ON THE STABILITY AND EVOLUTION
OF ECONOMIC EQUILIBRIUM

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Abstract. In an economic model of exchange of goods, an equilibrium of market prices and resultant holdings of the agents, if stable, can reconstitute itself in response to slight perturbations of those holdings, although with slightly adjusted prices and holdings. Results about that were obtained by Balasko in 1975, but here such stability properties are developed with utility functions more general than his in allowing the boundary of the goods orthant to come into play. This platform supports the investigation — for the first time — of how a market in equilibrium may evolve in continual response to exogenous additions and subtractions to holdings, a dynamical process which is conceptually different from tâtonnement. The study is undertaken moreover with agents being able to hold zero quantities of goods that are not indispensable to them. A one-sided differential equation characterizing the evolution of equilibrium prices and holdings is derived as well.

This advance is made possible by the application of methodology of variational analysis beyond classical differential analysis. It benefits also from viewing goods very generally, not just as commodities and not only for immediate disposal, with the agents then not necessarily being just consumers. It further relies on identifying a broader, but still convenient, criterion for the attainment of equilibrium from initial holdings in the Walrasian setting, one which gets away from requiring every agent to start out with a positive quantity of every good.

Key Words. General equilibrium, exchange equilibrium, market stability, evolutionary dynamics, evolution equation, shift stability, tâtonnement stability, variational analysis

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1 Introduction

Equilibrium in market models of exchange has a static character in most discussions. It corresponds to having achieved a situation in which agents are satisfied with their holdings and have no wish to buy or sell. When dynamics are brought up, the typical context is how an equilibrium might be identified through a negotiation process like Walrasian tâtonnement. But there is a different side to dynamics that deserves attention as well.

A market may be in equilibrium with respect to current prices and holdings, but those holdings could be affected by incremental additions and subtractions coming somehow from outside. It can be imagined that the agents would react by trading, and the balance of supply and demand would thereby be restored under slightly adjusted prices. That leads to the picture of an equilibrium evolving dynamically in both goods and prices in response to various influences. Could there be a sort of differential equation to describe such evolution in its dependence on the agents’ utilities?

One of the reasons why little thinking among economists has gone in this direction, perhaps, is the reputation equilibrium has for being unstable, even bizarrely so. That comes from results such as those of Mas-Colell [36], according to which just about any proposed collection of price vectors can be the set of equilibrium price vectors associated with some instance of initial holdings of the agents. With that being the case, it is hard to take seriously the idea of incremental adjustments being well behaved. However, results of Balasko [6] and Sattinger [46] going back to 1975 indicate that, under certain assumptions, good behavior is assured when equilibrium holdings are perturbed by it small-enough amounts.

Balasko’s stability results were extended by Keenan [31] in 1982, but reference to them can hardly be found elsewhere in equilibrium literature beyond later works of Balasko himself, e.g., [7, 8, 9]. They are not cited by Hirota [22] (1981), where an example indicating such localization influence was offered. More recently, the paper of Brown and Shannon [12] (2008), containing insights into the extent that “rationalization” of an economy from finite data can be carried out so as to promote tâtonnement, presents the convergence issue in terms of prices only; no localization in goods comes up. Anyway, no one up to now has contemplated dynamics of the kind targeted here as consequences of properties of stability. Balasko in [8, 9] applied his results only to dynamics in a context of tâtonnement.

The aim of this paper is to explore stability issues, including these, and their implications for evolution of equilibrium in a framework in which recent advances in variational analysis can be brought to bear. It is possible then to deal with resources on the boundary of the “goods orthant” instead of just in the interior, as has previously been dictated by the limitations of differential geometry. The utility functions we work with for this purpose, although twice

\[1\] For an update on this subject and its background in Sonnenschein-Mantel-Debreu theory, see the 2006 article of Rizvi [41].

\[2\] An earlier version of this paper, with virtually same results but a different introduction and somewhat different packaging, has been available since October 2013 under the title “The robust stability of every equilibrium in economic models of exchange even under relaxed standard conditions” and can found on-line at SSRN: https://ssrn.com/abstract=2462975 or http://dx.doi.org/10.2139/ssrn.2462975.

\[3\] The importance of stability properties in economic equilibrium was strongly emphasized early-on by Samuelson [45] (1941) and recently all the more by Kirman [33] (2011).
continuously differentiable as usual in stability studies (aside from some boundary possibilities), allow preferences among the agents to differ sharply. Some goods can have no influence at all on a particular agent’s utility while others are indispensable. Some can be attractive without being indispensable and be present initially and at equilibrium in either positive or zero amounts. The one restrictive aspect of our utility functions is concavity instead of quasi-concavity.\(^4\) \(^5\)

In the pattern put forward originally by Walras [49], agents enter a market wishing to trade their given holdings for holdings that they optimally prefer. The challenge is the existence of market prices under which such trading can take place with supply matching demand. With preferences coming from utility functions as general as just described, we have to develop early-on a convenient but more ample criterion for existence than the one ordinarily relied upon, in which a positive amount of every good must be present in the initial holdings of every agent.

But our dynamical focus also causes a shift from ordinary in our terminology of equilibrium in the Walrasian context. In the common notation where vectors representing the holdings of all the agents are denoted by \(\omega\) and vectors of prices by \(p\), we can think of initial holdings \(\omega^0\) leading to optimized holdings \(\omega\) facilitated by prices \(p\). In the works of Balasko [6, 7] and many others, the pair \((p, w^0)\) is called an equilibrium and \((p, \omega)\) is a special case, called a no-trade equilibrium, but we want to speak only of \((p, \omega)\) as an equilibrium, the one associated here with \(\omega^0\). That is in line with uses of the word in physics, game theory and elsewhere, in which a configuration deemed to give an “equilibrium” is kept separate from whatever initial conditions, hardly unique, may have led to it by some process or other. Moreover this approach to terminology seems essential for our purpose of investigating how an “equilibrium” might evolve, since that naturally has to concern trajectories \((p(t), \omega(t))\), not \((p(t), \omega^0(t))\).

It also helps to clarify what ought to be meant by an “equilibrium” being “stable.” In a simple example furnished by Dontchev and Rockafellar in [17], for instance, the same pair \((p, \omega)\) is associated with two different instances of initial holdings, \(\omega^0\) and \(\omega^1\), with \(\omega^0\) farther from \(\omega\) than \(\omega^1\). Small perturbations of \(\omega^1\) lead to small, locally unique, perturbations of \((p, \omega)\), but small perturbations of \(\omega^0\) have wild effects. Those effects do not, however, signify “instability of equilibrium” when \((p, \omega)\) is the designated equilibrium in this setting. Stability of equilibrium should not, from our perspective, be tied to the arbitrariness of the initial conditions from which the equilibrium might be reached.

With all this in mind we concentrate our efforts on the “equilibrium correspondence” that associates \((p, \omega)\) with \(\omega^0\) instead of on the set of pairs \((p, \omega^0)\) widely studied by others as the “equilibrium manifold.” We show that, in a neighborhood of the set of equilibrium pairs in our sense, the correspondence is single-valued with properties of semi-differentiability. True

\(^4\)Concavity plays a major role in the technology we apply. Perhaps it could be avoided with more effort, but anyway preferences coming from a quasi-concave utility can be approximated arbitrarily closely by those coming from a concave utility; see Kannai [30] (1977). Moreover, finite demand data cannot test the difference; see Brown and Shannon [12] (2008).

\(^5\)Incidentally, whether the strange sets of equilibrium price vectors detected by Mas-Collel [36] and others are possible under concavity instead of quasi-concavity is an unanswered question. Concavity imposes more regularity; cf. Mas-Colell [37] and Kannai [30]. It is interesting to speculate that an updated approach to the axiomatization of preferences which addresses comparative utility as in Kahneman and Tversky [29], or even just marginal utility, might lead directly to concave utility and its advantages.
differentiability as uncovered by Balasko [6] in his context can’t be expected in a context where orthant boundaries can have an active role. Continual perturbations in initial holdings may cause the continually adjusted equilibrium holdings to hit some part of the boundary, bounce off and then hit again. Such effects have to be taken into account in the potential dynamics of evolution of equilibrium, too.

Along with establishing that the equilibrium correspondence is locally stable in the sense of adjustments being well behaved, which we call shift stability, we furthermore demonstrate local stability with respect to Walrasian tâtonnement for the equilibrium pairs \((p, \omega)\), a property which we call tâtonnement stability. Due to boundary effects, our tâtonnement price trajectories \(p(t)\), in contrast to those of Balasko [6], have second derivatives that may only be one-sided because the right side of the differential equation is only semidifferentiable. The interest in tâtonnement stability lies of course in confirming that an economically plausible mechanism exists to restore equilibrium after a small perturbation.\(^6\)

When we come to evolution of equilibrium being described by trajectories \((p(t), w(t))\), even the first derivatives in time may only be one-sided because of boundary effects. Nonetheless we are able to derive a “one-sided” differential equation that describes how equilibrium must progress in time in response to exogenously induced perturbations of the agents holdings.

