Sensitivity analysis of monotone inclusions via the proto-differentiability of the resolvent operator

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ABSTRACT. This paper is devoted to the study of sensitivity to perturbation of parametrized variational inclusions involving maximal monotone operators in a Hilbert space. The perturbation of all the data involved in the problem is taken into account. Using the concept of proto-differentiability of a multifunction and the notion of semi-differentiability of a single-valued map, we establish the differentiability of the solution of a parametrized monotone inclusion. We also give an exact formula of the proto-derivative of the resolvent operator associated to the maximal monotone parameterized variational inclusion. This shows that the derivative of the solution of the parametrized variational inclusion obeys the same pattern by being itself a solution of a variational inclusion involving the semi-derivative and the proto-derivative of the associated maps. An application to the study of the sensitivity analysis of a parametrized primal-dual composite monotone inclusion is given. Under some sufficient conditions on the data, it is shown that the primal and the dual solutions are differentiable and their derivatives belong to the derivative of the associated Kuhn-Tucker set.

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1 Introduction

Generalized equations constitute an important topic in optimization theory and variational analysis. They consists in finding the zeros of the sum of a single-valued map and a multifunction (or a set-valued map). Generalized equations are also known in the literature as variational inclusions and contain as a particular case variational inequalities and complementarity problems. When the multifunction coincides with the normal cone mapping to a closed convex set, generalized equations reduce to variational inequalities of first kind (which express first-order optimality conditions associated with constrained optimization problems). For the particular case of the set-valued part being the subdifferential of a lower semicontinuous, proper and convex function, it gives rise to variational inequalities of second kind. In both cases, the involved multifunctions are maximal monotone operators. Generalized equations involving maximal monotone operators are common beyond the optimization literature. They provide an important tool for the modelling of many nonlinear phenomena in PDE's. The literature on this subject is abundant about different kinds of existence results, regularity of solutions, various extension of inclusions as well as many numerical aspects for approximating these problems. One of the most important issues in variational analysis is the study of sensitivity to perturbation of the solution to parametrized inclusions involving maximal monotone operators. The analysis of the continuity or differentiability properties of the solution map to

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generalized equations under perturbation of the involved data is essential in applied mathematics and engineering, not only for the knowledge of the behavior and the response of a system to perturbations, but also for the design of efficient algorithms in view of their numerical simulations. For a comprehensive reference on sensitivity analysis for continuous optimization problems we refer to the book by Bonnans and Shapiro [5]. In 1980, S.M. Robinson [22] studied parameterized variational inequalities. He proved a powerful implicit function theorem that states, under suitable second-order sufficient conditions, that the solution mapping of the standard nonlinear programming problem has a Lipschitz continuous single-valued localization around a reference point (see Theorem 2G.9 and Chapter 3 in [9] for an update statement). Dontchev and Hager in [7] extended Robinson's implicit function theorem and obtained a characterization of the pseudo-Lipschitz properties of the solution map associated to a perturbed generalized equation. In 2006, Dontchev and Rockafellar [8] revisited Robinson's theorem from the perspective of the recent tools developed in variational analysis. With the objective of studying the single-valuedness, the Lipschitz continuity and the differentiability properties of the solution map to a perturbed general variational inclusion around a reference point, Dontchev and Rockafellar gave several extensions of Robinson's theorem (see [9] Chapters 2, 3 and 4 for more details). To deal with the generalized differentiation of multifunctions, there exist in the literature of variational analysis many tools such as: "Bouligand" differentiability [22], the graphical derivative, the limiting coderivative [15, 28] etc. The proto-differentiability of multifunctions introduced by Rockafellar [25] proves to be an efficient tool for the study of sensitivity analysis of generalized equations. It is obtained from the Painlevé-Kuratowski set limits of the graphs of first-order difference quotient of the involved multifunctions. Going further into second-order analysis, R.T. Rockafellar [26] proved the equivalence between the proto-differentiability of the subdifferential of a lower semicontinuous proper and convex function and the twice epi-differentiability of this function (obtained from the Painlevé-Kuratowski set limits of the epigraph of second-order difference quotient). The proto-differentiability of the associated Moreau's proximity operator is also obtained (see also [28, Chapter 13]). A large class of proto-differentiable maps has been given in the works by Rockafellar [25, 26, 27, 28] and Poliquin-Rockafellar [19, 20].

In the whole paper $\mathcal{H}$ is a real Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. The focus is on the study of the sensitivity analysis, with respect to the parameter $t \in [0, \delta)$, with $\delta > 0$, of the *Variational Inclusion* given by

$$\mathrm{VI}\big(A(t,\cdot), B(t,\cdot), \xi(t)\big) \left\{ \begin{array}{l} \text{find } x(t) \in \mathcal{H} \text{ such that} \\ \xi(t) \in A(t, x(t)) + B(t, x(t)), \end{array} \right.$$

where

(i) $A : [0, \delta) \times \mathcal{H} \to \mathcal{H}$ is a single-valued map supposed to be:
• *uniformly Lipschitz continuous*, i.e.

$$\exists k \geq 0, \quad \forall t \in [0, \delta), \quad \forall x_1, x_2 \in \mathcal{H}, \quad \|A(t, x_2) - A(t, x_1)\| \leq k\|x_2 - x_1\|,$$

• and *uniformly strongly monotone*, i.e.

$$\exists \alpha > 0, \quad \forall t \in [0, \delta), \quad \forall x_1, x_2 \in \mathcal{H}, \quad \langle A(t, x_2) - A(t, x_1), x_2 - x_1 \rangle \geq \alpha \|x_2 - x_1\|^2.$$

(ii) $B : [0, \delta) \times \mathcal{H} \rightrightarrows \mathcal{H}$ is a parametrized maximal monotone operator, i.e. for all $t \in [0, \delta)$, $B(t, \cdot)$ is a maximal monotone operator.

(iii) $\xi : [0, \delta) \to \mathcal{H},\ t \mapsto \xi(t)$ is a given right hand-term.

In the whole paper, the parameter t is a *general parameter*, not necessarily the *time* as the notation may suggest. Since we are interested in the right-differentiability properties at $t = 0$, we just need to

have all maps defined on a neighborhood to the right of the origin, i.e. an interval $[0, \delta)$. We note that both operators A and B are perturbed in $\mathrm{VI}\big(A(t,\cdot), B(t,\cdot), \xi(t)\big)$, which requires special adjustments in the definitions of the semi-differentiability and proto-differentiability of the perturbed maps $A(t,\cdot)$ and $B(t,\cdot)$ (see Definitions 2.1 and 2.5 for more details). As we focus on the differentiability properties of the solution $t \mapsto x(t)$ with respect to data perturbations, we suppose that the perturbation parameter t lives in a one-dimensional space $[0, \delta) \subset \mathbb{R}_+$.
By assumptions (i) and (ii), the inclusion $\mathrm{VI}\big(A(t,\cdot), B(t,\cdot), \xi(t)\big)$ admits for all $t \in [0, \delta)$ a unique solution $x(t) \in \mathcal{H}$ given under the following resolvent equation

$$x(t) = [A(t,\cdot) + B(t,\cdot)]^{-1}(\xi(t)) := J_{A(t,\cdot),\, B(t,\cdot)}(\xi(t)). \tag{1.1}$$

