

VARIATIONAL ANALYSIS OF NASH EQUILIBRIUM

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Abstract

Tools of variational analysis are employed in studying the local stability of Nash equilibrium in a game-like framework of multi-agent optimization that emphasizes first- and second-order conditions for local optimality. The equilibrium is tied to a parameterized variational inequality for which properties of metric regularity can be derived.

Keywords: *Nash equilibrium, multi-agent optimization, local stability, variational inequalities, strong metric regularity*

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1 Introduction

Equilibrium problems have many formulations, but one of the most important is concerned with the “game” interactions of N “players” who seek to optimize individually and noncooperatively but can’t escape the effects of each other’s choices. This has furnished a valuable model for competitive situations in economics and operations management. Such a model may have more than one equilibrium, and the standard questions of applied mathematics about uniqueness and stability of solutions therefore need to be addressed, in particular for their importance in building a platform for computing an equilibrium. Answers to those questions will be obtained here with the help of finite-dimensional variational analysis. The perspective will be that of “multi-agent optimization” as a generalization of “single-agent optimization” in which a major role is given to first- and second-order conditions for optimality.

In multi-agent optimization, agent k for $k = 1, \dots, N$ has a set $C_k \subset \mathbb{R}^{n_k}$ from which to select a decision x_k and an objective $f_k(x_1, \dots, x_N)$ which is to be minimized with respect to that decision, but in the face of the objective being influenced by the decisions of the other agents. In traditional game-theory notation, where x_{-k} stands for all those other decisions, a Nash equilibrium is a combination

$$(\bar{x}_1, \dots, \bar{x}_N) \in C_1 \times \dots \times C_N \text{ such that } \bar{x}_k \in \underset{x_k \in C_k}{\operatorname{argmin}} f_k(x_k, \bar{x}_{-k}) \text{ for } k = 1, \dots, N. \quad (1.1)$$

To facilitate approaching the minimization problems of the agents through conditions for local optimality, it will be assumed throughout that the functions f_k are twice-continuously differentiable. The sets C_k will be taken to be nonempty, closed and *convex*. Although that would be a serious restriction from the angle of single-agent optimization, it’s natural here because such convexity is a precondition to most results on the *existence* of a Nash equilibrium. The differentiability of the gradients of the functions f_k and the convexity of the sets C_k will be essential also for path we are going to take in connecting equilibrium problems to one of solving “variational inequalities.”

The prime tool for making that connection is the *normal cone* $N_C(x)$ to a closed convex set $C \subset \mathbb{R}^n$ at a point $x \in C$, this being the closed convex cone in \mathbb{R}^n defined by

$$z \in N_C(x) \iff z \cdot [x' - x] \leq 0 \text{ for all } x' \in C. \quad (1.2)$$

The first-order necessary condition for a differentiable function f on \mathbb{R}^n to have a *local* minimum relative to C at \bar{x} is $-\nabla f(\bar{x}) \in N_C(\bar{x})$, and when f is convex this is also sufficient condition for a *global* minimum relative to C .

Definition 1.1 (variational equilibrium). *A combination*

$$(\bar{x}_1, \dots, \bar{x}_N) \in C_1 \times \dots \times C_N \text{ such that } -\nabla_{x_k} f_k(\bar{x}_k, \bar{x}_{-k}) \in N_{C_k}(\bar{x}_k) \text{ for } k = 1, \dots, N \quad (1.3)$$

will be called a variational Nash equilibrium.

When every $f_k(x_k, x_{-k})$ is convex with respect to x_k , such a variational Nash equilibrium (1.3) is equivalent to a classical Nash equilibrium (1.1). Otherwise, though, it is a broader concept able to encompass local instead of global minimization by the agents along with situations where having a local minimum is relaxed to a kind of stationarity. It can be argued in support of this broader view of “equilibrium” that global minimization by competing agents, as traditionally called for, is an unrealistic requirement. If even computers can have great difficulty in determining a global minimum and often have to make do with solving necessary conditions for optimality, why should agents have to meet such a standard? Moreover, equilibrium in other disciplines, such as physics, can refer to circumstances where a balance of conflicting forces is “infinitesimally delicate.”

Be that as it may, although our efforts will initially revolve around properties of a variational Nash equilibrium, we'll end up working with classical Nash equilibrium — but in a local sense in which the sets C_k are truncated $C_k \cap X_k$ with respect to neighborhoods X_k of the equilibrium components \bar{x}_k .