The variational analysis texts of Rockafellar and Wets [43] and Dontchev and Rockafellar [16] provide extensive background for the tools we employ, but an article of Dontchev and Rockafellar [17], already invoking those tools in economics, is especially important in that respect. It is from there that we distill the broadened existence result we need and also some key properties of the equilibrium correspondence. The approach to those issues in [16] depended on formulating the Walrasian model as a variational inequality problem which, along with prices and holdings as variables brought in Lagrange multipliers for budget constraints. Concavity of utility functions had a big influence in that, hence our attachment to concavity here. In particular it promotes duality in the analysis of the utility maximization problems of the agents.

Variational inequality models of economic equilibrium have likewise been central to the work of the three current authors in our previous papers [23, 24, 25, 26, 27].

\(^6\)Of course, Walrasian tâtonnement is not a trading mechanism and refers rather to a fictional negotiation process with artificially specified parameters. Early analysis of its stability utilized the differentiability of the right side of the differential equation, which is not available to us here. It was undertaken via the matrix in the linearization of the right side at an equilibrium. That approach, seen early in Hicks [20] (1939) and Metzler [39] (1945), led to the investigation of various matrix conditions with special meaning for economics, cf. Arrow and McManus [5] (1958). Other research, as in Arrow and Hahn [2] (1971), explored non-Walrasian adjustment processes which go beyond the elicitation of supply-demand information and enter into iterative trading; cf. Keisler [32] (1996) and its references. The complicated history of interpreting the original ideas of Walras [49] (1874) is discussed in that article as well. See also Walker [48] (1987).
2 Statement of assumptions and the main results

Proceeding toward a precise formulation, we take the nonnegative orthant $\mathbb{R}_{++}^{n+1}$ as the space of goods\(^7\) and suppose that the agents, indexed by \(i = 1, \ldots, r\), have preferences on it which are given by utility functions \(u_i\). To fully appreciate the equilibrium context we are aiming at, it is important to keep in mind that the goods can be very general, not just commodities destined for consumption. They can be anything physical, or perhaps even “rights,” that an agent might wish to acquire and are available for trading in fixed supply. The question of what an agent might do with them is separate and will be revisited later.

**Assumption A1** (utility fundamentals). Each utility function \(u_i\) on $\mathbb{R}_{++}^{n+1}$ is nondecreasing, concave\(^8\) and upper semicontinuous. It may take on $-\infty$, but if so, only at points on the boundary of $\mathbb{R}_{++}^{n+1}$. Relative to the set where it is finite, \(u_i\) is continuous.\(^9\)

An important provision will depend on classifying goods according to the interest that an agent has in them. A good will be called attractive for agent \(i\) if every increase in that good leads to a higher value of \(u_i\). It will be called indispensable for agent \(i\) if it is attractive and, at any point in which the quantity of that good (but not every attractive good) is zero, either \(u_i\) takes on $-\infty$ or \(u_i\) is finite but the marginal utility of the good is $+\infty$.\(^{10,11}\)

**Assumption A2** (indispensability). There is a good that is indispensable to all agents. Every good is indispensable to at least one agent.

The presence of a good that is indispensable to all agents will have an central role in what follows. It implies insatiability of all the utility functions, but it will have other major consequences as well. Classically standard assumptions, requiring the iso-surfaces of utility to curve away from orthant boundaries, actually force every good to be indispensable to every agent. We need just one such good and will work with it as a numéraire.

In our picture, some goods, far from being indispensable, can fail to be attractive at all to agent \(i\). Other goods can be attractive without being indispensable. Our next condition sharpens the distinction.

**Assumption A3** (unattractiveness). If a good is not attractive to agent \(i\), then it has no effect on the utility function \(u_i\).

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\(^7\)Having \(n + 1\) instead of \(n\) will shortly be seen to help in the presentation. We could just as well work with survival sets in the form of displaced orthants specifying various nonnegative lower bounds on the goods required by the agents. But that can be reduced to the basic orthant case by a change of variables, so for the sake of a simpler presentation we leave this as an obvious implicit enhancement.

\(^8\)The distinction between concave and quasi-concave is vital here. For results about approximating quasi-concave utility by concave utility while taking into account various properties of strictness and differentiability, such as enter below, see Kannai [30], Mas-Colell [37, Chapter 2], and recently Connell and Rasmusen [13].

\(^9\)Continuity on the interior of the orthant, where \(u_i\) is surely finite, is automatic from concavity, so this technical provision refers only to boundary behavior.

\(^{10}\)Marginal utility refers here to the one-sided directional derivative with respect to an increase in the good in question. That derivative exists from the concavity.

\(^{11}\)This provision encompasses Cobb-Douglas utilities, for example, which are “infintely steep” along the boundary of the goods orthant.
In this condition we forgo the possibility of goods that might have positive marginal utility up to some level but zero marginal utility thereafter.

**Assumption A4** (partial strong concavity). With respect to the suborthants of the goods space that are defined by

\[ \Omega_i = \{ \text{vectors in } \mathbb{R}^{n+1}_+ \text{ having positive components for goods indispensable to agent } i \} \],

the utility functions \( u_i \) are twice continuously differentiable.\(^{12}\) Furthermore, the Hessian matrices, formed by the second partial derivatives, are negative definite with respect to the goods that are attractive to agent \( i \).\(^ {13}\)

In contrast to A4, it would classically be standard — with concave instead of just quasi-concave utility — to insist on the entire Hessian being negative definite. This would be combined with limiting attention to the interior of the goods space; the orthant boundary would not be allowed to complicate the analysis. But for economic theory to be more realistic, it ought to be permitted to do so.

With these assumptions at our disposal, we can move to more specific symbolism and formulations of stability. We concentrate on a particular good that is indispensable to all agents, as guaranteed by A2, calling it *money* for short.\(^ {14}\)\(^ {15}\) We designate quantities of money in the hands of agent \( i \) by \( m_i \), and vectors giving quantities of the other goods by \( x_i \), so that the elements of the goods space have the form \((m_i, x_i)\) with \( m_i \in \mathbb{R}_+ \) and \( x_i \in \mathbb{R}^n_+ \). Initial holdings will have the notation \((m_i^0, x_i^0)\).

Prices will always be denominated in money. Since money has price 1 with respect to itself, we will only need to be concerned with price vectors \( p \in \mathbb{R}^n_+ \) for the remaining goods.

**Utility maximization problems.** The goal of agent \( i \) with respect to a price vector \( p \) and initial holdings \((m_i^0, x_i^0) \in \Omega_i\) is to maximize the utility \( u_i(m_i, x_i) \) over all goods vectors \((m_i, x_i)\) satisfying the budget constraint

\[ m_i + p \cdot x_i = m_i^0 + p \cdot x_i^0. \]
In dealing often with agents collectively, it will be expedient to use the “supervector” notation 

\((m, x)\) for \(m = (\ldots, m_i, \ldots), \ x = (\ldots, x_i, \ldots)\),

and similarly \((m^0, x^0)\) in the case of initial holdings, and so forth.\(^{16}\)

**Definition of equilibrium.** An equilibrium is a triple \((\bar{p}, \bar{m}, \bar{x})\) such that each \((\bar{m}_i, \bar{x}_i)\) component solves the utility maximization problem of agent \(i\) relative to \(\bar{p}\) when \((m^0_i, x^0_i) = (\bar{m}_i, \bar{x}_i)\).

More generally, \((\bar{p}, \bar{m}, \bar{x})\) is an equilibrium associated with initial holdings \((m^0, x^0)\) possibly different from \((\bar{m}, \bar{x})\), written

\[ (\bar{p}, \bar{m}, \bar{x}) \in E(m^0, x^0), \]

if each pair \((\bar{m}_i, \bar{x}_i)\) solves the utility maximization problem of agent \(i\) relative to \(\bar{p}\) and \((m^0_i, x^0_i)\), and moreover

\[ \sum_{i=1}^{r} \bar{x}_i = \sum_{i=1}^{r} x^0_i. \quad (3) \]

The goods equation (3) requires supply to equal demand in all the goods other than money. Money can be left out because the corresponding condition

\[ \sum_{i=1}^{r} \bar{m}_i = \sum_{i=1}^{r} m^0_i \quad (4) \]

follows at once from (3) and the budget constraints (2). Apart from the money feature, this definition is fairly ordinary but depicts what others have called a “no-trade” equilibrium. The “no-trade” label has been dropped here for reasons explained in the introduction; no other kind of configuration of goods and prices is an “equilibrium” in our setting aimed at uncovering the dynamics of evolution.