The main objective of this paper is to derive sufficient conditions on the data A, B and ξ ensuring the right-differentiability at $t = 0$ of the solution $x : [0, \delta) \to \mathcal{H}$ and to provide an explicit formula for its right-derivative $x'(0)$. We show exactly that the right-derivative of x at 0 is a solution of a variational inclusion involving the semi- and proto-derivatives of the operators A and B. More precisely, we prove that $x'(0)$ is a solution of the following variational inclusion

$$\xi'(0) \in D_s A\big(x(0)\big)\big(x'(0)\big) + D_p B\big(x(0)|x^*(0)\big)\big(x'(0)\big), \tag{1.2}$$

with $x^*(0) = \xi(0) - A(0, x(0)) \in B(0, x(0))$, $D_s A\big(x(0)\big)$ the semi-derivative of A at $x(0)$ and $D_p B\big(x(0)|x^*(0)\big)$ the proto-derivative of B at $x(0)$ relative to $x^*(0)$ (see Theorem 3.1 for more details).

As an application of the main result, we investigate the sensitivity analysis of a primal-dual composite variational inclusion involving parametrized maximal monotone and linear operators. Using Attouch-Théra duality theory [4], we associate to a composite monotone variational inclusion a dual problem. We recast the primal-dual inclusion as a problem of the form $\mathrm{VI}\big(A(t,\cdot), B(t,\cdot), \xi(t)\big)$ and apply the main result to prove that the primal and the dual perturbed solutions are right-differentiable at $t = 0$ and their right-derivatives belong to the derivative of the associated Kuhn-Tucker set (see Theorem 4.1 for more details).

Our aim in this paper is the first-order sensitivity analysis in the same spirit as the paper by Levy and Rockafellar [11] with a special focus on the set-valued part being a parametrized maximal monotone operator. For second-order analysis and its link with the twice epi-differentiability we refer the reader to [1, 6, 16, 19, 20, 24, 26, 27].

The paper is organized as follows. Section 2 is devoted to the main notations and definitions used throughout the paper. We recall some tools from the monotone operator theory, the variational convergence associated with sets and graph of operators as well as the semi- and proto-derivatives associated to single-valued and set-valued maps. In Section 3, we state and prove the main result (Theorem 3.1). In Section 4, as an application, we investigate the sensitivity analysis of a parameterized primal-dual composite monotone inclusion. We conclude this paper with some additional comments in Section 5.

2 Background on operator theory and variational convergence

Let us introduce first some notations and recall some backgrounds from variational analysis concerning the operator theory and the variational convergence. We say that a single-valued map $A : \mathcal{H} \to \mathcal{H}$ is k-Lipschitz and α-strongly monotone if

$$\|A(x) - A(y)\| \le k\|x - y\| \text{ and } \langle A(x) - A(y), x - y\rangle \ge \alpha\|x - y\|^2,$$

for all $x,\ y \in \mathcal{H}$ and for some $k > 0$ and $\alpha > 0$, respectively.
In what follows, we denote by $\mathcal{L}_{k,\alpha}(\mathcal{H})$ the set of single-valued operators $A : \mathcal{H} \to \mathcal{H}$ which are k-Lipschitz continuous and α-strongly monotone.

Remark 2.1 Let $A : \mathcal{H} \to \mathcal{H}$ be a given single-valued map.

(i) We note that if A is strongly monotone with modulus $\alpha > 0$, then the (monotone) inverse of A is Lipschitz continuous with constant $\frac{1}{\alpha}$.

(ii) A is α-strongly monotone if and only if $(A - \alpha \mathrm{Id})$ is monotone.

(iii) The operator A is said to be β-cocoercive, with $\beta > 0$, if

$$\langle Ax - Ay, x - y \rangle \geq \beta \|Ax - Ay\|^2, \ \ \forall x, \, y \in \mathcal{H},$$

which means that A^{-1} is β-strongly monotone.
It is easy to check that a β-cocoercive operator is $\frac{1}{\beta}$-Lipschitz continuous.

(iv) If $A \in \mathcal{L}_{k,\alpha}(\mathcal{H})$, then A is $\frac{\alpha}{k^2}$-cocoercive.

For a set-valued map $B : \mathcal{H} \rightrightarrows \mathcal{H}$, the *domain* of B is given by $\mathrm{Dom}(B) := \{x \in \mathcal{H} \mid B(x) \neq \emptyset\}$ and its *graph* is defined by $\mathrm{gph}\,(B) := \{(x, y) \in \mathcal{H} \times \mathcal{H} \mid y \in B(x)\}$. We denote by $B^{-1} : \mathcal{H} \rightrightarrows \mathcal{H}$ the set-valued map defined by

$$B^{-1}(y) := \{x \in \mathcal{H} \mid y \in B(x)\},$$

for all $y \in \mathcal{H}$.
The range of B is defined by

$$\mathrm{Rge}(B) = \bigcup_{x \in \mathcal{H}} B(x).$$

The set-valued map $B : \mathcal{H} \rightrightarrows \mathcal{H}$ is called monotone if and only if $\langle x^* - y^*, x - y \rangle \geq 0, \ \forall (x, x^*) \in \mathrm{gph}\,(B), \ \forall (y, y^*) \in \mathrm{gph}\,(B)$.
The set-valued map B is maximal monotone if and only if it is monotone and its graph is maximal in the sense of inclusion, i.e., $\mathrm{gph}\,(B)$ is not properly contained in the graph of any other monotone operator.
For a given set-valued map $B : \mathcal{H} \rightrightarrows \mathcal{H}$, the resolvent of B is given by

$$J_B = (\mathrm{Id} + B)^{-1}, \tag{2.1}$$

where Id stands for the identity operator on $\mathcal{H}$.
It is well known that if $B : \mathcal{H} \rightrightarrows \mathcal{H}$ is maximal monotone, then its resolvent J_B is a single-valued and nonexpansive mapping, i.e.

$$\|J_B(x) - J_B(y)\| \leq \|x - y\|, \ \forall x, \ y \in \mathcal{H}.$$

We could replace the identity operator in (2.1) by any single-valued operator $A \in \mathcal{L}_{k,\alpha}(\mathcal{H})$ to define the extended A-resolvent of B given by

$$J_{A,B} = (A + B)^{-1}. \tag{2.2}$$

We note that if $B : \mathcal{H} \rightrightarrows \mathcal{H}$ is maximal monotone and $A \in \mathcal{L}_{k,\alpha}(\mathcal{H})$, then $J_{A,B}$ is single-valued and $\frac{1}{\alpha}$-Lipschitz, i.e.

$$\|J_{A,B}(x) - J_{A,B}(y)\| \leq \frac{1}{\alpha}\|x - y\|, \ \ \forall x, \ y \in \mathcal{H}. \tag{2.3}$$

When $A = \mathrm{Id}$, the resolvent $J_{\mathrm{Id},B}$ defined in (2.2) coincides with J_B.