Methodology associated with “variational inequalities” will assist us in powerful ways. The *variational inequality problem* for a nonempty closed convex set $C \subset \mathbb{R}^n$ and a mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ seeks to

$$\text{find } \bar{x} \in C \text{ such that } -F(\bar{x}) \in N_C(\bar{x}). \quad (1.4)$$

The first-order optimality condition $-\nabla f(\bar{x}) \in N_C(\bar{x})$ obviously fits this with $F = \nabla f$, but the conditions for a variational Nash equilibrium in (1.3) can be posed simultaneously in the pattern of (1.4) as well, namely by taking

$$\bar{x} = (\bar{x}_1, \dots, \bar{x}_N) \in C = C_1 \times \dots \times C_N, \quad F(\bar{x}) = (\nabla_{x_1} f_1(\bar{x}_1, \bar{x}_{-1}), \dots, \nabla_{x_N} f_N(\bar{x}_N, \bar{x}_{-N})). \quad (1.5)$$

In our approach to the “stability” of a variational Nash equilibrium we build on this by introducing a parameter p on which the functions f_k are allowed to depend and then investigating how solutions to

$$(x_1, \dots, x_N) \in C_1 \times \dots \times C_N \text{ such that } -\nabla_{x_k} f_k(p, x_k, x_{-k}) \in N_{C_k}(x_k) \text{ for } k = 1, \dots, N \quad (1.6)$$

may behave with respect to perturbations of p away from a given \bar{p} . With p introduced similarly in basic setting of (1.4), we can recognize this as the study of how solutions to a variational inequality problem may depend on parameters in the problem.

This subject has a large literature that we can draw on. It treats the solution mapping S for a parameterized variational inequality problem of general type,

$$S(p) = \{x \mid -F(x, p) \in N_C(x)\}, \quad (1.7)$$

where p is a vector in \mathbb{R}^d . Among of its main ingredients are *implicit function theorems* identifying circumstances in which the potentially set-valued mapping S “localizes” to a single-valued mapping s . The history of that topic has been laid out in the book [1] along with recent advances, many of them not requiring necessarily requiring differentiability of F . Here, though, we will concentrate on F being \mathcal{C}^1 from $\mathbb{R}^d \times \mathbb{R}^n$ to \mathbb{R}^n and rely only on adaptations of the more general results to that case.

The classical implicit function theorem deals with solving an equation $F(p, x) = 0$ “locally for x as a function of p ” around a pair (\bar{p}, \bar{x}) for which $F(\bar{p}, \bar{x}) = 0$. A condition on the Jacobian $\nabla_x F(\bar{p}, \bar{x})$ guarantees the existence a \mathcal{C}^1 function s having $F(p, s(p)) = 0$ locally and $s(\bar{p}) = \bar{x}$. In the broader setting of the the solution mapping to a parameterized variational inequality as in (1.7), the parallel results yielding an implicit function can't claim its smoothness, due to unilateral effects coming from the convex set C . Instead, a Lipschitz property must be elicited with respect to a localization of the *graph* of S , that being the set

$$\text{gph } S = \{(p, x) \mid x \in S(p)\} \subset \mathbb{R}^d \times \mathbb{R}^n. \quad (1.8)$$

Definition 1.2 (strong metric regularity). *A set-valued mapping $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ is strongly metrically regular at $(\bar{p}, \bar{x}) \in \text{gph } S$ if there exist neighborhoods P of \bar{p} and X of \bar{x} such that $(P \times X) \cap \text{gph } S$ is the graph of a single-valued Lipschitz continuous mapping $s : P \rightarrow X$ with $s(\bar{p}) = \bar{x}$, which is then the implicit function obtained from the localization of $\text{gph } S$ to $(P \times X) \cap \text{gph } S$.*

This concept is associated with a landmark result of Robinson [6], a version of which will be stated shortly. Metric regularity properties of set-valued mappings come on several levels, cf. the book [1] and its references, but here only the “strong” kind will come into play.

For the applications to variational inequalities we are aiming at, eventually tied to aspects of Nash equilibrium, it will be important to consider not only the *basic* solution mapping S in (1.7) but also the *canonical* solution mapping

$$S^*(v, p) = \{x \mid v - F(p, x) \in N_C(x)\} \text{ around } (0, \bar{p}, \bar{x}) \in \text{gph } S^*. \quad (1.9)$$

Furthermore, solution mappings associated with the *linearization* of the parameterized variational inequality $-F(p, x) \in N_C(x)$ will be crucial as well. Linearization refers to the replacement of F by its first-order expansion around the reference pair (\bar{p}, \bar{x}) , which we'll denote by \bar{F} for convenience:

$$\bar{F}(p, x) = F(\bar{p}, \bar{x}) + \nabla_p F(\bar{p}, \bar{x})[p - \bar{p}] + \nabla_x F(\bar{p}, \bar{x})[x - \bar{x}]. \quad (1.10)$$