Some observations can immediately be made which will simplify the discussions to come. First,

\[ \text{in any equilibrium, all prices must be positive.} \quad (5) \]

This follows from A2 through the fact that if the price of some good were zero, then the maximization problem for an agent considering that good to be indispensable, or even just attractive, could not have a solution. Next,\(^{17}\)

\[ \text{for } p > 0 \text{ the problem of an agent } i \text{ has a unique solution,} \]

\[ \text{and it has to belong to the suborthant } \Omega_i \text{ defined by } (1). \quad (6) \]

Indeed, the budget constraint defines a compact set of goods vectors which meets the interior of \(\mathbb{R}^{n+1}\), where utility is surely finite. The upper semicontinuity of \(u_i\) in A1 guarantees then that the maximum is finitely attained. Quantities of goods that are not attractive are pushed to zero, since otherwise they would drag down the budget available for attractive goods, but goods with infinite marginal utility at zero are forced to be positive. The partial strict concavity guaranteed by A4 then provides the uniqueness.

\(^{16}\)The notation \((p, \omega)\) invoked in the introduction will thus be replaced in what follows by \((p, m, x)\), with \(p\) being of one less dimension than usual.

\(^{17}\)For us, a strong vector inequality refers to a strict inequality in each component.
On the platform of (5) and (6) we can introduce, with respect to price vectors \( p > 0 \), the demand mappings \( X_i \) defined by

\[
X_i(p; m_i^0, x_i^0) = \text{the corresponding unique optimal } x_i \text{ for agent } i,
\]

observing that the associated optimal money amount will then be

\[
m_i = M_i(p; m_i^0, x_i^0) = m_i^0 + p[x_i^0 - X_i(p; m_i^0, x_i^0)],
\]

while the excess demand mapping \( Z \) for the non-money goods will be given by

\[
Z(p; m^0, x^0) = \sum_{i=1}^{r} [X_i(p; m_i^0, x_i^0) - x_i^0].
\]

In this notation we can say for the correspondence \( E \) in the definition of equilibrium that, with respect to \( \bar{x}_i = X_i(p; m_i^0, x_i^0) \) and \( \bar{m}_i = M_i(p; m_i^0, x_i^0) \),

\[
(\bar{p}, \bar{m}, \bar{x}) \in E(m^0, x^0) \iff \bar{p} > 0, \ Z(\bar{p}; m^0, x^0) = 0.
\]

In particular, \((\bar{p}, \bar{m}, \bar{x})\) is an equilibrium when this holds with \((m^0, x^0) = (\bar{m}, \bar{x})\).

The following result about the existence of an equilibrium, the proof of which will come later, provides important underpinning for our investigations of stability.

**Ample Existence Theorem.** Under assumptions A1, A2, A3, A4, there exists for every instance of initial holdings \((m^0, x^0)\) such that \((m_i^0, x_i^0) \in \Omega_i \) for all agents \( i \) at least one equilibrium \((\bar{p}, \bar{m}, \bar{x}) \in E(m^0, x^0)\). Every equilibrium \((\bar{p}, \bar{m}, \bar{x})\) likewise has \((\bar{m}_i, \bar{x}_i) \in \Omega_i \) for all agents \( i \).

An existence result that bears closely on this one has recently been obtained in [17] in terms the initial holdings being “amply survivable.” That condition, couched in money terms, is far weaker than the customary assumption that initial goods belong to survival set interiors.\(^{18}\) It will be shown to be satisfied in the present setting by all the instances of initial holdings under consideration.

Two kinds of stability of an equilibrium will be considered in this framework. The first has to do with perturbations and the second with tâtonnement. The neighborhoods appearing in the definitions can be regarded as closed balls with respect to the Euclidean norm \( || \cdot || \).

**Definition of shift stability.** An equilibrium \((\bar{p}, \bar{m}, \bar{x})\) is **shift-stable** if there are neighborhoods \( N_0 \) of \((\bar{m}, \bar{x})\) and \( N_1 \) of \((\bar{p}, \bar{m}, \bar{x})\) such that

\[
\text{each } (m^0, x^0) \in N_0 \text{ yields a unique equilibrium } (p, m, x) \in N_1,
\]

and the corresponding localized equilibrium mapping

\[
E : (m^0, x^0) \in N_0 \mapsto E(m^0, x^0) = (p, m, x) \in N_1, \text{ having } E(\bar{m}, \bar{x}) = (\bar{p}, \bar{m}, \bar{x}),
\]

\(^{18}\)Apart from that strict positivity assumption there are other, more subtle equilibrium-supporting conditions in [1] and later in [18], [19], involving “irreducibility.” However, these are all rather unwieldy in comparison with “ample survivability.”

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is Lipschitz continuous. The equilibrium is semidifferentially shift stable if $E$ is not only Lipschitz continuous but also possesses, with respect to all choices of $(m^{0'}, x^{0'})$, the one-sided directional derivative\(^{19}\)

\[
DE(m^0, x^0; m^{0'}, x^{0'}) = \lim_{h \to 0^+} \frac{1}{h} \left[ E(m^0 + hm^{0'}, x^0 + hx^{0'}) - E(m^0, x^0) \right]. \tag{12}
\]

The one-sided limit in (12) with $h \to 0^+$ refers to $h$ tending to 0 only from above; the classical two-sided directional derivative would have $h \to 0$ with no such restriction. The companion derivative expression with the opposite one-sided limit is tacitly covered by this as well, because

\[
\lim_{h \to 0^-} \frac{1}{h} \left[ E(m^0 + hm^{0'}, x^0 + hx^{0'}) - E(m^0, x^0) \right] = -DE(m^0, x^0; -m^{0'}, -x^{0'}). \tag{13}
\]

By virtue of the Lipschitz continuity in the definition of shift stability, we get a sort of Taylor expansion of the localized equilibrium mapping:

\[
E(m^0 + hm^{0'}, x^0 + hx^{0'}) = E(m^0, x^0) + hDE(m^0, x^0; m^{0'}, x^{0'}) + o(h).
\]

This corresponds to the differentiability of $E$ at $(m^0, x^0)$ if and only if $DE(m^0, x^0; m^{0'}, x^{0'})$ is linearly dependent on $(m^{0'}, x^{0'})$. Without that linearity it still signals semidifferentiability,\(^{20}\) which serves as the tool enabling us to handle the one-sided effects on equilibrium distributions of goods that may be caused by the orthant boundary coming into play.

Note that the possibility of there being more than one equilibrium associated with initial holdings $(m^0, x^0)$ chosen from the neighborhood $N_0$ of $(\bar{m}, \bar{x})$ is not ruled out by shift stability. The requirement is only that there cannot be a second equilibrium within the specified neighborhood $N_1$ of $(\bar{p}, \bar{m}, \bar{x})$. For $(m^0, x^0) \notin N_0$ there might be multiple equilibria in $N_1$, but none could be an equilibrium associated with $(\bar{m}, \bar{x})$.

In turning next to tâtonnement, backed by shift stability, we follow Arrow and Hurwicz [3] in posing it in terms of an ordinary differential equation; see also [4]. Versions in discrete time could similarly be laid out along the lines in [21] and [47], but in our opinion the case of continuous time puts the ideas in sharper focus.\(^{21}\)

**Definition of tâtonnement stability.** An equilibrium $(\bar{p}, \bar{m}, \bar{x})$ is tâtonnement-stable if there is a neighborhood $N$ of $\bar{p}$ and a neighborhood $N_0$ of $(\bar{m}, \bar{x})$ such that, for all $p^0 \in N$ and all $(m^0, x^0) \in N_0$ having $E(m^0, x^0) = (\bar{p}, \bar{m}, \bar{x})$,\(^{22}\) the differential equation of tâtonnement, namely\(^{23}\)

\[
\dot{p}(t) = Z(p(t); m^0, x^0) \quad \text{for } t \geq 0 \text{ with } p(0) = p^0,
\]

\(^{19}\)The primes here do not, themselves, refer to derivatives.

\(^{20}\)See [43, Chapter 7] for more on semidifferentiability.

\(^{21}\)Anyway, the issue here is a conceptual property of stability of an equilibrium. It deserves to be seen in simpler terms without getting into a myriad possible variants, which perhaps would not add much in overall understanding to stability theory. See Kitti [34] for a comprehensive discussion of the efforts that have been put into the discrete-time case and the latest accomplishments in that direction.

\(^{22}\)In the sense of (11); this $N_0$ shrinks the one there, if necessary.