In the following definition, we introduce the notion of *semi-differentiability* of a single-valued map.

Definition 2.1 (Semi-differentiability) *Let $A : [0,\delta) \times \mathcal{H} \to \mathcal{H}$ be a parameterized single-valued map and let $x \in \mathcal{H}$. If the limit*

$$D_s A(x)(\omega) := \lim_{\substack{\tau \to 0 \\ \omega' \to \omega}} \frac{A(\tau, x + \tau\omega') - A(0,x)}{\tau}$$

exists in $\mathcal{H}$ for all $\omega \in \mathcal{H}$, we say that A is semi-differentiable at x. In that case, $D_s A(x) : \mathcal{H} \to \mathcal{H}$ is a single-valued map called the semi-derivative of A at x.

If the single-valued map A is t-independent, Definition 2.1 recovers the classical notion of semi-differentiability originally introduced in [18]. We refer also to [9] (Chapter 2 for alternative characterizations and calculus rules for the semi-differentiability).

In what follows, we denote by $\mathcal{U}_{k,\alpha}(\mathcal{H})$ the set of all parameterized single-valued maps $A : [0,\delta) \times \mathcal{H} \to \mathcal{H}$ such that A is *uniformly k-Lipschitz continuous* and *uniformly α-strongly monotone* (with respect to its second argument).

Remark 2.2 If $A \in \mathcal{U}_{k,\alpha}(\mathcal{H})$ is semi-differentiable at $x \in \mathcal{H}$, then $D_s A(x) \in \mathcal{U}_{k,\alpha}(\mathcal{H})$.

Let $(C_\tau)_{\tau>0}$ be a parameterized family of subsets of $\mathcal{H}$. The *outer* and the *inner limits* of $(C_\tau)_{\tau>0}$ when $\tau \to 0$ are defined respectively by

$$\begin{aligned}
\limsup C_\tau &:= \{x \in \mathcal{H} \mid \exists (t_\nu)_\nu \to 0,\ \exists (x_\nu)_\nu \to x,\ \forall \nu \in \mathbb{N},\ x_\nu \in C_{t_\nu}\}, \\
\liminf C_\tau &:= \{x \in \mathcal{H} \mid \forall (t_\nu)_\nu \to 0,\ \exists (x_\nu)_\nu \to x,\ \exists N \in \mathbb{N},\ \forall \nu \geq N,\ x_\nu \in C_{t_\nu}\}
\end{aligned}$$

In the whole paper, note that all limits (inner and outer limits) are taken with respect to $\tau \to 0$ and for the strong topology. For the ease of notations, when no confusion is possible, the notation $\tau \to 0$ will be removed.

Note that the following inclusion holds true in general:

$$\liminf C_\tau \subset \limsup C_\tau.$$

The Painlevé-Kuratowski convergence is defined by this inclusion being an equality. The next definition is in this sense.

Definition 2.2 (Painlevé-Kuratowski convergence) *A parameterized family $(C_\tau)_{\tau>0}$ of subsets of $\mathcal{H}$ is said to be convergent in the sense of Painlevé-Kuratowski if*

$$\limsup C_\tau \subset \liminf C_\tau.$$

In that case, we denote by $\lim C_\tau := \liminf C_\tau = \limsup C_\tau$.

Definition 2.3 (Graph convergence) *A parameterized family $(B_\tau)_{\tau>0}$ of set-valued maps on $\mathcal{H}$ graph converges to the set-valued map $B : \mathcal{H} \rightrightarrows \mathcal{H}$ as $\tau \to 0$ if*

$$\limsup \operatorname{gph}(B_\tau) = \liminf \operatorname{gph}(B_\tau) = \operatorname{gph}(B),$$

i.e. $(\operatorname{gph}(B_\tau))_{\tau>0}$ *converges in the sense of Painlevé-Kuratowski to* $\operatorname{gph}(B)$.
We write $\operatorname{gph-lim} B_\tau = B$.

Let $B : \mathcal{H} \rightrightarrows \mathcal{H}$ be a given set-valued map. Rockafellar introduced in [25] the notion of proto-differentiability of B at a point $x \in \mathrm{Dom}(B)$ relative to a point $x^* \in B(x)$ by using the graph convergence of the following first-order difference quotient

$$\Delta_\tau B(x|x^*)(\omega) := \frac{B(x+\tau\omega) - x^*}{\tau}, \tag{2.4}$$

with $\tau > 0$ and $\omega \in \mathcal{H}$.

Definition 2.4 (Proto-differentiability) *The set-valued map $B : \mathcal{H} \rightrightarrows \mathcal{H}$ is proto-differentiable at $x \in \mathcal{H}$ relative to $x^* \in B(x)$ if $(\Delta_\tau B(x|x^*))_{\tau>0}$ defined in* (2.4) *graph converges. The proto-derivative of B at x relative to x^* is the set-valued map whose graph is the limit set, and is denoted by*

$$D_p B(x|x^*) := \text{gph-lim}\, \Delta_\tau B(x|x^*).$$

B is said to be proto-differentiable at $x \in \mathrm{Dom}(B)$ if for every $x^* \in B(x)$, B is proto-differentiable at x relative to x^*.

As noticed by Rockafellar in [25] (see also [6]), there is a link between the proto-differentiability and the contingent cone. In fact, we have

$$\mathrm{gph}\left(\Delta_\tau B(x|x^*)\right) = \frac{\mathrm{gph}\,(B) - (x, x^*)}{\tau}.$$

We define the Bouligand tangent (contingent) and the adjacent tangent cones of $\mathrm{gph}\,(B)$ at (x, x^*) respectively by

$$T_{\mathrm{gph}\,(B)}(x, x^*) = \limsup \frac{\mathrm{gph}\,(B) - (x, x^*)}{\tau} \text{ and } \widetilde{T}_{\mathrm{gph}\,(B)}(x, x^*) = \liminf \frac{\mathrm{gph}\,(B) - (x, x^*)}{\tau} \tag{2.5}$$

The proto-differentiability of B at x relative to x^* is equivalent to the derivability of $\mathrm{gph}\,(B)$ at (x, x^*) which means that the two cones, defined in (2.5), coincide and the graph of the proto-derivative $D_p B(x|x^*)$ is the common cone. For more details we refer to [25].
One of the important properties of the proto-derivative $D_p B(x|x^*) : \mathcal{H} \rightrightarrows \mathcal{H}$ is that its graph is closed and satisfies

$$0 \in D_p B(x|x^*)(0) \text{ and } D_p B(x|x^*)(\lambda\omega) = \lambda D_p B(x|x^*)(\omega), \text{ for every } \omega \in \mathcal{H} \text{ and } \lambda > 0. \tag{2.6}$$

In order to take into account the perturbation in the set-valued part of the variational inclusion $\mathrm{VI}\big(A(t,\cdot), B(t,\cdot), \xi(t)\big)$, the notion of proto-differentiability introduced in Definition 2.4, could be extended easily to the case where the set-valued map B depends on the parameter $t \in [0, \delta)$.
Let $B : [0, \delta) \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a parameterized set-valued map. For all $\tau > 0$, $x \in \mathcal{H}$ and $x^* \in B(0, x)$, the difference quotient in (2.4) is replaced by the following

$$\Delta_\tau B(x|x^*)(\omega) := \frac{B(\tau, x+\tau\omega) - x^*}{\tau}. \tag{2.7}$$

Definition 2.5 (Proto-differentiability: the t-dependent case) *Let $B : [0, \delta) \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a parameterized set-valued map. We say that B is proto-differentiable at $x \in \mathcal{H}$ relative to $x^* \in B(0, x)$ if $(\Delta_\tau B(x|x^*))_{\tau>0}$ graph converges. In that case, we denote by*

$$D_p B(x|x^*) := \text{gph-lim}\, \Delta_\tau B(x|x^*)$$

the set-valued map $D_p B(x|x^*) : \mathcal{H} \rightrightarrows \mathcal{H}$ is called the proto-derivative of B at x relative to x^*.

