \neq \emptyset\}, \end{aligned} \tag{2.19}$$

with $Y(x,u,v)$ being compact. Here, however, we need still better understanding of how $Y(x,u,v)$ depends on (x,u,v). We can get that next by repeating the argument that's behind (2.19) and extracting from it the claimed outer semicontinuity and local boundedness of Y at $(\tilde{x},\tilde{u},\tilde{v})$ as a representative element of $((\mathcal{X}_0\times\mathcal{U}_0)^{\alpha_0}_\varphi \times I\!R^n) \cap \operatorname{dom} Y$. This means starting from

$$(v^\nu,y^\nu) \in \partial\varphi(x^\nu,u^\nu) \;\; \text{with} \;\; (x^\nu,u^\nu,v^\nu) \to (\tilde{x},\tilde{u},\tilde{v}) \;\; \text{and} \;\; \varphi(x^\nu,u^\nu) \to \varphi(\tilde{x},\tilde{u}), \tag{2.20}$$

and demonstrating that the y^ν sequence must be bounded with all its cluster points $\tilde{y}$ satisfying $(\tilde{v},\tilde{y}) \in \partial\varphi(\tilde{x},\tilde{u})$. That cluster point property is automatic from boundedness according to the definition of $\partial\varphi$, so we can concentrate on establishing the boundedness.

If boundedness failed, we could, by selection, reduce to having (2.20) with $0 < |y^\nu| \nearrow \infty$. Then, in taking $\lambda^\nu = 1/|y^\nu|$ we could further suppose that $\lambda^\nu y^\nu \to y$ with $|y| = 1$. In that case $(\lambda^\nu v^\nu, \lambda^\nu y^\nu) \to (0, y)$ belonging to $\partial^\infty \varphi(\tilde{x}, \tilde{u})$ by the definition of that set of horizon subgradients [19, 8.3]. But that's precluded by our assumption (b). This argument confirms not only (2.19) for a neighborhood $(\mathcal{X}_0 \times \mathcal{U}_0)_\varphi^{\alpha_0}$, but also the outer semicontinuity and local boundedness of Y on $(\mathcal{X}_0 \times \mathcal{U}_0)_\varphi^{\alpha_0} \times \mathbb{R}^n$.

The next step is applying Proposition 2.6 to φ and the compact set $C = \{\, (\bar{v}, \bar{y}) \,|\, \bar{y} \in Y(\bar{x}, \bar{u}, \bar{v})\}$ within $\partial\varphi(\bar{x}, \bar{u})$. First, we get $\operatorname{conv} C \subset \partial\varphi(\bar{x}, \bar{u})$, hence $\operatorname{conv} Y(\bar{x}, \bar{u}, \bar{v}) \subset Y(\bar{x}, \bar{u}, \bar{v})$, i.e., the convexity of $Y(\bar{x}, \bar{u}, \bar{v})$ as well as of C. From Proposition 2.6(a) and (b) we get open convex sets $\mathcal{W} \ni (\bar{x}, \bar{u})$ and $\mathcal{Z} \supset C$ along with $\alpha > \varphi(\bar{x}, \bar{u})$ and r for which

$$\begin{array}{c} \varphi(x', u') \geq \varphi(x, u) + (v, y)\cdot[(x', u') - (x, u)] - \frac{r}{2}|(x', u') - (x, u)|^2 \\ \text{for all } (x', u') \in \mathcal{W} \text{ when } (x, u, v, y) \in (\mathcal{W}_\varphi^\alpha \times \mathcal{Z}) \cap \operatorname{gph} \partial\varphi, \end{array} \tag{2.21}$$

with $r < \bar{r}$ if the prox-regularity in assumption (a) is $< \bar{r}$ for every $\bar{y}$.

We can think of C as $Y_+(\bar{x}, \bar{u}, \bar{v})$ for the mapping $Y_+(x, u, v) = \{v\} \times Y(x, u, v)$, which inherits from Y its outer semicontinuity and local boundedness relative to $(\mathcal{X}_0 \times \mathcal{U}_0)_\varphi^{\alpha_0} \times \mathbb{R}^n$ for the neighborhood $(\mathcal{X}_0 \times \mathcal{U}_0)_\varphi^{\alpha_0}$ in (2.19). For (x, u, v) near enough to $(\bar{x}, \bar{u}, \bar{v})$ there, we'll have $Y_+(x, u, v) \subset \mathcal{Z}$. That will yield the convexity of $Y_+(x, u, v) = \mathcal{Z} \cap \partial\varphi(x, u) \cap (\{v\} \times \mathbb{R}^m)$ by Proposition 2.5(b) and with it the convexity of $Y(x, u, v)$. Without loss of generality we can have the open convex neighborhoods in (2.21) be small enough for that to hold and also be of the form