With respect to this we will be interested in the solution mapping

$$\bar{S}^*(v, p) = \{x \mid v - \bar{F}(p, x) \in N_C(x)\} \text{ around } (0, \bar{p}, \bar{x}) \in \text{gph } \bar{S}^* \quad (1.11)$$

and two of its “submappings”, the first being $\bar{S} : p \mapsto \bar{s}^*(0, p)$, with

$$\bar{S}(p) = \{x \mid -\bar{F}(p, x) \in N_C(x)\} \text{ around } (\bar{p}, \bar{x}) \in \text{gph } \bar{S}, \quad (1.12)$$

and the second being $\bar{S}_0 : v \mapsto \bar{S}^*(v, \bar{p})$, with

$$\bar{S}_0(v) = \{x \mid v - \bar{F}(\bar{p}, x) \in N_C(x)\} \text{ around } (0, \bar{x}) \in \text{gph } \bar{S}_0, \quad (1.13)$$

where $\bar{F}(\bar{p}, x) = F(\bar{p}, \bar{x}) + \nabla_x F(\bar{p}, \bar{x})[x - \bar{x}]$. In this picture S will be partnered by \bar{S} , while S^* will be partnered by \bar{S}^* . The simplest mapping \bar{S}_0 , to be called the *auxiliary* solution mapping, will be the key to confirming properties of the others.

We will be interested in the strong metric regularity of all of these mappings at the indicated reference elements in their graphs. Definition 1.2 covers the basic solution mapping S directly, and the corresponding statement for \bar{S} is then obvious with \bar{F} replacing F . The implicit function in the case of \bar{S} will be denoted by \bar{s} . For S^* the property of strong metric regularity is articulated with respect also to a neighborhood V of 0, so that the graph of the corresponding implicit function, to be denoted by s^* , is the localization $(V \times P \times X) \cap \text{gph } S^*$. Likewise for \bar{S}^* with implicit function \bar{s}^* , again just having \bar{F} in place of F . Finally, for strong metric regularity of \bar{S}_0 the graph of the implicit function \bar{s}_0 is $(V \times X) \cap \text{gph } \bar{S}_0$.

Theorem 1.3 (implicit functions for variational inequalities). *The solution mapping S^* is strongly metrically regular at $(0, \bar{p}, \bar{x})$ if and only if the solution mapping \bar{S}^* is strongly metrically regular at $(0, \bar{p}, \bar{x})$, in which case the implicit function \bar{s}^* serves as a first-order approximation to the implicit function s^* :*

$$s^*(v, p) = \bar{s}^*(v, p) + o(v, p - \bar{p}). \quad (1.14)$$

Moreover the strong metric regularity of the auxiliary solution mapping \bar{S}_0 at $(0, \bar{x})$ is both necessary and sufficient for this equivalence, in consequence of which the solution mappings S and \bar{S} are strongly metrically regular at (\bar{p}, \bar{x}) with the implicit function \bar{s} serving as a first-order approximation to the implicit function s :

$$s(p) = \bar{s}(p) + o(p - \bar{p}), \text{ where furthermore } \bar{s}(p) = \bar{s}_0(-\nabla_p F(\bar{p}, \bar{x})[p - \bar{p}]). \quad (1.15)$$

This simplifies, and at the same time elaborates, the implicit function of Robinson [6] along lines elaborated in the book [1, Section 2B]. The relationship with the classical implicit function theorem

is instructive. Solving $F(p, x) = 0$ corresponds to solving $-F(p, x) \in N_C(x)$ when $C = \mathbb{R}^n$, hence $N_C(x) = \{0\}$ for all x . The basic solution mapping S in that case yields in Theorem 1.3 the classical implicit function s (although promising only its Lipschitz continuity). The mapping \bar{S} and its implicit function \bar{s} are concerned in the same way with solving the linear equation

$$F(\bar{p}, \bar{x}) + \nabla_p F(\bar{p}, \bar{x})[p - \bar{p}] + \nabla_x F(\bar{p}, \bar{x})[x - \bar{x}] = 0$$

for x in terms of p . On the other hand, \bar{S}_0 and \bar{s}_0 solve

$$F(\bar{p}, \bar{x}) + \nabla_x F(\bar{p}, \bar{x})[x - \bar{x}] = v$$

for x in terms of v . Strong metric regularity of \bar{S} and \bar{S}_0 corresponds then to the Jacobian $\nabla_x F(\bar{p}, \bar{x})$ having full rank n and leads the formula $\bar{s}_0(v) = \nabla_x F(\bar{p}, \bar{x})^{-1}[v - F(\bar{p}, \bar{x})]$. The first-order approximation in (1.15) then mirrors a differentiation formula for the implicit function s ,

Beyond this familiar situation, Theorem 1.3 doesn't readily furnish a "pointwise" criterion at (\bar{p}, \bar{x}) for strong metric regularity of the auxiliary mapping \bar{S}_0 analogous to the full rank condition just at (\bar{p}, \bar{x}) , at least not directly. However, an optimization-specialized criterion on those lines will be developed in the next section.