\(^{23}\)We employ $\dot{p}(t)$ for the derivative of $p(t)$ in order to preserve primed symbols like $p'$ for other uses.
Indeed, (14) implies even that two price trajectories $p_i(t) = X_i(p(t); m^0, x^0)$ and $m_i(t) = M_i(p(t); m^0, x^0)$ converge then to $\bar{x}_i$ and $\bar{m}_i$. It is strongly tâtonnement stable if, for such neighborhoods, there is a constant $\mu > 0$ such that

$$(p' - p) \cdot (Z(p'; m^0, x^0) - Z(p; m^0, x^0)) \leq -\mu ||p' - p||^2$$

for all $p', p \in N$, $(m^0, x^0) \in N_0$. (14)

According to (13), the price of a good rises at a rate equal to the current excess demand for that good (which amounts to falling if that is negative).24 An equilibrium price vector $\bar{p}$, in having $Z(\bar{p}; m^0, x^0) = 0$, furnishes a stationary point for the differential equation (13): starting from $p^0 = \bar{p}$ one would get $p(t) \equiv \bar{p}$ as a solution.

The property in (14), which in the weaker form of having

$$(p' - p) \cdot (Z(p'; m^0, x^0) - Z(p, m^0, x^0)) < 0$$

when $p' \neq p$, (15)

is known as “monotonicity” among economists,25 has already been recognized as guaranteeing the convergence of $p(t)$ to $\bar{p}$, which is indeed easy to prove,26 as for instance in the textbook of Hildenbrand and Kirman [21, page 237]. (The convergence of the demands $x_i(t)$ then follows from the continuity of the mappings $X_i^0$, which will come out later.) From the stronger property in (14) an additional conclusion readily follows about the convergence rate:

$$||p(t) - \bar{p}|| \leq e^{-\mu t} ||p^0 - \bar{p}||.$$ (16)

Indeed, (14) implies even that two price trajectories $p_1(t)$ and $p_2(t)$ starting from different initial states $\bar{p}_1^0$ and $\bar{p}_2^0$ have $||p_1(t) - p_2(t)|| \leq e^{-\mu t} ||p^0_1 - p^0_2||$.

The question of what aspects of utility might induce the “monotonicity” of excess demand has received attention from a number of researchers over the years; see the article of Quah [40] (2000) and its references. However, the results in that literature are not applicable in our context. They concern an excess demand mapping which differs from the one in (9) by being defined in terms of a wealth parameter that suppresses the role of initial holdings and their proximity to equilibrium holdings.

It has to be emphasized that tâtonnement, as formulated here, does not represent a process in which distributions of goods get adjusted in real time. Rather it is conceived as a scheme for exchange of information in “virtual time” by means of which a Walrasian broker or auctioneer determines prices that will bring supply and demand into balance.27

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24 It is easy to make the rates depend instead on different proportionality coefficients for different goods. However, this requires no additional mathematics because it really amounts only to changing the units of measurement for the goods other than money, e.g., prices per pound becoming prices per kilo. Arrow and Hurwicz observed this already in [3]. Other formulations in which the right side of (13) depends in extra ways on $p(t)$ have been explored by MacKenzie [35].

25 There is an unfortunate conflict with long-established terminology in mathematics, according to which (15) is the strict monotonicity of $-Z(\cdot; m^0, x^0)$, not $Z(\cdot; m^0, x^0)$. Plain monotonicity would have $\leq$ in place of $<$, whereas (14) is the strong monotonicity of $-Z(\cdot; m^0, x^0)$. For an introduction to the remarkable theory of such monotonicity, which comprises a valuable and much applied branch of convex analysis, see [42] and in greater detail [43, Chapter 12].

26 A simple tactic is to show by differentiation that $||p(t) - \bar{p}||^2$ is a decreasing function of $t$ that must go to 0.

27 This is made especially clear by Uzawa in [47]; see also the text ofMas-Colell [37, page 621].
Robust Stability Theorem. Under assumptions A1, A2, A3, A4, every equilibrium \((\bar{p}, \bar{m}, \bar{x})\) is both semidifferentially shift-stable and strongly tâtonnement-stable. Moreover, in this setting the demand mappings \(X_i\) and the excess demand mapping \(Z\) are themselves Lipschitz continuous with one-sided directional derivatives, hence semidifferentiable.

This will be proved below. It should be noted that the combination of the two stability properties says more about tâtonnement stability than might be apparent from the definition, where \((m^0, x^0)\) is restricted to having \(E(m^0, x^0) = (\bar{p}, \bar{m}, \bar{x})\). For any choice of \((m^0, x^0)\) near enough to \((\bar{p}, \bar{m}, \bar{x})\), still in the range of uniqueness of equilibrium but with \(E(m^0, x^0) \neq (\bar{p}, \bar{m}, \bar{x})\), the process can activated anyway, with the only difference that it will converge instead to \(E(m^0, x^0)\).

A result related to the theorem’s assertions about shift stability has recently been obtained in [17, Theorem 3], but not in the same framework. It will be crucial to our proof.

The stability properties confirmed in the Robust Stability Theorem may help further, beyond the insights of Balasko and others building on his work, to alleviate worries about the fragility of equilibrium. They also open new horizons for exploration.

The idea that the goods acquired by the agents through trading yield general “holdings” with many conceivable attractions besides immediate “consumption” is crucial for contemplating a broad range of additional possibilities. From this angle, as explained in the introduction, equilibrium does not need be viewed statically. It can interpreted as modeling an observable phenomenon over time in which supply and demand, with respect to maintaining the agents’ holdings, stay close to being in balance, but the balance continually shifts due to the influence of various factors, both internal and external.\(^{28}\)

Those factors could have many forms. A good that stands for a consumable commodity, for which the holding is a sort of stockpile, might be subject to a rate of consumption dictated by an agent’s needs, or for that matter, deterioration due to environmental circumstances. Money, as a good, could be taken from an agent through exogenously instituted taxation, or on the other hand enhanced by an ongoing subsidy. And so forth.\(^{29}\) The localized (or truncated) mapping \(E : (m^0, x^0) \mapsto (p, m, x)\) described in the Robust Stability Theorem can help to clarify what should then happen in such circumstances.

Evolutionary Dynamics Theorem. Suppose \((p(t), m(t), x(t))\) is an equilibrium evolving over some time interval in \(t\) in response to additions to \(m(t)\) and \(x(t)\) coming in at continuous rates \(m_+(t)\) and \(x_+(t)\).\(^{30}\) Then \((p(t), m(t), x(t))\) depends locally Lipschitz continuously on \(t\) and has right derivatives \((\dot{p}^+(t), \dot{m}^+(t), \dot{x}^+(t))\) that satisfy the one-sided differential equation

\[
(\dot{p}^+(t), \dot{m}^+(t), \dot{x}^+(t)) = DE(m(t), x(t); m_+(t), x_+(t)), \quad (p(0), m(0), x(0)) = (\bar{p}, \bar{m}, \bar{x}). \tag{18}
\]

For a locally Lipschitz continuous function \(y(t)\) (which necessarily has a derivative \(\dot{y}(t)\) almost

\(^{28}\)This vision can be found in the 1941 paper of Samuelson [45] in the era before equilibrium had achieved an adequate mathematical formulation.

\(^{29}\)This extended view of holdings and their potential persistence also underlies our work with financial market modeling in [27].

\(^{30}\)Some of the components in these input rate vectors could be negative, thereby acting as subtractions.
everywhere), one-sided differentiability refers to the existence of the right or left limits

\[ \dot{y}^+(t) = \lim_{h \to 0^+} \frac{1}{h} [y(t + h) - y(h)], \quad \dot{y}^-(t) = \lim_{h \to 0^-} \frac{1}{h} [y(t + h) - y(h)]. \]  

(19)

Differentiability corresponds to having \( \dot{y}^+(t) = \dot{y}^-(t) \).

The details of the argument behind this result will appear later. One-sided, instead of two-sided, differentiability of the trajectory is unavoidable in this result, because some goods components in the \( x(t) \) trajectory could start at 0, or drop to zero at a later time, only to eventually rise up and perhaps again drop down.

This result offers the interesting prospect that, within the confines of the model, both the prices and holdings in an equilibrium will evolve in time according to a fixed rule, dictated only by the utility functions of the agents, in response to internally/externally driven inputs. For instance, what might be expected if agents had their money holdings “controlled” by a government through subsidies or taxation? Of course, the limitations of the idea are indeed many, and most important among them is the absence in this formulation of any modeling of uncertainty over the future.