$$\mathcal{W}_\varphi^\alpha = (\mathcal{X} \times \mathcal{U})_\varphi^\alpha \subset (\mathcal{X}_0 \times \mathcal{U}_0)_\varphi^{\alpha_0} \qquad \mathcal{Z} = \mathcal{V} \times \mathcal{Y}, \tag{2.22}$$

with $\mathcal{X}$, $\mathcal{U}$, $\mathcal{V}$, $\mathcal{Y}$, all convex. That gives us (2.18). In the case of (2.18) with $u' = u$, we have

$$\begin{array}{c} \varphi(x', u) \geq \varphi(x, u) + v\cdot[x' - x] - \frac{r}{2}|x' - x|^2, \ \forall x' \in \mathcal{X}, \\ \text{when } (x, u) \in (\mathcal{X} \times \mathcal{U})_\varphi^\alpha, \ v \in \mathcal{V}, \ y \in Y(x, u, v), \end{array}$$

which implies $v \in \partial_x \varphi(x, u) \cap \mathcal{V}$. This verifies that (2.19) for $(\mathcal{X} \times \mathcal{U})_\varphi^\alpha$ must hold with equality, hence (2.17). The convexity of $\partial_x\varphi(x, u) \cap \mathcal{V}$ follows then from Proposition 2.5(b) and the convexity of $\mathcal{V}$, inasmuch as $\varphi(\cdot, u)$ inherits prox-regularity from φ under the subgradient formula in (2.17). □

Corollary 2.8 (uniform prox-regularity of partial functions)**.** *For φ as in Theorem 2.7, there exist open convex neighborhoods $\mathcal{X}$ of $\bar{x}$, $\mathcal{V}$ of $\bar{v}$ and $\mathcal{U}$ of $\bar{u}$ along with $\alpha > \varphi(\bar{x}, \bar{u})$ and r such that, for each $u \in \mathcal{U}$, the partial function $\varphi^u(x) = \varphi(x, u)$ satisfies the prox-regularity condition*

$$\begin{array}{l} \varphi^u(x') \geq \varphi^u(x) + v\cdot[x' - x] - \frac{r}{2}|x' - x|^2, \ \forall x' \in \mathcal{X}, \\ \textit{when } (x, v) \in (\mathcal{X} \times \mathcal{V}) \cap \operatorname{gph} \partial\varphi^u \ \textit{with } \varphi^u(x) < \alpha, \end{array} \tag{2.23}$$

moreover with $r < \bar{r}$ if for every $\bar{y} \in Y(\bar{x}, \bar{u}, \bar{v})$ the assumed prox-regularity holds at a level $< \bar{r}$.

Proof. This comes out of (2.17) and (2.18), inasmuch as $\partial\varphi^u(x) = \partial_x\varphi(x, u)$. □

3 Main proofs and counterexamples

This section takes up the task of proving the theorems stated in Section 1 with the help of the background and additional results in Section 2, while also offering further insights through examples. A closer look at variational sufficiency comes first.

Recall that the strong variational sufficient condition for local optimality in problem $(\bar{P})$ is fulfilled at $\bar{x}$ in partnership with a multiplier vector $\bar{y} \in \overline{Y} = \{\, \bar{y} \mid (0, \bar{y}) \in \partial\varphi(\bar{x}, 0)\}$ if there is an elicitation level $e \geq 0$ such that the function

$$\varphi_e(x,u) = \varphi(x,u) + \frac{e}{2}|u|^2, \quad \text{with } \partial\varphi_e(x,u) = \partial\varphi(x,u) + (0, eu), \tag{3.1}$$

is variationally s-convex at $(\bar{x}, 0)$ for $(0, \bar{y})$ at a level $s > 0$. In the terminology (1.7), strong variational suffiency at $\bar{x}$ without mention of a particular $\bar{y}$ refers to having this condition fulfilled for every $\bar{y} \in \overline{Y}$, but conceivably this could be with different levels of e and s that depend on $\bar{y}$. In fact, as demonstrated next, the same e and s can be made to work, even with the same neighborhoods.