Remark 2.3 Let $B : [0, \delta) \times \mathcal{H} \rightrightarrows \mathcal{H}$ be a parameterized set-valued map, $x \in \mathcal{H}$ and $x^* \in B(0, x)$. Then, B is proto-differentiable at x relative to x^* if and only if B^{-1} is proto-differentiable at x^* relative to x. In that case, it holds that

$$D_p(B^{-1})(x^*|x) := (D_pB(x|x^*))^{-1}.$$

Remark 2.4 Let us point out that contrary to the t-independent case, the proto-derivative $D_pB(x|x^*)$ defined in Definition 2.5 for the t-dependent case may not satisfy the properties (2.6) in general. The following example is in this sense.

Example 2.1 *Let us consider the following set-valued map $B : [0, \delta) \times \mathbb{R} \rightrightarrows \mathbb{R}$, $(t, x) \mapsto B(t, x)$ defined by*

$$B(t,x) = \begin{cases} -1 & \textit{if} \quad x < t \\ [-1,1] & \textit{if} \quad x = t \\ 1 & \textit{if} \quad x > t \end{cases}$$

For $x = 0$ and $x^* = 0 \in B(0, 0)$, we have

$$\Delta_\tau B(0|0)(\omega) = \begin{cases} \frac{-1}{\tau} & \text{if} \quad \omega < 1 \\ [\frac{-1}{\tau}, \frac{1}{\tau}] & \text{if} \quad \omega = 1 \\ \frac{1}{\tau} & \text{if} \quad \omega > 1. \end{cases}$$

Hence,

$$D_pB(0|0)(\omega) = \begin{cases} \emptyset & \text{if} \quad \omega \neq 1 \\ \mathbb{R} & \text{if} \quad \omega = 1 \end{cases}$$

We note that in this case $0 \notin D_pB(0|0)(0)$.

Remark 2.5 If a parameterized single-valued map $A : [0, \delta) \times \mathcal{H} \to \mathcal{H}$ is semi-differentiable at $x \in \mathcal{H}$, then A is proto-differentiable at x for $A(0, x)$ with $D_pA(x|A(0, x)) = D_sA(x)$.

One can easily prove the following result. In the t-independent case, we recover [28, p.331-333].

Proposition 2.1 *Let $A : [0, \delta) \times \mathcal{H} \to \mathcal{H}$ and $B : [0, \delta) \times \mathcal{H} \rightrightarrows \mathcal{H}$ be two parameterized single-valued and set-valued maps, respectively. Let $x \in \mathcal{H}$ and $x^* \in A(0, x) + B(0, x)$. If A is semi-differentiable at x, then $A + B$ is proto-differentiable at x relative to x^* if and only if B is proto-differentiable at x relative to $x^* - A(0, x)$. In that case it holds that*

$$D_p(A + B)(x|x^*) = D_sA(x) + D_pB(x|x^* - A(0, x)).$$

Lemma 2.1 *Suppose that $B : [0, \delta) \times \mathcal{H} \rightrightarrows \mathcal{H}$ is a parametrized maximal monotone operator. Let $\tau > 0$, $x \in \mathcal{H}$ and $x^* \in B(0, x)$. Then the following operator defined by*

$$\begin{array}{rccl} \Delta_\tau B(x|x^*) : & \mathcal{H} & \rightrightarrows & \mathcal{H} \\ & \omega & \mapsto & \Delta_\tau B(x|x^*)(\omega) := \dfrac{B(\tau, x + \tau\omega) - x^*}{\tau} \end{array}$$

is also maximal monotone.

Proof. Let $(\omega_1, \omega_1^*),\ (\omega_2, \omega_2^*) \in \operatorname{gph}(\Delta_\tau B(x|x^*))$. We have

$$\omega_i^* \in \Delta_\tau B(x|x^*)(\omega_i) \Longleftrightarrow x^* + \tau\omega_i^* \in B(\tau, x + \tau\omega_i),\ i = 1, 2.$$

The monotonicity of $\Delta_\tau B(x|x^*)$ is a direct consequence of the monotonicity of $B(\tau,\cdot)$.
Let $(\zeta,\zeta^*)\in\mathcal{H}\times\mathcal{H}$ such that

$$\langle \omega^*-\zeta^*,\omega-\zeta\rangle\geq 0,\ \forall(\omega,\omega^*)\in\mathrm{gph}\,(\Delta_\tau B(x|x^*)).$$

Hence,

$$\big\langle (x^*+\tau\omega^*)-(x^*+\tau\zeta^*),(x+\tau\omega)-(x+\tau\zeta)\big\rangle\geq 0,\ \forall(\omega,\omega^*)\in\mathrm{gph}\,(\Delta_\tau B(x|x^*)).$$

Since $B(\tau,\cdot)$ is maximal monotone, we deduce that $x^*+\tau\zeta^*\in B(\tau,x+\tau\zeta)$, i.e. $\zeta^*\in\Delta_\tau B(x|x^*)(\zeta)$. Therefore, $\Delta_\tau B(x|x^*)$ is maximal monotone. ■

Remark 2.6 If the space $\mathcal{H}$ is of finite dimensions, then it is well-known that the class of maximal monotone operators is closed with respect to the Painlevé-Kuratowski set convergence. This means that if $B:[0,\delta)\times\mathcal{H}\rightrightarrows\mathcal{H}$ is a parametrized maximal monotone operator and proto-differentiable at $x\in\mathcal{H}$ relative to $x^*\in B(0,x)$ with $\dim(H)<+\infty$, then by Lemma 2.1 $\Delta_\tau B(x|x^*)$ is maximal monotone and hence its graph-limit $D_pB(x|x^*)$ is also a maximal monotone operator.
The question of the preservation of the maximal monotonicity property under the graph-convergence limit in an infinite dimensional space remains open. It would be interesting to give a sufficient condition ensuring that the graph limit (in the sense of Painlevé-Kuratowski) of a sequence of maximal monotone operators is still maximal monotone.

The following proposition characterizes the graph convergence of maximal monotone operators in terms of the pointwise convergence of their resolvents (for a proof see [2]).