2 Adaptation to local optimality

Continuing still with the parameterized variational inequality (1.7), we move toward optimization by specializing F to a gradient mapping:

$$F(p, x) = \nabla_x f(p, x) \text{ for a function } f \in \mathcal{C}^2. \quad (2.1)$$

Then $F \in \mathcal{C}^1$ and the conclusions of Theorem 1.3 hold with the Jacobians of F reflecting second-derivatives of f :

$$\nabla_x F(p, x) = \nabla_{xx}^2 f(p, x), \quad \nabla_p F(p, x) = \nabla_{xp}^2 f(p, x).$$

In fact the linearization \bar{F} of F around (\bar{p}, \bar{x}) corresponds to the quadratic expansion of f around (\bar{p}, \bar{x}) , i.e., the function

$$\begin{aligned} \bar{f}(p, x) = & f(\bar{p}, \bar{x}) + \nabla_x f(\bar{p}, \bar{x}) \cdot [x - \bar{x}] + \frac{1}{2}[x - \bar{x}] \cdot \nabla_{xx}^2 f(\bar{p}, \bar{x})[x - \bar{x}] \\ & + \nabla_p f(\bar{p}, \bar{x}) \cdot [p - \bar{p}] + [x - \bar{x}] \cdot \nabla_{xp}^2 f(\bar{p}, \bar{x})[p - \bar{p}] + \frac{1}{2}[p - \bar{p}] \cdot \nabla_{pp}^2 f(\bar{p}, \bar{x})[p - \bar{p}], \end{aligned} \quad (2.2)$$

so that

$$\bar{F}(p, x) = \nabla_x f(\bar{p}, \bar{x}) + \nabla_{xp}^2 f(\bar{p}, \bar{x})[p - \bar{p}] + \nabla_{xx}^2 f(\bar{p}, \bar{x})[x - \bar{x}].$$

The various mappings in Theorem 1.3 relate then to solving first-order conditions for optimality in the minimization problems for f and \bar{f} :

$$\begin{aligned} S(p) &= \{x \mid -\nabla_x f(p, x) \in N_C(x)\}, \\ S^*(v, p) &= \{x \mid v - \nabla_x f(p, x) \in N_C(x)\}, \\ \bar{S}(p) &= \{x \mid -\nabla_x \bar{f}(p, x) \in N_C(x)\}, \\ \bar{S}^*(v, p) &= \{x \mid v - \nabla_x \bar{f}(p, x) \in N_C(x)\}, \\ \bar{S}_0(v) &= \{x \mid v - \nabla_x \bar{f}(\bar{p}, x) \in N_C(x)\}. \end{aligned} \quad (2.3)$$

The consequences of Theorem 1.3 for such mappings have value in their own right, but we want now to look beyond first-order conditions at actual local optimality in the following way.

Definition 2.1 (strongly stable local optimality). *In the framework of (2.1) and (2.3) the solution mapping S provides strongly stable local optimality at (\bar{p}, \bar{x}) if strong metric regularity holds there relative to the neighborhoods P of \bar{p} and X of \bar{x} for which the associated implicit function s has the additional property that*

$$s(p) = \operatorname{argmin}_{x \in C \cap X} f(p, x) \text{ for every } p \in P. \quad (2.4)$$

Likewise for the other mappings, where the additional properties demanded beyond strong metric regularity are, respectively,

$$\begin{aligned} \bar{s}(p) &= \operatorname{argmin}_{x \in C \cap X} \bar{f}(p, x) \text{ for every } p \in P, \\ s^*(v, p) &= \operatorname{argmin}_{x \in C \cap X} \{f(p, x) - v \cdot x\} \text{ for every } (v, p) \in V \times P, \\ \bar{s}^*(v, p) &= \operatorname{argmin}_{x \in C \cap X} \{\bar{f}(p, x) - v \cdot x\} \text{ for every } (v, p) \in V \times P, \\ \bar{s}_0(v) &= \operatorname{argmin}_{x \in C \cap X} \{\bar{f}(x) - v \cdot x\} \text{ for every } v \in V. \end{aligned} \quad (2.5)$$

Note that in (2.4) not only is $s(p)$ asserted to be the unique optimal solution in the minimization over $C \cap X$, but also, there can't be any other locally optimal solution to the minimization over C that lies in the interior of X . That's because any such locally optimal solution x' would have to satisfy the first-order condition for local optimality and therefore belong to $S(p)$, in which case (p, x') would be in $(P \times X) \cap \operatorname{gph} S$. But according to the definition of strong metric regularity, the latter is the graph of s , so this is impossible unless $x' = s(p) = x$. Similarly in the other cases.