Proposition 3.1 (uniformity in strong variational sufficiency). *Strong variational sufficiency for local optimality at $\bar{x}$ under the basic constraint qualification (1.5) corresponds to having open convex sets $\mathcal{X} \times \mathcal{U} \ni (\bar{x}, 0)$ and $\mathcal{V} \times \mathcal{Y} \supset \{0\} \times \overline{Y}$, parameter levels $e \geq 0$, $s > 0$, $\alpha > \varphi(\bar{x}, 0)$, and a closed proper s-convex function $\psi \leq \varphi_e$ on $\mathcal{X} \times \mathcal{U}$ such that*

$$\begin{aligned} &(x,u,v,y) \in \operatorname{gph} \partial\psi \cap [(\mathcal{X} \times \mathcal{U}) \times (\mathcal{V} \times \mathcal{Y})] \\ &\quad \iff \quad (x,u,v,y) \in \operatorname{gph} \partial\varphi_e \cap [(\mathcal{X} \times \mathcal{U})^{\alpha}_{\varphi_e} \times (\mathcal{V} \times \mathcal{Y})], \\ &\text{and } \psi(x,u) = \varphi_e(x,u) \text{ for such vectors } (x,u,v,y), \end{aligned} \tag{3.2}$$

with φ then having the properties in Theorem 2.7 for $\bar{u} = 0$, $\bar{v} = 0$, and $r = -s$. This furthermore entails uniform variational s-convexity of the partial functions $\varphi^u(x) = \varphi(x,u)$ in the sense of having for all $u \in \mathcal{U}$, with respect to $\psi^{u,e}(x) := \psi(x,u) - \frac{e}{2}|u|^2$, for which $\psi^{u,e} \leq \varphi^u$ on $\mathcal{X}$,

$$\begin{aligned} &(x,v) \in \operatorname{gph} \partial\psi^{u,e} \cap (\mathcal{X} \times \mathcal{V}) \\ &\quad \iff \quad (x,v) \in \operatorname{gph} \partial\varphi^u \cap (\mathcal{X}^{\alpha}_{\varphi^u} \times \mathcal{V}), \\ &\text{and } \psi^{u,e}(x) = \varphi^u(x) \text{ for such vectors } (x,v). \end{aligned} \tag{3.3}$$

Proof. For each $\bar{y} \in \overline{Y}$ we are given the existence of open neighborhoods $\mathcal{X}_{\bar{y}} \times \mathcal{U}_{\bar{y}}$ of $(\bar{x}, 0)$ and $\mathcal{V}_{\bar{y}} \times \mathcal{Y}_{\bar{y}}$ of $(0, \bar{y})$, parameter values $e_{\bar{y}} \geq 0$, $s_{\bar{y}} > 0$, $\alpha_{\bar{y}} > \varphi(\bar{x}, 0)$, and a closed proper $s_{\bar{y}}$-convex function $\psi_{\bar{y}}$ on $\mathcal{X}_{\bar{y}} \times \mathcal{U}_{\bar{y}}$ such that

$$\begin{aligned} &(x,u,v,y) \in \operatorname{gph} \partial\psi_{\bar{y}} \cap [(\mathcal{X}_{\bar{y}} \times \mathcal{U}_{\bar{y}}) \times (\mathcal{V}_{\bar{y}} \times \mathcal{Y}_{\bar{y}})] \\ &\quad \iff \quad (x,u,v,y) \in \operatorname{gph} \partial\varphi_{e_{\bar{y}}} \cap [(\mathcal{X}_{\bar{y}} \times \mathcal{U}_{\bar{y}})^{\alpha_{\bar{y}}}_{\varphi_{e_{\bar{y}}}} \times (\mathcal{V}_{\bar{y}} \times \mathcal{Y}_{\bar{y}})], \\ &\text{and } \psi_{\bar{y}}(x,u) = \varphi_{e_{\bar{y}}}(x,u) \text{ for such vectors } (x,u,v,y). \end{aligned} \tag{3.4}$$