Proposition 2.2 *Let $B:[0,\delta)\times\mathcal{H}\rightrightarrows\mathcal{H}$ be a parametrized maximal monotone operator which is proto-differentiable at $x\in\mathcal{H}$ relative to $x^*\in B(0,x)$ such that $D_pB(x|x^*)$ is also maximal monotone. The following equivalences hold:*

(i) gph-lim $\Delta_\tau B(x|x^*)=D_pB(x|x^*)$;

(ii) $\forall\lambda>0$ and $\forall\omega\in\mathcal{H}$, $J_{\lambda\Delta_\tau B(x|x^*)}(\omega)\to J_{\lambda D_pB(x|x^*)}(\omega)$ (pointwise);

(iii) for some $\lambda_0>0$ and $\forall\omega\in\mathcal{H}$, $J_{\lambda_0\Delta_\tau B(x|x^*)}(\omega)\to J_{\lambda_0 D_pB(x|x^*)}(\omega)$ (pointwise).

3 Sensitivity analysis of variational inclusions involving maximal monotone operators

Let $M:[0,\delta)\times\mathcal{H}\rightrightarrows\mathcal{H}$ be a parameterized set-valued map. For the ease of notations, we denote by $M^{-1}:[0,\delta)\times\mathcal{H}\rightrightarrows\mathcal{H},\ (t,x)\mapsto M^{-1}(t,x)$ the parameterized set-valued map defined by

$$M^{-1}(t,x):=(M(t,\cdot))^{-1}(x),\ \text{for all}\ (t,x)\in[0,\delta)\times\mathcal{H}.$$

Let $A:[0,\delta)\times\mathcal{H}\to\mathcal{H}$ and $B:[0,\delta)\times\mathcal{H}\rightrightarrows\mathcal{H}$ two given t-dependent single-valued and set-valued maps, respectively. For the simplicity of notations we set

$$\begin{array}{rccl}\mathrm{J}_{A,B}: & [0,\delta)\times\mathcal{H} & \rightrightarrows & \mathcal{H}\\ & (t,x) & \mapsto & \mathrm{J}_{A,B}(t,x):=J_{A(t,\cdot),B(t,\cdot)}(x).\end{array}$$

Using the notations introduced above, we note that $\mathrm{J}_{A,B}=(A+B)^{-1}$. From Remarks 2.3, 2.5 and Proposition 2.1, one can easily conclude the following lemma.

Lemma 3.1 *Let $A : [0,\delta) \times \mathcal{H} \to \mathcal{H}$, $B : [0,\delta) \times \mathcal{H} \rightrightarrows \mathcal{H}$ and $x \in \mathcal{H}$. Suppose that A is semi-differentiable at $v := \mathrm{J}_{A,B}(0,x)$. Then, $\mathrm{J}_{A,B}$ is proto-differentiable at x relative to v if and only if B is proto-differentiable at v relative to $v_0 := x - A(0,v) \in B(0,v)$. In that case, it holds that*

$$D_p\mathrm{J}_{A,B}(x|v) = \Big(D_sA(v) + D_pB(v|v_0)\Big)^{-1} = \mathrm{J}_{D_sA(v),D_pB(v|v_0)}.$$

Now we return to the initial motivation of the present paper, that is, the sensitivity analysis, with respect to the parameter $t \in [0,\delta)$, of the general nonlinear variational inclusion given by

$$\mathrm{VI}\big(A(t,\cdot),B(t,\cdot),\xi(t)\big) \begin{cases} \text{find } x(t) \in \mathcal{H} \text{ such that} \\ \xi(t) \in A(t,x(t)) + B(t,x(t)), \end{cases}$$

where $A : [0,\delta) \times \mathcal{H} \to \mathcal{H}$ is such that $A \in \mathcal{U}_{k,\alpha}(\mathcal{H})$, $B : [0,\delta) \times \mathcal{H} \rightrightarrows \mathcal{H}$ is a parametrized maximal monotone operator and $\xi : [0,\delta) \to \mathcal{H}$ is a given function. The above variational inclusion admits for all $t \in [0,\delta)$ a unique solution $x(t) \in \mathcal{H}$ given by

$$x(t) = J_{A(t,\cdot),B(t,\cdot)}(\xi(t)) = \mathrm{J}_{A,B}(t,\xi(t)). \tag{3.1}$$

In order to state the next theorem, we recall the notion of right-differentiability (or one-sided differentiability) of a function $w : [0,\delta) \to \mathcal{H}$, at $t = 0$.

Definition 3.1 *For a given function $w : [0,\delta) \to \mathcal{H}$, $t \mapsto w(t)$, the right derivative of w at $t = 0$ is the limit*

$$w'_+(0) := \lim_{t \downarrow 0} \frac{w(t) - w(0)}{t},$$

when this limit exists. For the ease of notation, we use $w'(0)$ instead of $w'_+(0)$.

The following theorem provides sufficient conditions on A, B and ξ under which $x : [0,\delta) \to \mathcal{H}$ is right-differentiable at $t = 0$ and provides an explicit formula for $x'(0)$.

Theorem 3.1 *Let $A : [0,\delta) \times \mathcal{H} \to \mathcal{H}$ such that $A \in \mathcal{U}_{k,\alpha}(\mathcal{H})$, $B : [0,\delta) \times \mathcal{H} \rightrightarrows \mathcal{H}$ a parametrized maximal monotone operator and let $\xi : [0,\delta) \to \mathcal{H}$ be a function. Consider the function $x : [0,\delta) \to \mathcal{H}$ unique solution of $\mathrm{VI}\big(A(t,\cdot),B(t,\cdot),\xi(t)\big)$ given in (3.1). If the following assertions are satisfied:*

(i) *ξ is right-differentiable at $t = 0$;*

(ii) *A is semi-differentiable at $x(0)$;*

(iii) *B is proto-differentiable at $x(0)$ relative to $x^*(0) := \xi(0) - A(0,x(0)) \in B(0,x(0))$ and its proto-derivative $D_pB(x(0)|x^*(0))$ is maximal monotone;*

then $x : [0,\delta) \to \mathcal{H}$ is right-differentiable at $t = 0$ and its right-derivative is given by

$$x'(0) = J_{D_sA(x(0)),\, D_pB(x(0)|x^*(0))}(\xi'(0)),$$

which means that $x'(0)$ is the unique solution of the following variational inclusion

$$\xi'(0) \in D_sA(x(0))(x'(0)) + D_pB(x(0)|x^*(0))(x'(0)).$$

Proof. Since A is semi-differentiable at $x(0) = \mathrm{J}_{A,B}(0,\xi(0))$ and B is proto-differentiable at $x(0)$ relative to $x^*(0) := \xi(0) - A(0,x(0)) \in B(0,x(0))$, we deduce from Lemma 3.1 that $\mathrm{J}_{A,B}$ is proto-differentiable at $\xi(0)$ relative to $x(0)$ and its proto-derivative is given by

$$D_p\mathrm{J}_{A,B}(\xi(0)|x(0)) = \mathrm{J}_{D_sA(x(0)),D_pB(x(0)|x^*(0))}.$$