The Evolutionary Dynamics Theorem does not directly address the existence of a trajectory \((p(t), m(t), x(t))\) as described, the reason being the one-sidedness in the differential equation. Dealing with that will require mathematical innovations. But the complications in (18) with one-sidedness stem from letting the quantities of attractive goods sometimes be 0. Around an equilibrium \((\bar{p}, \bar{m}, \bar{x})\) with everything positive, the differential equation (18) loses its one-sided aspects and takes the ordinary form

\[
(p(t), \dot{m}(t), \dot{x}(t)) = DE(m(t), x(t); m_+(t), x_+(t)), \quad (p(0), m(0), x(0)) = (\bar{p}, \bar{m}, \bar{x}),
\]

in which \( E \) is continuously differentiable instead of just semidifferentiable. The standard theory of differential equations is applicable then, and the existence of trajectory of evolution over a time interval \([0, \varepsilon)\), at least, passes from just being a conjecture.

### 3 The arguments behind the theorems

**Proof of the Ample Existence Theorem.** We rely for this on specializing the existence result of Dontchev and Rockafellar [17, Theorem 1]. The key to that result, besides the utility conditions in A1, is the following replacement for the usual assumption that all agents start with positive quantities of every good. The initial holdings \((m^0, x^0)\) give *ample survivability* if the agents \( i \) have choices \((\hat{m}_i, \hat{x}_i)\) with \( u_i(\hat{m}_i, \hat{x}_i) > -\infty \) such that

a) \( \hat{x}_i \leq x_i^0 \) but \( \hat{m}_i < m_i^0 \), and  

b) \( \sum_{i=1}^r \hat{x}_i < \sum_{i=1}^r x_i^0. \)

The interpretation is that the agents could, if they wished, survive without any trading at all and do so with individual surpluses of money and collective surpluses in every other good.

To justify the claim made in the Ample Existence Theorem, we need to verify that

all choices of \((m^0, x^0)\) with \((m^0_i, x^0_i) \in \Omega_i\) give ample survivability.  

(20)
The argument is elementary and merely depends on the observing the extent to which various components in \((m^0, x^0)\) can be lowered slightly to get new holdings \((\hat{m}, \hat{x})\) such that \((\hat{m}_i, \hat{x}_i)\) still lies in \(\Omega_i\). This is evidently possible when the money holdings of all agents are positive and each other good is possessed in positive quantity by at least one agent. The definition of \(\Omega_i\) in (1) ensures this through the indispensability in A2. The assertions in the theorem about every equilibrium \((\bar{p}, \bar{m}, \bar{x})\) come from the observations made in (5) and (6).

**Proof of the Robust Stability Theorem.** The assertions about shift-stability will be an application of the stability result of Dontchev and Rockafellar [17, Theorem 3].

An immediate consequence of (20), in combination with the fact noted earlier in (6) about solutions to the agents’ optimization problems, is that

\[
\text{in any equilibrium } (\bar{p}, \bar{m}, \bar{x}), \text{ the holdings } (\bar{m}, \bar{x}) \text{ as } (m^0, x^0) \text{ give ample survivability. (21)}
\]

Through this, the parametric stability result of Dontchev and Rockafellar [17, Theorem 3] can applied to an equilibrium with respect to its own holdings. The result then asserts the shift stability of the equilibrium. (The cited result is applicable to any initial holdings \((m^0, x^0)\) in some small-enough neighborhood of \((m, \hat{x})\) as long as \((m^0, x^0)\) gives ample survivability. Shift stability of an equilibrium, not defined or considered in [17], needs \((m^0, x^0) = (\bar{m}, \bar{x})\) itself to give ample survivability, and the guarantee of that is what is new here.)

The existence of one-sided derivatives of the localized equilibrium mapping is provided by [17, Theorem 3] as well.

Because \(u_i\) is concave\(^{32}\) and differentiable on the convex set \(\Omega_i\), where any solution must lie, a condition both necessary and sufficient for \((m_i, x_i)\) to be optimal for a given \(p > 0\) can be given in terms of the gradient of \(u_i\) at \((m_i, x_i)\) and a Lagrange multiplier \(\lambda_i\) for the budget constraint:

\[
(m_i, x_i) \geq (0, 0), \quad \lambda_i(1, p) - \nabla u_i(m_i, x_i) \geq (0, 0), \quad (m_i, x_i) \cdot [\lambda_i(1, p) - \nabla u_i(m_i, x_i)] = 0, \quad (22)
\]

where

\[
m_i = m_i^0 + p(x_i^0 - x_i). \quad (23)
\]

The so-called complementary slackness conditions (22), expressed in a manner typical in optimization, say that for each good the corresponding components of the nonnegative vectors \((m_i, x_i)\) and \(\lambda_i(1, p) - \nabla u_i(m_i, x_i)\) cannot both be positive; at least one or the other must be 0. Specializations can be gleaned from the categorization of goods in our model by attractiveness and indispensability. Let the goods other than money be indexed by \(j = 1, \ldots, n\), so that

\[
x_i = (\ldots, x_{ij}, \ldots) \text{ with } x_{ij} \geq 0, \quad p = (\ldots, p_j, \ldots) \text{ with } p_j > 0.
\]

\(^{31}\) The format in [17] is that of survival sets \(U_i\) not necessarily of orthant type, but the cited result depends on having orthant-like structure locally around the equilibrium under investigation. That is true automatically here for the same reasons that have been laid out in deriving (20).

\(^{32}\) Plain quasi-concavity of the utility function \(u_i\) would not suffice for this.
Since indispensable goods, including money, occur only in positive amounts in $\Omega_i$, we can reduce (22) through assumptions A2 and A3 to

$$\lambda_i = \frac{\partial u_i}{\partial m_i}(m_i, x_i) \quad \text{(hence } \lambda_i > 0),$$

$$\lambda_i p_j = \frac{\partial u_i}{\partial x_{ij}}(m_i, x_i) \quad \text{for indispensable goods } j \text{ of agent } i,$$

$$\lambda_i p_j \geq \frac{\partial u_i}{\partial x_{ij}}(m_i, x_i) \quad \text{for attractive but not indispensable goods } j,$$

with equality holding when $x_{ij} > 0$,

$$x_{ij} = 0 \quad \text{for goods } j \text{ that are not attractive for agent } i.$$  \hspace{1cm} (24)

The conditions for solutions to these problems for $i = 1, \ldots, r$ to constitute an equilibrium $(p, m, x)$ associated with $(m^0, x^0)$ are the combination of (23) and (24) with

$$\sum_{i=1}^r x_{ij} = \sum_{i=1}^r x_{ij}^0 \quad \text{for } j = 1, \ldots, n. \hspace{1cm} (25)$$

The remainder of the proof, which is concerned with tâtonnement stability, must delve deeper into the variational analysis through which the results in [17] that we have been applying were themselves derived. Some background in [16], concerning solution mappings associated with variational inequality models for expressing optimality conditions and equilibrium, will be essential.\textsuperscript{33} To make things easier for readers not familiar with that subject, we start with a brief overview.

**Variational inequalities.** The variational inequality associated with a nonempty, closed, convex set $C \subset \mathbb{R}^N$ and a mapping $f : C \rightarrow \mathbb{R}^N$ with parameter $p \in \mathbb{R}^n$ takes the form finding $w \in C$ such that

$$-f(p, w) \in N_C(w), \hspace{1cm} (26)$$

where $N_C(w)$ is the normal cone to $C$ at $w$, defined by

$$v \in N_C(w) \iff w \in C \text{ and } v \cdot (w' - w) \leq 0 \text{ for all } w' \in C. \hspace{1cm} (27)$$

The normal cone $N_C(w)$ at any $w \in C$ is closed and convex. It always contains $v = 0$, and that is its only element when $w$ is an interior point of $C$, which is true of course for every $w$ in the special case when $C = \mathbb{R}^N$. The variational inequality reduces then to the vector equation $f(p, w) = 0$, and this is the sense in which variational inequality models expand on equation models. Vectors $v \neq 0$ necessarily exist in $N_C(w)$ when $w$ is a boundary point of $C$. They can be of any length and are the outward normals to the (closed) supporting half-spaces to $C$ at $w$.

The solution mapping associated with (26), which may be set-valued (i.e., a relation, or a correspondence in terminology common to economics literature), is

$$S : p \mapsto \{ w \mid -f(p, w) \in N_C(w) \}. \hspace{1cm} (28)$$

Results in [16] generalize the classical implicit function theorem for equations by providing criteria under which, in localization around a pair $(\bar{p}, \bar{w})$ with $\bar{w} \in S(\bar{p})$, the mapping $S$ is single-valued.