The key observation is that these same elements serve equally then to describe the strong variational sufficiency at $\bar{x}$ not only for $\bar{y}$, but also for every $\bar{y}' \in \overline{Y} \cap \mathcal{Y}_{\bar{y}}$. The compactness of $\overline{Y}$, assured by the constraint qualification, allows us to select finitely many $\bar{y}_k \in \overline{Y}$ such that the open neighborhoods $\mathcal{Y}_{\bar{y}_i}$ cover $\overline{Y}$. Then for $e = \max_k e_{\bar{y}_k} \geq 0$ and $s = \min_k s_{\bar{y}_k} > 0$ we have, for every $\bar{y} \in \overline{Y}$, that φ_e is variationally s-convex at $(\bar{x}, 0)$ with respect to $(0, \bar{y})$. In particular, for any $\bar{s} \in (0, s)$, we have $\operatorname{cnv} \varphi_e(\bar{x}, 0 \,|\, 0, \bar{y}) > \bar{s}$ for every $(0, \bar{y}) \in C = \{0\} \times \overline{Y} = \partial\varphi(\bar{x}, 0)$, a convex set by Theorem 2.7, and are able that way to apply Proposition 2.6(b) to reach the uniform condition in (3.2), with a slightly lower $s > 0$ if necessary.

Since prox-regularity at level $r = -s$ is implied by such variational s-convexity, Theorem 2.7 is applicable, as claimed, for the case of $\bar{u} = 0$, $\bar{v} = 0$, and $\alpha > \varphi_e(\bar{x}, 0) \geq \varphi(\bar{x}, 0)$, that carries over to φ_e in yielding

$$\begin{aligned} &\forall u \in \mathcal{U},\ \exists x \in \mathcal{X} \text{ with } \varphi_e(x,u) < \alpha \text{ and, for every such } x, \\ &\partial_x \varphi_e(x,u) \cap \mathcal{V} = \{\, v \in \mathcal{V} \mid \exists y,\ (v,y) \in \partial\varphi_e(x,u)\}. \end{aligned} \tag{3.5}$$

We likewise have for $(x,u) \in \mathcal{X} \times \mathcal{U}$ that $\partial_x \psi(x,u) = \{ v \in \mathcal{V} \,|\, \exists y,\ (v,y) \in \partial\psi(x,u)\}$ if the constraint qualification holds that $(0,y) \in \partial^\infty \psi(x,u)$ only for $y = 0$. That does hold because the convexity of ψ reduces it to u belonging to $\operatorname{int}\{ u \,|\, \exists x,\ \psi(x,u) < \infty\}$, which includes $\mathcal{U}$ through (3.5) since here $\psi(x,u) \le \varphi_e(x,u)$. These observations about partial subgradients, in combination with noting that $\psi(x,u) = \varphi_e(x,u)$ corresponds to $\psi^{u,e}(x) = \varphi^u(x)$, translate (3.2) into (3.3) for fixed u. □

Proof of Theorem 1.1. In all three conditions (A), (B) and (C), we have the basic constraint qualification (1.5) combined with φ being prox-regular at $(\bar{x},0)$ for every $(0,\bar{y})$ with $\bar{y} \in \overline{Y}$, so we are always in the setting of Theorem 2.7 and have at our disposal all the subgradient and prox-regularity properties of φ there for $\bar{v} = 0$ and $\bar{u} = 0$.

(A)⇒(B): Along with the prox-regularity of φ at $(\bar{x},0)$ for every $(0,\bar{y}) \in \partial\varphi(\bar{x},0)$, i.e., $\bar{y} \in \overline{Y}$, we have the Lipschitz continuity of $M_\delta(v,u)$ with respect to $(v,u) \in \mathcal{V} \times \mathcal{U}$ for a Lipschitz constant κ. Fix any $\bar{y} \in \overline{Y}$. To reach (B), it must be shown that, for $e > 0$ high enough, the function $\varphi_e(x,u) = \varphi(x,u) + \frac{e}{2}|u|^2$ is variationally strongly convex at $(\bar{x},0)$ for the subgradient $(0,\bar{y})$. By Theorem 2.2 as applied to φ_e in place of f, that's equivalent to the tilt stability of the function

$$\overline{\varphi}_e(x,u) := \varphi(x,u) - \bar{y}{\cdot}u + \frac{e}{2}|u|^2 \tag{3.6}$$

at $(\bar{x},0)$. According to Theorem 2.4, we can therefore confirm it by demonstrating that $\overline{\varphi}_e$ has a local minimum at $(\bar{x},0)$ and satisfies

$$(0,0) \in \partial^2_* \overline{\varphi}_e(\bar{x},0;0,0)(\xi,\omega) \ \text{ only for } \ (\xi,\omega) = (0,0). \tag{3.7}$$