For the ease of notations we set $\widetilde{A} = D_sA(x(0))$ and $\widetilde{B} = D_pB(x(0)|x^*(0))$, with $x^*(0) = \xi(0) - A(0, x(0))$. Hence,

$$D_p\mathrm{J}_{A,B}(\xi(0)|x(0))(\omega) = \mathrm{J}_{\widetilde{A},\widetilde{B}}(\omega), \text{ for every } \omega \in \mathcal{H}.$$

In particular,

$$D_p\mathrm{J}_{A,B}(\xi(0)|x(0))(\xi'(0)) = \mathrm{J}_{\widetilde{A},\widetilde{B}}(\xi'(0)).$$

Let $t_\nu \to 0$ as $\nu \to +\infty$. Using the definition of the proto-differentiability, there exists a sequence $(\zeta_\nu, \zeta_\nu^*)_{\nu\in\mathbb{N}} \to (\xi'(0), \mathrm{J}_{\widetilde{A},\widetilde{B}}(\xi'(0)))$ as $\nu \to +\infty$ such that

$$(\zeta_\nu, \zeta_\nu^*) \in \operatorname{gph}\left(\Delta_{t_\nu}\mathrm{J}_{A,B}(\xi(0)|x(0))\right) \text{ for } \nu \text{ large enough.}$$

Using (2.7), we get for ν large enough

$$J_{A(t_\nu,\cdot),\, B(t_\nu,\cdot)}(\xi(0) + t_\nu\zeta_\nu) = x(0) + t_\nu\zeta_\nu^*. \tag{3.2}$$

Let us show that $x : [0, \delta) \to \mathcal{H}$ is right-differentiable at $t = 0$. In fact, using (2.3), (3.1) and (3.2), we have

$$\begin{aligned}\left\|\frac{x(t_\nu) - x(0)}{t_\nu} - \mathrm{J}_{\widetilde{A},\widetilde{B}}(\xi'(0))\right\| &= \left\|\frac{\mathrm{J}_{A,B}(t_\nu, \xi(t_\nu)) - x(0)}{t_\nu} - \mathrm{J}_{\widetilde{A},\widetilde{B}}(\xi'(0))\right\| \\ &= \left\|\frac{J_{A(t_\nu,\cdot),\, B(t_\nu,\cdot)}(\xi(t_\nu)) - J_{A(t_\nu,\cdot),\, B(t_\nu,\cdot)}(\xi(0) + t_\nu\zeta_\nu)}{t_\nu} + \zeta_\nu^* - \mathrm{J}_{\widetilde{A},\widetilde{B}}(\xi'(0))\right\| \\ &\leq \frac{1}{\alpha}\left\|\frac{\xi(t_\nu) - \xi(0)}{t_\nu} - \zeta_\nu\right\| + \|\zeta_\nu^* - \mathrm{J}_{\widetilde{A},\widetilde{B}}(\xi'(0))\|.\end{aligned}$$

Since $\zeta_\nu \to \xi'(0)$ and $\zeta_\nu^* \to \mathrm{J}_{\widetilde{A},\widetilde{B}}(\xi'(0))$, we obtain the right-differentiability of $x(\cdot)$ at $t = 0$ and

$$x'(0) = J_{D_sA(x(0)),\, D_pB(x(0)|x^*(0))}(\xi'(0)),$$

which completes the proof of Theorem 3.1. ■

Remark 3.1 If the Hilbert space $\mathcal{H}$ is of finite dimensions, then the maximal monotonicity of the proto-derivative $D_pB(x(0)|x^*(0))$ in assumption (iii) of Theorem 3.1 is superfluous (see Remark 2.6).

Remark 3.2 For the case $B(t, \cdot) = \partial f(t, \cdot)$ where $f \in \Gamma_0(\cdot, \mathcal{H})$ the set of all parametrized lower semicontinuous, convex and proper extended real-valued functions $f : [0, \delta) \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$, $(t, x) \mapsto f(t, x)$, there exists a link between the proto-differentiability of the subdifferential $\partial f(t, \cdot)$ and the twice epi-differentiability of $f(t, \cdot)$. The link between these two notions is tied to Attouch's theorem [3] (Theorem 3.66). The notion of twice epi-differentiability was introduced by Rockafellar in [26] and was adapted to a parametrized lower semicontinuous, convex and proper extended real-valued functions in [1] (see Definition 3.9). For the t-dependent case $f(t, \cdot)$, using the notion of convergent supporting hyperplane, it is shown in Theorem 4.7 [1], that the twice epi-differentiability of $f(t, \cdot)$ is equivalent to the proto-differentiability of its subdifferential $B(t, \cdot) = \partial f(t, \cdot)$. The notion of convergent supporting hyperplane to the second-order difference quotient plays an important role and is shown to be equivalent to the properness of the second epi-derivative of $f(t, \cdot)$ (see Proposition 4.12 in [1]).
On the other hand, much efforts have been devoted in the literature to identify a large class of proto-differentiable mappings (see the works of Rockafellar [25, 26, 27], Poliquin-Rockafellar [19, 20, 21], Levy-Rockafellar [11, 13, 12] and Levy-Poliquin-Thibault [14]). The class of fully amenable functions constitutes an important class in optimization with a subdifferential which is proto-differentiable. This class consists of compositions of convex functions with a mapping of class C^2 satisfying a basic constraint qualification. For more details, we refer to [28].

4 Application to primal-dual composite monotone inclusions

In this section, we investigate the sensitivity analysis of primal-dual composite variational inclusions involving maximal monotone and linear operators. Let us consider the following primal inclusion

$$(\mathcal{P}) \begin{cases} \text{find } u(t) \in \mathcal{U} \text{ such that} \\ p(t) \in A_1(t,u(t)) + S(t,u(t)) + L^*(t) J_{A_2(t,\cdot),T(t,\cdot)}\Big(q(t) + L(t)u(t)\Big), \end{cases}$$

and its associated dual inclusion

$$(\mathcal{D}) \begin{cases} \text{find } v(t) \in \mathcal{V} \text{ such that} \\ q(t) \in A_2(t,v(t))) + T(t,v(t)) - L(t) J_{A_1(t,\cdot),S(t,\cdot)}\Big(p(t) - L^*(t)v(t)\Big), \end{cases}$$

where $\mathcal{U}$ and $\mathcal{V}$ are two real Hilbert spaces, $A_1 : [0,\delta) \times \mathcal{U} \to \mathcal{U}$, $A_2 : [0,\delta) \times \mathcal{V} \to \mathcal{V}$ are two single-valued maps, $S : [0,\delta) \times \mathcal{U} \rightrightarrows \mathcal{U}$, $T : [0,\delta) \times \mathcal{V} \rightrightarrows \mathcal{V}$ are two set-valued maps. For each $t \in [0,\delta)$, $L(t) \in \mathcal{L}(\mathcal{U},\mathcal{V})$ (space of bounded linear operators from $\mathcal{U}$ to $\mathcal{V}$), $p(t) \in \mathcal{U}$ and $q(t) \in \mathcal{V}$. Here $L^*(t)$ stands for the adjoint operator associated to $L(t) \in \mathcal{L}(\mathcal{U},\mathcal{V})$. We suppose furthermore that $A_1 \in \mathcal{U}_{k_1,\alpha_1}(\mathcal{U})$, $A_2 \in \mathcal{U}_{k_2,\alpha_2}(\mathcal{V})$ (with some positive constants $k_i > 0,\ \alpha_i > 0,\ i = 1,2$) and that $S : [0,\delta) \times \mathcal{U} \rightrightarrows \mathcal{U}$ and $T : [0,\delta) \times \mathcal{V} \rightrightarrows \mathcal{V}$ are two parametrized maximal monotone operators.
The dual inclusion $(\mathcal{D})$ is obtained from $(\mathcal{P})$ by using Attouch-Théra duality theory (see [4] for more details). We note the symmetry between the two problems $(\mathcal{P})$ and $(\mathcal{D})$ in the following sense:

Primal	p	A_1	S	L^*	A_2	T	L	u
Dual	q	A_2	T	$-L$	A_1	S	$-L^*$	v

The primal and dual variables are $u(t) \in \mathcal{U}$ and $v(t) \in \mathcal{V}$, respectively.
The Kuhn-Tucker set associated to the primal-dual problem $(\mathcal{P}) - (\mathcal{D})$ is given by

$$\mathcal{W} = \Big\{ \big(u(t),v(t)\big) \in \mathcal{U} \times \mathcal{V} : p(t) - L^*(t)v(t) \in A_1(t,u(t)) + S(t,u(t)) \text{ and } \\ q(t) + L(t)u(t) \in A_2(t,v(t)) + T(t,v(t)) \Big\}.$$

To study the sensitivity analysis of the primal-dual problem $(\mathcal{P}) - (\mathcal{D})$, we introduce the following new input and output variables

$$\xi(t) = (p(t),q(t)) \in \mathcal{U} \times \mathcal{V} \text{ and } x(t) = (u(t),v(t)) \in \mathcal{U} \times \mathcal{V}. \tag{4.1}$$

We set $\mathcal{H} = \mathcal{U} \times \mathcal{V}$ and we introduce the following operators

$$\begin{array}{rrcl} A : & [0,\delta) \times \mathcal{H} & \to & \mathcal{H} \\ & (t,u,v) & \mapsto & A(t,x) := \Big(A_1(t,u), A_2(t,v)\Big), \end{array} \tag{4.2}$$

and

$$\begin{array}{rrcl} B : & [0,\delta) \times \mathcal{H} & \rightrightarrows & \mathcal{H} \\ & (t,u,v) & \mapsto & B(t,x) := \Big(S(t,u) + L^*(t)v, T(t,v) - L(t)u\Big), \end{array} \tag{4.3}$$

with $x = (u,v) \in \mathcal{H}$. It is clear that solving the primal-dual problem $(\mathcal{P}) - (\mathcal{D})$ is equivalent to solve the following variational inclusion

$$\mathrm{VI}\big(A(t,\cdot),B(t,\cdot),\xi(t)\big) \begin{cases} \text{find } x(t) \in \mathcal{H} \text{ such that} \\ \xi(t) \in A(t,x(t)) + B(t,x(t)), \end{cases}$$

with the input/output variables $\xi(\cdot)$ and $x(\cdot)$ defined in (4.1) and the operators A and B defined in (4.2)-(4.3).

Lemma 4.1 *If $A_1 \in \mathcal{U}_{k_1,\alpha_1}(\mathcal{U})$ and $A_2 \in \mathcal{U}_{k_2,\alpha_2}(\mathcal{V})$ (for some positive constants $k_i > 0,\ \alpha_i > 0,\ i = 1,2$), then the operator A defined in* (4.2) *is in $\mathcal{U}_{k,\alpha}(\mathcal{H})$ with $\alpha = \min(\alpha_1, \alpha_2)$ and $k = \max(k_1, k_2)$.*

Proof. The proof is straightforward. ■

Lemma 4.2 *If $S : [0, \delta) \times \mathcal{U} \rightrightarrows \mathcal{U}$ and $T : [0, \delta) \times \mathcal{V} \rightrightarrows \mathcal{V}$ are two parametrized maximal monotone operators, then the operator B defined in* (4.3) *is also a parametrized maximal monotone operator.*

Proof. We write B as the sum of a linear continuous antisymmetric operator and a maximal monotone operator. In fact,

$$B(t, x) = \Lambda(t, x) + \Gamma(t, x),$$

with $\Lambda(t, x) = \Big(L^*(t)v, -L(t)u\Big)$ and $\Gamma(t, x) = \Big(S(t, u), T(t, v)\Big)$.
By assumptions Γ is a parametrized maximal monotone operator. The operator Λ is linear continuous (with respect to its second argument), hence a parametrized maximal monotone on $\mathcal{H}$ with full domain. By a classical result $\Lambda + \Gamma$ is maximal monotone on $\mathcal{H}$ (with respect to its second argument), which means that B is a parametrized maximal monotone operator. ■

Lemma 4.3 *If A_1 and A_2 are semi-differentiable respectively at $u \in \mathcal{U}$ and $v \in \mathcal{V}$, then the operator A defined in* (4.2) *is semi-differentiable at $x = (u, v) \in \mathcal{H}$ and $D_sA(x) = \Big(D_sA_1(u), D_sA_2(v)\Big)$.*

Proof. The proof is straightforward. ■

Lemma 4.4 *Suppose the following*

(i) *S is proto-differentiable at $u \in \mathcal{U}$ relative to $u^* \in S(0, u)$;*

(ii) *T is proto-differentiable at $v \in \mathcal{V}$ relative to $v^* \in T(0, v)$;*

(iii) *The linear operators $L(\cdot) \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ and $L^*(\cdot) \in \mathcal{L}(\mathcal{V}, \mathcal{U})$ are right-differentiable at $t = 0$;*

then the operator B defined in (4.3) *is proto-differentiable at $x = (u, v)$ relative to $x^* = (L^*(0)v + u^*, -L(0)u + v^*) \in B(0, x)$ and its proto-derivative is given by*

$$D_pB(x|x^*)(\omega) = \Big(L^{*'}(0)v + L^*(0)\omega_2 + D_pS(u|u^*)(\omega_1),\ -L'(0)u - L(0)\omega_1 + D_pT(v|v^*)(\omega_2)\Big), \quad (4.4)$$

for every $\omega = (\omega_1, \omega_2) \in \mathcal{U} \times \mathcal{V} = \mathcal{H}$.