\textsuperscript{33}Variational analysis as laid out in [43] has also been the key to our other papers on economic equilibrium, namely [23], [24], [25] and [27].
and Lipschitz continuous, moreover with one-sided derivatives having a specific formula. The best case, which will be in play here, centers on $C$ being polyhedral, i.e., expressible as the intersection of a finite collection of closed half-spaces. A useful object then is the critical cone to $C$ at a point $w \in C$ with respect to a normal $v \in N_C(w)$, which is the polyhedral cone

$$K(w, v) = \{ w' \in T_C(w) \mid v \cdot w' = 0 \},$$

where $T_C(w)$ is the tangent cone to $C$ at $w$, equal to the polar of $N_C(w)$. All these cones are important in the study of optimality conditions, and to a large extent the passage from equations to variational inequalities is motivated by modeling circumstances that involve first-order optimality conditions associated with inequality constraints, such as the nonnegativity of goods in our economic setting.

The form of generalized implicit function theorem for (26) that was basic in Dontchev and Rockafellar [17], and will be basic here again, refers to the smallest linear subspace $K^+(w, v)$ containing the critical cone $K(w, v)$ as well as the largest linear subspace, $K^-(w, v)$ contained within $K(w, v)$. Under the assumption that the function $f$ in (26) is continuously differentiable, it focuses on a particular solution $w \in S(p)$ and invokes for the normal vector $v = -f(p, w)$ the criterion that

$$w' \in K^+(w, -f(p, w)), \quad \nabla_w f(p, w) w' \perp K^-(p, -f(p, w)), \quad w' \cdot \nabla_w f(p, w) w' \leq 0 \implies w' = 0,$$

where $\nabla_w f(p, w)$ denotes the $N \times N$ Jacobian of $f$ with respect to the $w$ argument. The conclusion then is that the solution mapping $S$ does have a single-valued Lipschitz continuous localization around $(p, w)$ for which the one-sided derivatives relative to vectors $p'$ exist and are given by

$$DS(p; p') = \text{the unique solution $w'$ to the auxiliary variational inequality}$$

$$-[\nabla_p f(p, w)p' + \nabla_w f(p, w)w'] \in N_{K(p, w)}(w').$$

This is from [16, Theorem 2E.8]. Note that in the equation case, with $C = IR^N$ and $f(p, w) = 0$, the critical cone and its associated subspaces are all just $IR^N$ itself. The criterion to be invoked reverts then to having $\nabla_w f(p, w)w' = 0$ imply $w' = 0$, or in other words, the full rank condition on the Jacobian matrix $\nabla_w f(p, w)$, as in the classical implicit function theorem.

For utilizing this general perturbation theory here, the target is the excess demand mapping $Z$ in (9) and specifically the monotonicity-type property we claim for it in (14). That property will be deduced from a formula for one-sided derivatives of $Z$. Clearly from (9), the key ingredient in that has to be formulas for one-sided derivatives of the agents’ demand mappings $X_i$ in (7). From now on the initial holdings $(m^0, x^0)$ will be fixed, so in working with these mappings we can pass to simpler notation:

$$X_i^0(p) = X_i(p; m_i^0, x_i^0), \quad Z^0(p) = \sum_{i=1}^r [X_i^0(p) - x_i^0] = Z(p; m^0, x^0). \quad (29)$$

It has already been noted that (through ample survivability) $X_i^0(p)$ is a uniquely determined goods vector in $\Omega_i$ for every price vector $p > 0$, and indeed that it is the unique solution to
the conditions in (24) with \( m_i \) given by (23) (and \( \lambda_i \) given by the first line in (24)). We are involved, in other words, with solving these conditions for \( x_i \) as a function of \( p \). If it were not for the third line in (24), we could view this from the classical perspective of solving a system of equations and try to apply the implicit function theorem. The inequality complication would drop away, of course, if we could be sure that the demand vector \( x_i \) would be \( > 0 \) in all its components, but allowing goods that are attractive but not indispensable to have zero demand for some combinations of prices is an important goal of our efforts.

It will help to reconfigure our task as the analysis of the enlarged mapping

\[
S_i^0 : p \mapsto \{ (m_i, x_i, \lambda_i) \text{ satisfying (23)--(24) } \}. \tag{30}
\]

From that analysis, the properties we require of \( X_i^0 \), as a component mapping, will be easy to extract. By interpreting \( S_i^0 \) as the solution mapping associated with a “variational inequality” problem, we will have available the above extension of the implicit function theorem, which can handle the inequality condition in (24).

In the case to which we want to apply this, the solution mapping will be \( S_i^0 \), already known to be single-valued. This case identifies (23)--(24) with the variational inequality

\[
-f_i(p, w_i) \in N_{C_i}(w_i) \text{ for } \begin{cases} w_i = (m_i, x_i, \lambda_i) \in C_i = \mathbb{R}^{n+1} \times \mathbb{R}, \\ f_i(p, w_i) = -\nabla u_i(m_i, x_i) - \lambda_i(1, p), m_i - m_i^0 - p(x_i^0 - x_i) \end{cases}. \tag{31}
\]

We will be analyzing this relative to an arbitrary \( p > 0 \) and \( (m_i, x_i, \lambda_i) = w_i = S_i^0(p) \). Then \( (m_i, x_i) \in \Omega_i \), and since the analysis is local, the fact that \( \nabla u_i \) is undefined at points of \( \mathbb{R}_+^{n+1} \) outside \( \Omega_i \) will not matter. The analysis will utilize the Jacobian expressions

\[
\nabla_p f_i(p, w_i)p' = [\lambda_i(0, p'), p'(x_i - x_i^0)], \\
\nabla_{w_i} f_i(p, w_i)w'_i = [-\nabla^2 u_i(m_i, x_i)(m'_i, x'_i) + \lambda'_i(1, p), -m'_i - p x'_i], \tag{32}
\]

where \( \nabla^2 u_i \) is the matrix of second partial derivatives of \( u_i \). It will involve us not only with the normal cone \( N_{C_i}(w_i) \), but also its polar, the tangent cone \( T_{C_i}(w_i) \), and the “critical cone”

\[
K_i(p, w_i) = \{ w'_i = (m'_i, x'_i, \lambda'_i) \in T_{C_i}(w_i) \mid f_i(p, w_i)w'_i = 0 \}. \tag{33}
\]

Because \( C_i \) is a polyhedral convex set (actually a cone itself), the critical cone \( K_i(p, w_i) \) is polyhedral convex as well. The theorem about solution mappings to variational inequalities over polyhedral sets that we are going to apply requires us also to look at

\[
K_i^+(p, w_i) = K_i(p, w_i) - K_i(p, w_i) = \text{smallest subspace } \supseteq K_i(p, w_i), \\
K_i(p, w_i) \cap K_i^{-}(p, w_i) = \text{largest subspace } \subseteq K_i(p, w_i). \tag{34}
\]

**Perturbation Result to be Applied** (as specialized from [16, Theorem 2E.8]). Under the criterion that

\[
w'_i \in K_i^+(p, w_i), \quad \nabla_{w_i} f_i(p, w_i)w'_i \perp K_i^-(p, w_i), \quad w'_i \nabla_{w_i} f_i(p, w_i)w'_i \leq 0 \implies w'_i = 0, \tag{35}
\]

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\(^{34}\)Again, this formulation could not be reached without utility being concave instead of just quasi-concave.
the solution mapping \( S_i^0 \) for the variational inequality (31) is Lipschitz continuous in a neighborhood of \( p \) and semidifferentiable there with one-sided directional derivatives given by

\[
DS_i^0(p; p') = \text{the unique solution } w_i' \text{ to the auxiliary variational inequality} \\
\quad -[\nabla p f_i(p, w_i)p' + \nabla w_i f_i(p, w_i)w_i'] \in N_{K_i(p, w_i)}(w_i').
\]

(36)

The next step is to work out the details of this in our context of (31). The cone \( K_i(p, w_i) \) and subspaces \( K_i^+(p, w_i) \) and \( K_i^-(p, w_i) \) come out, in expression with respect to the goods \( j \), as

\[
K_i(p, w_i) = IR \times \Pi_{j=1}^n K_{ij}(p_j, m_i, x_{ij}, \lambda_i) \times IR,
\]

where

\[
K_{ij}(p_j, m_i, x_{ij}, \lambda_i) = \begin{cases} 
IR & \text{if } x_{ij} > 0, \\
\mathbb{R} & \text{if } x_{ij} = 0 \text{ and } (\partial u_i/\partial x_{ij})(m_i, x_i) = \lambda_ip_j, \\
\{0\} & \text{if } x_{ij} = 0 \text{ and } (\partial u_i/\partial x_{ij})(m_i, x_i) < \lambda_ip_j.
\end{cases}
\]