Here $\partial\overline{\varphi}_e(x,u) = \partial\varphi(x,u) + (0, eu - \bar{y})$ and $\partial^2_*\overline{\varphi}_e(\bar{x},0;0,0)(\xi,\omega) = \partial^2_*\varphi(\bar{x},0;0,0)(\xi,\omega) + (0,e\omega)$, so that showing (3.7) comes down to showing

$$(0,-e\omega) \in \partial^2_* \varphi_e(\bar{x},0;0,0)(\xi,\omega) \ \text{ only for } \ (\xi,\omega) = (0,0). \tag{3.8}$$

To establish the local minimum for e sufficiently high, we invoke the prox-regularity in (2.18) of Theorem 2.7 for $\bar{v} = 0$ and $\bar{u} = 0$ to see first that

$$\begin{gathered}\varphi(M_\delta(0,u),u) \ge \varphi(\bar{x},0) + (0,\bar{y}){\cdot}[(M_\delta(0,u),u) - (\bar{x},0)] - \tfrac{r}{2}|(M_\delta(0,u),u) - (\bar{x},0)|^2,\\ \text{where } \ \varphi(x,u) \ge \varphi(M_\delta(0,u),u) \ \text{ and } \ \bar{x} = M_\delta(0,0), \ \text{ hence } \ |M_\delta(0,u) - \bar{x}| \le \kappa|u|.\end{gathered}$$

We get from this that

$$\overline{\varphi}_e(x,u) - \overline{\varphi}_e(\bar{x},0) \ge \frac{e}{2}|u|^2 - \frac{r}{2}(\kappa^2|u|^2 + |u|^2) = \frac{e - r(\kappa^2+1)}{2}|u|^2.$$

To ensure the local minimum, it suffices therefore to have

$$e > r(\kappa^2 + 1). \tag{3.9}$$

Turning now to verifying the strict second-order subdifferential condition in (3.8), recall from the definition of that mapping in (2.12), in its articulation for φ, that

$$\begin{aligned}&(\xi,\omega,0,-e\omega) \in \operatorname{gph}\partial^2_*\varphi(\bar{x},0\,|\,0,\bar{y}) \iff\\ &\qquad \exists\,(x^\nu,u^\nu,v^\nu,y^\nu) \to (\bar{x},0,0,\bar{y}) \text{ in } \operatorname{gph}\partial\varphi\\ &\qquad \text{and } (\xi^\nu,\omega^\nu,\mu^\nu,\eta^\nu) \to (\xi,\omega,0,-e\omega),\ \tau^\nu \searrow 0, \text{ with}\\ &\qquad (x^\nu + \tau^\nu\xi^\nu, u^\nu + \tau^\nu\omega^\nu, v^\nu + \tau^\nu\mu^\nu, y^\nu + \tau^\nu\eta^\nu) \in \operatorname{gph}\partial\varphi, \text{ while}\\ &\qquad \varphi(x^\nu,u^\nu) \to \varphi(\bar{x},0) \ \text{ and } \ \varphi(x^\nu + \tau^\nu\xi^\nu, u^\nu + \tau^\nu\omega^\nu) \to \varphi(\bar{x},0).\end{aligned} \tag{3.10}$$

In these circumstances we have $x^\nu = M_\delta(v^\nu, u^\nu)$ and $x^\nu + \tau^\nu\xi^\nu = M_\delta(v^\nu + \tau^\nu\mu^\nu, u^\nu + \tau^\nu\omega^\nu)$, hence

$$|\xi^\nu| = \frac{1}{\tau^\nu}\Big|M_\delta(v^\nu + \tau^\nu, u^\nu + \tau^\nu\omega^\nu) - M_\delta(v^\nu, u^\nu)\Big| \leq \kappa|(\mu^\nu, \omega^\nu)|,$$

and in the limit

$$|\xi| \leq \kappa|\omega|. \tag{3.11}$$

On the other hand, the prox-regularity coming from Theorem 2.7 with $r > 0$ in (2.18) gives us from (3.10), through the associated monotonicitiy of $\partial\varphi$ at level $s = -r$ in Theorem 2.1, that

$$\begin{aligned}\Big((v^\nu + \tau^\nu\mu^\nu, y^\nu + \tau^\nu\eta^\nu) - (v^\nu, y^\nu)\Big)\cdot\Big((x^\nu + \tau^\nu\xi^\nu, u^\nu + \tau^\nu\omega^\nu) - (x^\nu, u^\nu)\Big) \\ \geq -r|(x^\nu + \tau^\nu\xi^\nu, u^\nu + \tau^\nu\omega^\nu) - (x^\nu, u^\nu)|^2.\end{aligned}$$