Proof. As in the proof of Lemma 4.2, we decompose B as

$$B(t, x) = \Lambda(t, x) + \Gamma(t, x),$$

with $\Lambda(t, x) = \Big(L^*(t)v, -L(t)u\Big)$ and $\Gamma(t, x) = \Big(S(t, u), T(t, v)\Big)$.
Assumption (iii) implies that the operator Λ is semi-differentiable and its semi-derivative is given by

$$D_s\Lambda(x)(\omega) = \Big(L^{*'}(0)v, -L'(0)u\Big) + \Big(L^*(0)\omega_2, -L(0)\omega_1\Big),$$

which can be rewritten as

$$D_s\Lambda(x)(\omega) = \Lambda'(0, x) + \Lambda(0, \omega).$$

On the other hand, assumptions (i) and (ii) imply that Γ is proto-differentiable at x relative to $x^* - (L^*(0)v, -L(0)u) = (u^*, v^*)$ and its proto-derivative is given by

$$D_p\Gamma\big(x|x^* - (L^*(0)v, -L(0)u)\big)(\omega) = \Big(D_pS(u|u^*)(\omega_1)\,,\, D_pT(v|v^*)(\omega_2)\Big),$$

for every $\omega = (\omega_1, \omega_2) \in \mathcal{U} \times \mathcal{V} = \mathcal{H}$.
Using Proposition 2.1, B is proto-differentiable at $x = (u, v)$ relative to $x^* = (L^*(0)v + u^*, -L(0)u + v^*) \in B(0, x)$ and its proto-derivative is given by formula (4.4). ■
We derive the following theorem.

Theorem 4.1 *Let $A_1 \in \mathcal{U}_{k_1,\alpha_1}(\mathcal{U})$, $A_2 \in \mathcal{U}_{k_2,\alpha_2}(\mathcal{V})$, $S : [0, \delta) \times \mathcal{U} \rightrightarrows \mathcal{U}$ and $T : [0, \delta) \times \mathcal{V} \rightrightarrows \mathcal{V}$ two parametrized maximal monotone operators. Let $p : [0, \delta) \to \mathcal{U}$ and $q : [0, \delta) \to \mathcal{V}$ be two given functions. We consider the primal and dual functions $u : [0, \delta) \to \mathcal{U}$ and $v : [0, \delta) \to \mathcal{V}$ solutions respectively of $(\mathcal{P})$ and $(\mathcal{D})$. If the following assumptions are satisfied:*

(i) *p and q are right-differentiable at $t = 0$;*

(ii) *A_1 and A_2 are semi-differentiable respectively at $u(0)$ and $v(0)$;*

(iii) *S is proto-differentiable at $u(0) \in \mathcal{U}$ relative to $u^*(0) \in S(0, u(0))$ with $D_pS(u(0), u^*(0))$ maximal monotone;*

(iv) *T is proto-differentiable at $v(0) \in \mathcal{V}$ relative to $v^*(0) \in T(0, v(0))$with $D_pT(v(0), v^*(0))$ maximal monotone;*

(v) *The linear bounded operators $L(\cdot) \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ and $L^*(\cdot) \in \mathcal{L}(\mathcal{V},\mathcal{U})$ are right-differentiable at $t = 0$;*

then the primal and dual functions $u : [0, \delta) \to \mathcal{U}$ and $v : [0, \delta) \to \mathcal{V}$ solutions respectively of $(\mathcal{P})$ and $(\mathcal{D})$ are right-differentiable at $t = 0$ with

$$\begin{cases} p'(0) - L^{*'}(0)v(0) - L^{*'}(0)v'(0) \in D_sA_1\big(u(0)\big)\big(u'(0)\big) + D_pS\big(u(0)|u^*(0)\big)\big(u'(0)\big) \\ q'(0) + L'(0)u(0) + L(0)u'(0) \in D_sA_2\big(v(0)\big)\big(v'(0)\big) + D_pT\big(v(0)|v^*(0)\big)\big(v'(0)\big) \end{cases}$$

Proof. We recast the primal-dual composite inclusion $(\mathcal{P}) - (\mathcal{D})$ to a variational inclusion of the form $\mathrm{VI}\big(A(t,\cdot), B(t,\cdot), \xi(t)\big)$ where $\mathcal{H} = \mathcal{U} \times \mathcal{V}$, ξ, x, A and B are defined in (4.1), (4.2) and (4.3), respectively. We use Theorem 3.1 and Lemmas 4.1, 4.2, 4.3 and 4.4 to conclude. By Lemma 4.1, the operator A defined in (4.2) belongs to $\mathcal{U}_{k,\alpha}(\mathcal{H})$ with $\alpha = \min(\alpha_1, \alpha_2)$ and $k = \max(k_1, k_2)$. Lemma 4.2 entails that the operator B defined in (4.3) is a parametrized maximal monotone operator on $\mathcal{H}$. Assumption (i) implies that ξ is right-differentiable at $t = 0$. Assumption (ii) and Lemma 4.3 imply that A is semi-differentiable at $x(0) = (u(0, v(0))$. Using assumptions (iii)-(iv)-(v) and Lemma 4.4, we obtain the proto-differentiability of the operator B at $x(0) = (u(0), v(0))$ relative to $x^*(0) = (L^*(0)v(0) + u^*(0), -L(0)u(0) + v^*(0)) \in B(0, x(0))$. Theorem 3.1 allows us to conclude. ■

5 Concluding remarks

Many problems in physics, engineering and economics can be formulated as solving classical nonlinear equations. When dealing with constraints in convex optimization, a unified model is given by maximal monotone generalized equations, which consists in finding the zeros of the sum of a single-valued map and a maximal monotone operator. When the data, involved in the problem, are known only with a certain precision, an important question is how to get informations on the rates of change of the solutions with respect to parameter perturbations. In this paper, we investigated the sensitivity analysis of a maximal monotone inclusion by using the proto-differentiability of the associated resolvent map. The sensitivity analysis in this context has to be understood in the sense of differentiable properties of the perturbed solution with respect to the one dimensional perturbation parameter $t \in [0, \delta)$. More precisely, we showed in this note, under suitable assumptions, that the derivative of the solution of $\mathrm{VI}\big(A(t,\cdot), B(t,\cdot), \xi(t)\big)$ at $t = 0$ is the unique solution of $\mathrm{VI}\big(D_sA(x(0)), D_pB\big(x(0)|x^*(0)\big), \xi'(0)\big)$, with $x^*(0) = \xi(0) - A(0, x(0)) \in B(0, x(0))$, $D_sA\big(x(0)\big)$ the semi-derivative of A at $x(0)$ and $D_pB\big(x(0)|x^*(0)\big)$ the proto-derivative of B at $x(0)$ relative to $x^*(0)$. Many issues remain open and require further investigations. This includes for example replacing the perturbation parameter $t \in [0, \delta)$ by an abstract multidimensional parameter $p \in \mathbb{R}^n$ (in the same lines as [10, 11]) and without requiring the semi- and proto- differentiability of the involved operators in the product space $(p, x) \in \mathbb{R}^n \times \mathcal{H}$. It would be interesting to carry out the same analysis by replacing the Painlevé-Kuratowski convergence with the bounded Hausdorff convergence. It is well known that these two notions of convergence coincide in finite dimensional spaces. In infinite dimensional spaces, the bounded Hausdorff convergence has the great advantage of being associated with a metrizable topology (Attouch-Wets topology) which is not the case of the Painlevé-Kuratowski convergence. An other question of great interest is the investigation of the parabolic twice epi-differentiability, defined by parabolic difference quotients, and its link with the proto-differentiability for the t-dependent case (see [28] Chapter 13).

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