(37)

\[
K_i^+(p, w_i) = IR \times \Pi_{j=1}^n K_{ij}^+(p_j, m_i, x_{ij}, \lambda_i) \times IR,
\]

where

\[
K_{ij}^+(p_j, m_i, x_{ij}, \lambda_i) = \begin{cases} 
IR & \text{if } x_{ij} > 0, \\
\mathbb{R} & \text{if } x_{ij} = 0 \text{ and } (\partial u_i/\partial x_{ij})(m_i, x_i) = \lambda_ip_j, \\
\{0\} & \text{if } x_{ij} = 0 \text{ and } (\partial u_i/\partial x_{ij})(m_i, x_i) < \lambda_ip_j.
\end{cases}
\]

(38)

\[
K_i^-(p, m_i, x_i, \lambda_i) = IR \times \Pi_{j=1}^n K_{ij}^-(p_j, m_i, x_{ij}, \lambda_i) \times IR,
\]

where

\[
K_{ij}^-(p_j, m_i, x_{ij}, \lambda_i) = \begin{cases} 
IR & \text{if } x_{ij} > 0, \\
\{0\} & \text{if } x_{ij} = 0 \text{ and } (\partial u_i/\partial x_{ij})(m_i, x_i) = \lambda_ip_j, \\
\{0\} & \text{if } x_{ij} = 0 \text{ and } (\partial u_i/\partial x_{ij})(m_i, x_i) < \lambda_ip_j.
\end{cases}
\]

(39)

In (37), (38) and (39) the same three categories of indices \( j \) are involved, and it will be convenient to speak of them as categories 1, 2 and 3. Unattractive goods \( j \) are clearly always in category 3, which gives \( \{0\} \) in every case. Let \( J^+ \) refer to all the indices \( j \) in categories 1 and 2, and let \( J^- \) refer to those only in category 1. In these terms we proceed toward verifying (35), which can written as

only \((m_j', x_j', \lambda_j') = (0, 0, 0)\) satisfies the conditions

\[
(m_j', x_j', \lambda_j') = w_i' \in K_i^+(p, w_i), \\
[-\nabla^2 u_i(m_i, x_i)(m_j', x_j') + \lambda_j'(1, p), -m_j' - px_j'] \perp K_i^-(p, w_i), \\
(m_j', x_j', \lambda_j') \cdot [-\nabla^2 u_i(m_i, x_i)(m_j', x_j') + \lambda_j'(1, p), -m_j' - px_j'] \leq 0.
\]

(40)

The first of the three conditions in (40) narrows our attention to cases of \((m_j', x_j', \lambda_j')\) having \(x_{ij}' = 0\) for all \( j \not\in J^+ \), while the second further narrows it to \( \lambda_j' = 0 \) and \( m_j' + \bar{p}x_j' = 0 \), along with having the components of the vector \( \nabla^2 u_i(m_i, \bar{m}_i, \bar{x}_i)(m_j', x_j') \) be 0 except possibly for some of them that belong to indices \( j \not\in J^- \). The quadratic expression in the third condition reduces then to

\[-(m_j', x_j') \cdot \nabla^2 u_i(m_i, \bar{m}_i, \bar{x}_i)(m_j', x_j') \leq 0.
\]

Because of the negative definiteness coming from our assumption A4, this expression cannot be \( \leq 0 \) unless \( m_j' = 0 \) and \( x_{ij}' = 0 \) for all attractive goods \( j \). But \( x_{ij}' = 0 \) already for unattractive goods, so we conclude that (40) does hold, and with it the properties of \( S_i^0 \) listed in the Perturbation Result above.

It follows then that the demand mapping \( X_i^0 \) is likewise locally Lipschitz continuous and semidifferentiable. More specifically, we have from (36) through (32) that

\[
DS_i^0(p; p') \text{ is the unique solution } (m_j', x_j', \lambda_j') \text{ to the variational inequality} \\
\quad -[\lambda_i(0, p') + \lambda_j'(1, p - \nabla^2 u_i(m_i, x_i)(m_j', x_j'), -p' \cdot (x_i - x_i^0) - m_j' - px_j'] \\
\quad \in N_{K_i(p, w_i)}(w_i') \text{ with } w_i = (m_i, x_i, \lambda_i), \ w_i' = (m_j', x_j', \lambda_j'),
\]

(41)
where the details of the cone $K_i(p, w_i)$ are in (37). The one-sided directional derivatives of $X^0_i$ are given then by
\[
DX^0_i(p; p') = x'_i \quad \text{for the } (m'_i, x'_i, \lambda'_i) \text{ in (41)}.
\] (42)

It is evident now that the excess demand mapping $Z^0$ in (29) is Lipschitz continuous locally as well, and semidifferentiable with its one-sided derivatives given by
\[
DZ^0(p; p') = \sum_{i=1}^r x'_i \text{ where } x'_i = DX^0_i(p; p').
\] (43)

This brings us to the stage where we have confirmed all of the claims of the Robust Stability Theorem except for the “monotonicity” property (14). Directional derivatives will help with that, as follows. The inequality in (14) can be equivalently be rewritten (with a change of variables that alters the meaning of $p'$) as
\[
-\mu ||p'||^2 \geq ([p + p'] - p) \cdot (Z^0(p + p') - Z^0(p)) = \int_0^1 p' \cdot DZ^0(p + tp'; p')dt,
\]

inasmuch as the (Lipschitz continuous) function $z(t) = Z^0(p + tp')$ is differentiable for almost every $t$ with $\dot{z}^+(t) = DZ^0(p + tp'; p')$. Our task in these terms is reduced to demonstrating the existence of $\mu > 0$ for which
\[
-\mu ||p'||^2 \geq p' \cdot DZ^0(p; p') = p' \cdot \sum_{i=1}^r DX^0_i(p; p') \text{ when } p \in N, \ p + p' \in N, \ p' \neq 0,
\] (44)

provided that the ball $N$ around the equilibrium price vector $\bar{p}$ and the ball $N_0$ around the equilibrium holdings $(\bar{m}, \bar{p})$ in (14) are chosen small enough. Here by (42) we have
\[
p' \cdot \sum_{i=1}^r DX^0_i(p; p') = \sum p' \cdot x'_i \text{ with } x'_i \text{ from } (m'_i, x'_i, \lambda'_i) \text{ solving (41)}
\] (45)

and can make that the platform for our analysis.

It is important now to notice a sort of uniformity in the local behavior of the sets in (37), (38) and (39), namely that
\[
K_i^- (\bar{p}, \bar{w}_i) \subset K_i^- (p, w_i) \subset K_i (p, w_i) \subset K_i^+ (p, w_i) \subset K_i^+ (\bar{p}, \bar{w}_i)
\]
for $(p, w_i)$ near enough to $(\bar{p}, \bar{w}_i)$.

This is evident from the formulas for these sets and the continuity of the partial derivatives of $u_i$. A follow-up to this observation, taking advantage of the fact that making $p$ be close to $\bar{p}$ also makes $w_i = S^0(p)$ be close to $\bar{w}_i = S^0(\bar{p})$ through the continuity of $S^0_i$, is that
\[
\begin{align*}
\text{if } (m'_i, x'_i, \lambda'_i) \text{ solves in (41) with } p \text{ near enough to } \bar{p}, \text{ then } (m'_i, x'_i, \lambda'_i) \in K_i^+ (\bar{p}, \bar{w}_i), \\
- [\lambda_i(0, p') + \lambda'_i(1, p) - \nabla^2 u_i(m_i, x_i)(m'_i, x'_i), -p' - x'_i - x^- - m'_i - p'x'_i] \perp K_i^- (\bar{p}, \bar{w}_i), \\
(m'_i, x'_i, \lambda'_i) \cdot \left(- [\lambda_i(0, p') + \lambda'_i(1, p) - \nabla^2 u_i(m_i, x_i)(m'_i, x'_i), -p' - x'_i - x^- - m'_i - p'x'_i] \right) = 0.
\end{align*}
\] (47)

In view of (38) and (39), we must then have
\[
p' \cdot (x_i - x^-_i) + m'_i + p \cdot x'_i = 0 \text{ for all agents } i, \text{ and}
\]
\[
x'_{ij} = 0 \text{ when a good } j \text{ is unattractive to agent } i.
\] (48)
The equation in the third condition of (47) reduces in this case to

\[-\lambda_i x'_i p' + \lambda'_i (m'_i + x'_i p) - (m'_i, x'_i) \nabla^2 u_i(m_i, x_i)(m'_i, x'_i) = 0.\]

Since \(\lambda_i > 0\), we can rewrite this, using the first line of (48), as

\[p'_i x'_i = \lambda_i^{-1} [(m'_i, x'_i) \nabla^2 u_i(m_i, x_i)(m'_i, x'_i) + \lambda'_i p'[x_i - x'_i]],\]  

where moreover \((m'_i, x'_i) \nabla^2 u_i(m_i, x_i)(m'_i, x'_i) < 0\) unless \((m'_i, x'_i) = (0, 0)\) through the second line of (48) and the negative definiteness of the submatrix of \(\nabla^2 u_i(m_i, x_i)\) with respect to the attractive goods for agent \(i\) in our assumption A4. Hence

\[p' \sum_{i=1}^r x'_i = \sum_{i=1}^r (m'_i, x'_i) \left[\frac{1}{\lambda_i} \nabla^2 u_i(m_i, x_i)(m'_i, x'_i) + \sum_{i=1}^r \frac{\lambda'}{\lambda_i} p'[x_i - x'_i],\right]\]

where the first sum on the right is < 0 unless \((m'_i, x'_i) = (0, 0)\) for all \(i\).