This reduces to $(\mu^\nu, \eta^\nu)\cdot(\xi^\nu, \omega^\nu) \geq -r|(\xi^\nu, \omega^\nu)|^2$, which in passing to the limit in (3.10) turns into $(0, -e\omega)\cdot(\xi, \omega) \geq -r|(\xi, \omega)|^2$, or in other words

$$r|\xi|^2 \geq (e - r)|\omega|^2. \tag{3.12}$$

In comparing this with (3.11), which entails $r|\xi|^2 \leq r\kappa^2|\omega|^2$, we obtain $0 \geq [e - r(\kappa^2 + 1)]|\omega|^2$. The conclusion is that, as long as e is large enough to satisfy (3.9), we get from (3.12) that $\omega = 0$ and then from (3.11) that also $\xi = 0$, so that (3.8) is correct.

(B)$\Rightarrow$(C): Proposition 3.1 furnishes in (3.3) variational strong convexity in x component that through Theorems 2.2 and 2.3(c) produces, for every $u \in \mathcal{U}$, tilt stability in $v \in \mathcal{V}$ with Lipschitz constant s^{-1}. That aspect of (C) in turn underpins, through Theorem 2.3(a)(b), to M_δ being the localization $M^{\delta,\alpha}$ of M in (1.8) for sufficiently low α, and $m_\delta(v, u)$ having the properties with respect to v that are indicated in the theorem.

For the continuity of $M_\delta(v, u)$ in u and the additional properties of $m_\delta(v, u)$ with respect to u claimed under (B), further argument is needed. Consider the replacement of φ by φ_e in the formulas (1.1) for $M_\delta(v, u)$ and $m_\delta(v, u)$. This would have no effect on $M_\delta(v, u)$ but would add $\frac{e}{2}|u|^2$ to $m_\delta(v, u)$; let that modified function be denoted by $m^e_\delta(v, u)$. In those terms we have, for $u \in \mathcal{U}$,

$$m^e_\delta(v, u) = \min_{|x-\bar{x}|<\delta} \{\, \varphi_e(x, u) - v\cdot[x - \bar{x}] \,\}, \ \text{ attained for } \ x = M_\delta(v, u) = M^{\delta,\alpha}(v, u).$$

Then, from the definition of $M^{\delta,\alpha}(v, u)$ in (1.8) and via (1.3), we have some y such that $(v, y) \in \partial\varphi(x, u)$, hence $(v, y_e) \in \partial\varphi_e(x, u)$ for $y_e = y + eu$, all under the umbrella of (3.2), so that actually $\varphi_e(x, u) = \psi(x, u)$. This reveals that

$$m^e_\delta(v, u) = \min_{|x-\bar{x}|<\delta} \{\, \psi(x, u) - v\cdot[x - \bar{x}] \,\}, \ \text{ attained for } \ x = M_\delta(v, u). \tag{3.13}$$

Since ψ is in particular convex, this inf-projection formula implies that $m^e_\delta(v, u)$ is convex in $u \in \mathcal{U}$ and moreover by duality [19, 11.39] that $\partial_u m^e_\delta(v, u) = Y_e(x, u, v) = Y(x, u, v) + eu$ for $x = M_\delta(v, u)$. Translating this back to $m_\delta(v, u)$, we get the claimed formula (1.9).

The continuity of $M_\delta(v, u)$ in u (which becomes continuity in (v, u) when combined with the Lipschitz continuity in v) also comes out of (3.13), as follows. Consider the strongly convex function $f_v(x, u)$ defined by

$$f_v(x, u) = \psi(x, u) - v\cdot[x - \bar{x}] \ \text{ when } \ |x - \bar{x}| \leq \delta, \ \text{ otherwise } \ f_v(x, u) = \infty.$$

Observe from (3.13) and the strong convexity that the singleton $x = M_\delta(v,u)$ is $\operatorname{argmin} f_v(\cdot,u)$. According to [19, 7.44] in describing parametric minimization under convexity, the mapping $u \mapsto \operatorname{argmin} f_v(x,u)$ is continuous on the interior of the convex set U consisting of the vectors u such that $f_v(\cdot,u)$ is proper. That set U obviously includes the vectors u where $\operatorname{argmin} f_v(\cdot,u)$ is nonempty, so its interior includes the open neighborhood $\mathcal{U}$ of 0 where we have singleton $M_\delta(v,u)$. $\square$

The example presented next demonstrates that the strong variational sufficiency in (B) is inadequate for having the full stability in (A), even when φ itself is already a strongly convex function, in which case no elicitation is even needed.

Example 3.2 (full stability absent despite strong convexity). *For $(x,u) \in I\!R \times I\!R$ and $(\bar{x},\bar{u}) = (0,0)$, the function $\varphi(x,u) = \frac{3}{4}|(x,u)|^{\frac{4}{3}} + |(x,u)| - x$ is strongly convex on any bounded convex neighborhood of $(0,0)$, where it achieves its global minimum value of 0. In particular it is prox-regular globally with the basic constraint qualification (1.5) satisfied. However, in the notation (1.1) for any $\delta > 0$, it has $M_\delta(v,u)$ being locally Lipschitz continuous in v but not locally Lipschitz continuous in (v,u). Full stability is thus lacking.*

Detail. For a convex function φ, $\partial^\infty\varphi(x,u)$ is the normal cone to the effective domain of φ at (x,u). Here that domain is $I\!R \times I\!R$, so the basic constraint qualification (1.5) is surely satisfied. Local strong convexity of φ is the property dual to its conjugate φ^* being differentiable with $\nabla\varphi$ locally Lipschitz continuous, cf. [18, Proposition 3.5], and here the conjugate function is $\varphi^*(v,y) = \frac{1}{4}\operatorname{dist}^4(v,y \mid D)$, where $D =$ unit disk around $(1,0)$, which does fit that prescription. The strong convexity in x for fixed u yields global tilt stability and with it the single-valuedness of $M_\delta(v,u)$ with $M_\delta(0,0) = 0$, where M_δ can be replaced by M through globality. But it will be seen that $M(0,u)$ fails to be Lipschitz continuous around $u = 0$. At (v,u) apart from $(0,0)$, φ is continuously differentiable with $x = M(v,u)$ if and only if $\frac{\partial\varphi}{\partial x}(x,u) = v$. In terms of the auxiliary variable $z = \sqrt{x^2+u^2} = |(x,u)|$, we have

$$\varphi(x,u) = k(z) - x \ \text{ for } \ k(z) = \frac{3}{4}z^{\frac{4}{3}} + z,$$

hence $\frac{\partial\varphi}{\partial x}(x,u) = k'(z)\frac{\partial z}{\partial x} - 1 = (z^{\frac{1}{3}}+1)(x/z) - 1$, so that $x = M(0,u) \Leftrightarrow x = z/(1+z^{\frac{1}{3}})$. From this and the definition of z we get $u^2 = z^2 - x^2 = z^2 - (z^2/(1+z^{\frac{1}{3}})^2) = [z^2/(1+z^{\frac{1}{3}})^2][(1+z^{\frac{1}{3}})^2 - 1] = x^2[(1+z^{\frac{1}{3}})^2 - 1]$. Therefore $u^2/x^2 = (1+z^{\frac{1}{3}})^2 - 1 \to 0$ as $(x,u) \to (0,0)$, which is equivalent to $M(0,u)/u \to \infty$ as $u \to 0$, where $M(0,0) = 0$. Thus, $M(0,u)$ lacks Lipschitz continuity around $u = 0$, as claimed. $\square$

It might be wondered whether it's really necessary in (B) to require strong variational sufficiency at $\bar{x}$ to hold for all multiplier vectors $\bar{y}$. Maybe if it holds for one it automatically holds for all? Here is a counterexample to that in the context of classical nonlinear programming.

Example 3.3 (strong variational sufficiency for some multipliers but not all). *In the case of φ in (P) that corresponds to the problem of minimizing $f_0(x_1,x_2,x_3,x_4)$ subject to $f_i(x_1,x_2,x_3,x_4) \le 0$ for $i = 1,2,3,4$, where*

$$\begin{aligned}
f_0(x_1,x_2,x_3,x_4) &= x_3 + \tfrac{1}{2}x_4^2\\
f_1(x_1,x_2,x_3,x_4) &= x_1 - x_3\\
f_2(x_1,x_2,x_3,x_4) &= -x_1 - x_3\\
f_3(x_1,x_2,x_3,x_4) &= x_2 - x_3 - \tfrac{1}{2}x_4^2\\
f_4(x_1,x_2,x_3,x_4) &= -x_2 - x_3 - \tfrac{1}{2}x_4^2,
\end{aligned}$$

with canonical perturbation parameters u_i, namely

$$\varphi(x,u) = f_0(x) + \delta_{\boldsymbol{R}^4_-}(f_1(x)+u_1, f_2(x)+u_2, f_3(x)+u_3, f_4(x)+u_4),$$

the strong variational sufficient condition is satisfied at $\bar{x} = (0,0,0,0)$ for just some of the associated multiplier vectors $\bar{y}$.