The crux of the matter emerges as making sure that the negativity of the quadratic sum in (50) cannot be overpowered by the second sum. The quadratic sum has the form

\[(m', x') A(m, x, \lambda)(m', x') \text{ for } m' = (\ldots, m'_i, \ldots), \ x' = (\ldots, x'_i, \ldots),\]

where \(A(m, x, \lambda)\) is a negative definite matrix depending continuously on \((m, x, \lambda)\), which in turn is comprised of elements \((m_i, x_i, \lambda_i) = S^0(p) = S_i(p; m^0_i, x^0_i)\) that depend continuously on \(p\) and also on \((m^0, x^0)\).\(^{35}\) Its eigenvalues can therefore be bounded locally away from 0:

there exist \(\varepsilon > 0\) and closed balls \(N\) at \(\bar{p}\) and \(N_0\) at \((\bar{m}, \bar{x})\) such that

\[\sum_{i=1}^r (m'_i, x'_i) \left[\frac{1}{\lambda_i} \nabla^2 u_i(m_i, x_i)(m'_i, x'_i) \leq -\varepsilon \|(m', x')\|^2 \text{ when } p \in N, \ (m^0, x^0) \in N_0.\]

An upper estimate of the size of the second sum in (50) must next come into play. For this we return to the conditions in (47). With \((\ldots, \lambda_i, \ldots)\) denoted by \(\lambda'\), let

\[W(p; m^0, x^0) = \{ (m', x', \lambda', p') \text{ satisfying the first two conditions in (47) for all } i \}.\]

Because \(K_i^+(\bar{p}, \bar{w}_i)\) and \(K_i^-(\bar{p}, \bar{w}_i)\) are linear subspaces, \(W(p; m^0, x^0)\) is a linear subspace as well. We claim

there exist \(\rho > 0\) and balls \(N'\) at \(\bar{p}\) and \(N'_0\) at \((\bar{m}, \bar{x})\) such that

\[\|(p', \lambda')\| \leq \rho \|(m', x')\| \]

for all \((p', m', x', \lambda') \in W(p; m^0, x^0)\) when \(p \in N', \ (m^0, x^0) \in N'_0.\)

If this were not true, there would be sequences of elements

\[(p^k, m^0_k, x^k, \lambda^k) \in W(p^k; m^0_k, x^0_k) \text{ with } p^k \in N', \ (m^0_k, x^0_k) \in N'_0,\]

such that \(\|(p^k, \lambda^k)\| = 1\) for all \(k\) and \(\|(m^0_k, x^0_k)\| \to 0.\)

\(^{35}\)The continuity of \(S_i(p; m^0_i, x^0_i)\) with respect to \((m^0_i, x^0_i)\) is seen from the optimality conditions (23)–(24) for the optimization problem of agent \(i\), which involve \((m^0_i, x^0_i)\) only through (23). A limit of solutions to these conditions coming from a convergent sequence of such initial holdings must be another solution.
Passing to convergent subsequences, we would arrive in the limit at elements
\[(p^*, m^*, x^*, \lambda^*) \in W(p^*; m_{0*}^*, x_{0*}^*) \text{ with } p^* \in N', \ (m_{0*}^*, x_{0*}^*) \in N'_0, \]
such that \((p^*, \lambda^*) \neq (0, 0)\) but \((m^*, x^*) = (0, 0)\).

This is an impossible situation for the following reason. It entails
\[-[\lambda_i^*(0, p^*) + \tilde{\lambda}_i^*(1, p^*), -p^*(x_i^* - x_i^0)] \perp K_i^-(\tilde{p}, \tilde{m}_i, \tilde{x}_i, \tilde{\lambda}_i),\]
which first implies \(\tilde{\lambda}_i^* = 0\) and then that \(p_j^* = 0\) for all goods \(j\) having \(x_{ij}^* > 0\). But then \(p_j^* = 0\) for all goods \(j\), yielding a contradiction because our assumptions make it impossible for any \(j\) to have \(x_{ij}^* = 0\) for every agent \(i\). Thus, (52) is confirmed.

Putting (52) now to use, and noting that \(||(p', \lambda')|| \leq \rho||(m', x')||\) implies that \(||p'|| \leq \rho||(m', x')||\) and \(|\lambda_i'| \leq \rho||(m', x')||\) for all \(i\), as well as \(||x_i - x_i^0|| \leq ||x - x^0||\), we get the upper bound
\[\sum_{i=1}^{r} \lambda_i' p_i' [x_i - x_i^0] \leq \rho \left[ \sum_{i=1}^{r} \frac{1}{\lambda_i} \right] ||x - x^0|| ||(m', x')||^2,\]
which holds when \(p\) is close enough to \(\tilde{p}\) and \((m^0, x^0)\) is close enough to \((\tilde{m}, \tilde{x})\). Since \(\lambda \to \tilde{\lambda}\) and \(x \to \tilde{x}\) as \(p \to \tilde{p}\) because \((m_i, \tilde{x}_i, \tilde{\lambda}_i) = S_i(\tilde{p}; m^0, x^0)\) for all \(i\), there is also a local upper bound
\[\left[ \sum_{i=1}^{r} \frac{1}{\lambda_i} \right] ||x - x^0|| \leq \nu ||\tilde{x} - x^0||.\]

Putting all this together with (51), we obtain from (50) and (52) that
\[p' \sum_{i=1}^{r} x_i' \leq -\left(\varepsilon - \nu \rho ||\tilde{x} - x^0||\right)|| (m', x') ||^2 \leq -\rho^{-2} \left(\varepsilon - \nu \rho ||\tilde{x} - x^0||\right)||p'||^2\]
when \(p\) is close enough to \(\tilde{p}\). The coefficient \(\mu = \rho^{-2}(\varepsilon - \nu \rho||\tilde{x} - x^0||)\) is sure to be \(> 0\) when \(x^0\) is close enough to \(\tilde{x}\). We have already determined that having \((m', x') = (0, 0)\) is incompatible with \(p' \neq 0\), so the desired conclusion, supporting the existence of a neighborhood \(N\) as in (44), has been reached.

**Proof of the Evolutionary Dynamics Theorem.** We are dealing with a time-dependent equilibrium, \((p(t), m(t), x(t)) = E(m(t), x(t))\), affected somehow by additions to \(m(t)\) and \(x(t)\) that enter at rates \(m_+(t)\) and \(x_+(t)\). The evolution can be understood through consideration of time increments \(h > 0\) that cause \(m(t)\) and \(x(t)\) to shift approximately to \(m(t) + hm_+(t)\) and \(x(t) + hx_+(t)\). These could be out of equilibrium and no longer be matched by \(p(t)\), but on the basis of the Robust Stability Theorem an adjusted equilibrium
\[(p(t + h), m(t + h), x(t + h)) = E(m(t) + hm_+(t), x(t) + hx_+(t)),\] identifiable by tâtonnement, will exist uniquely nearby. Then also
\[(p(t + h), m(t + h), x(t + h)) = E(m(t + h), x(t + h)),\]
along with \((p(t), m(t), x(t)) = E(m(t), x(t))\), so the corresponding rate of change in the components of the equilibrium is

\[
\frac{1}{h}[(p(t+h), m(t+h), x(t+h)) - (p(t), m(t), x(t))] = \frac{1}{h}[E(m(t) + hm(t), x(t) + hx(t)) - E(m(t), x(t))].
\]

In taking the limit of this as \(h \to 0^+\) we obtain, on the right, the one-sided directional derivative \(DE(m(t), x(t); m_+(t), x_+(t))\).

This establishes in particular that the trajectory does have right derivatives. A parallel argument equally well serves to establish that the trajectory has left derivatives. Even though the theorem does not deal with left derivatives, the existence of both, along with the local boundedness of the expression on the right side of the differential equation makes these both one-sided derivatives be locally bounded, and the local Lipschitz continuity in \(t\) then follows for the trajectory.

References