Detail. The constraints require $x_3 \geq |x_1|$ and $x_3 \geq |x_2| - \frac{1}{2}x_4^2$, so it's clear that $\bar{x} = (0,0,0,0)$ is locally optimal with all four constraints being active. The subspace orthogonal to the four active constraint gradients is $S = (0,0,0,I\!R)$, while the Lagrangian function is

$$\begin{aligned} L(x,y) &= x_3 + \tfrac{1}{2}x_4^2 + y_1(x_1 - x_3) - y_2(x_1 + x_3) + y_3(x_2 - x_3 - \tfrac{1}{2}x_4^2) - y_4(x_2 + x_3 + \tfrac{1}{2}x_4^2) \\ &= x_1(y_1 - y_2) + x_2(y_3 - y_4) + x_3(1 - y_1 - y_2 - y_3 - y_4) + \tfrac{1}{2}(1 - y_3 - y_4)x_4^2. \end{aligned}$$

The KKT conditions require $y_i \geq 0$ along with $y_1 = y_2$, $y_3 = y_4$, and $y_1 + y_2 + y_3 + y_4 = 1$. There is just one degree of freedom, so the set of multiplier vectors $\bar{y}$ associated with $\bar{x}$ can be described as

$$\overline{Y} = \{\, \tfrac{1}{2}(1-\theta, 1-\theta, \theta, \theta) \,|\, 0 \leq \theta \leq 1 \}.$$

The classical strong second-order sufficient condition for local optimality, which is known from [15, Example 1] to be equivalent to strong variational sufficiency, is satisfied at $\bar{x}$ for $\bar{y} \in Y$ if and only if the hessian $H = \nabla^2_{xx}L(\bar{x}, \bar{y}(\theta))$ is positive-definite relative to $S = (0,0,0,I\!R)$. The 4×4 matrix H has entries h_{ij} that are 0 except possibly for h_{44}, with the strong second-order sufficient condition be satisfied if and only if $h_{44} > 0$. But $h_{44} = 1 - 2\theta$. Thus we have the condition satisfied for the multiplier vectors $\bar{y}$ corresponding to $\theta \in [0, \frac{1}{2})$, but not for the ones corresponding to $\theta \in [\frac{1}{2}, 1]$. □

Proof of Theorem 1.2. From (C) we have the uniform Lipschitz continuity of $M_\delta(v,u)$ in $v \in \mathcal{V}$ for each $u \in \mathcal{U}$. To reach (A), we also need, for a possibly smaller δ and neighborhood $\mathcal{V} \times \mathcal{U}$ of $(0,0)$, the uniform Lipschitz continuity of $M_\delta(v,u)$ in $u \in \mathcal{U}$ for each $v \in \mathcal{V}$. Here we can replace $M_\delta(v,u)$ by $M^{\delta,\alpha}(v,u)$ on the basis of the relationship (1.8) in Theorem 1.1. The focus then is on having the mappings $M_v^{\delta,\alpha}$ in (1.10) be Lipschitz continuous around $u = 0$ uniformly with respect to v in some neighborhood of $v = 0$.

Each mapping $M_v^{\delta,\alpha}$ is already single-valued and continuous around $u = 0$ for v near to 0, so Lipschitz continuity is identical here to the Aubin property, although that ordinarily would be weaker. A criterion for M_v^δ to have the Aubin property at $u = 0$ with respect to $x = M_v^{\delta,\alpha}(0)$ is available from [2, Theorem 4B.2]; it supposes $\operatorname{gph} M_v^{\delta,\alpha}$ to be locally closed around (u,x) for $x = M_v^{\delta,\alpha}(u)$, but we know that does hold under the prox-regularity, because the optimal value $m_\delta(v,u) = \varphi(M_v^{\delta,\alpha}(u), u)$ approaches $m_\delta(v,0) = \varphi(M_v^{\delta,\alpha}(0), 0)$ as $u \to 0$. The criterion in question is the uniform boundedness of the inner norms $|DM_v^{\delta,\alpha}(u)|^-$ on some neighborhood of $u = 0$; such a bound then provides a Lipschitz constant. Here we furthermore want uniformity with respect to v in a neighborhood of $v = 0$. The finiteness of the upper limit in (1.12) guarantees all of that. □

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