In the terminology of [5] and [2] the vectors v give *tilt* perturbations of the objectives, and the concept in Definition 2.1 for S^* is the *full stability* of the local minimum of $f(\bar{p}, x)$ relative to C at \bar{x} . Both [5] and [2] cope with broader choices of f and C than here by operating on a higher technical level. But they will anyway be top sources of facts in our simpler setting.

We are headed next toward establishing a new result, parallel to Theorem 1.3, in which the stable local optimality in Definition 2.1 is incorporated.

Theorem 2.2 (implicit functions with local optimality). *In the specialization to optimization in (2.1) and (2.3), S^* provides strongly stable local optimality at $(0, \bar{p}, \bar{x})$ if and only if \bar{S}^* provides strongly stable local optimality at $(0, \bar{p}, \bar{x})$, which in turn holds if and only if \bar{S}_0 provides strongly stable local optimality at $(0, \bar{x})$. Then S and \bar{S} provide strongly stable local optimality at (\bar{p}, \bar{x}) .*

The proof of Theorem 2.2 will entail producing at the same time a ‘‘pointwise’’ criterion for the stability to hold for \bar{S}_0 . It will be in the form of a second-order condition for local optimality at \bar{x} in the minimization of $f(\bar{p}, x)$ over $x \in C$. In order to formulate that, we have to develop some background in second-order variational analysis.

The background concerns *coderivatives of set-valued mappings*, a topic explained in much detail in [8] with many references. There is no need to delve very deeply into it here, because all we require for now are coderivatives of the set-valued mapping $N_C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ that associates with each $x \in C$ the normal cone $N_C(x)$ (but associates the empty set to any $x \notin C$). The coderivative $D^*N_C(\bar{x}|\bar{z})$ at $(\bar{x}, \bar{z}) \in \operatorname{gph} N_C$ is the set-valued mapping from \mathbb{R}^n to \mathbb{R}^n that assigns to w the (possibly empty) set of all u such that $(u, -w)$ belongs to the normal cone in $\mathbb{R}^n \times \mathbb{R}^n$ to $\operatorname{gph} N_C$ at (\bar{x}, \bar{z}) . The normals are the generalized ones of variational analysis [8] instead of just convex analysis [7], inasmuch as $\operatorname{gph} N_C$ typically won't be a convex subset of $\mathbb{R}^n \times \mathbb{R}^n$.

Because N_C is the subgradient mapping $\partial\delta_C$ associated with the indicator function δ_C , passing to its coderivatives amounts to a kind of second-order differentiation of δ_C . For this reason $D^*N_C(\bar{x}|\bar{z})$ has also been denoted by $\partial^2\delta_C(\bar{x}|\bar{z})$ and called a *generalized hessian* of δ_C at \bar{x} for \bar{z} .

Definition 2.3 (strong second-order condition for local optimality). *In the problem of minimizing $f(\bar{p}, x)$ over $x \in C$, the strong second-order condition for local optimality will be said to be satisfied at \bar{x} under the first-order condition $-\nabla_x f(\bar{p}, \bar{x}) \in N_C(\bar{x})$ if*

$$w \cdot \nabla_{xx}^2 f(\bar{p}, \bar{x})w + w \cdot u > 0 \quad \forall u \in D^*N_C(\bar{x}|\bar{z})(w) \text{ when } w \neq 0, \text{ where } \bar{z} = -\nabla_x f(\bar{p}, \bar{x}). \quad (2.6)$$

This condition (in which the parameterization in p doesn't really enter although we retain \bar{p} for consistency with what comes later) first appeared in [5], where it was shown to be the natural generalization of a more understandable condition for polyhedral C . That condition involves a subcone of the tangent cone $T_C(\bar{x})$ (which itself is polar to the normal cone $N_C(\bar{x})$).

Theorem 2.4 (polyhedral version of the strong second-order condition, [5, Theorem 4.5]). *When C is polyhedral, the strong second-order condition reduces to*

$$w \cdot \nabla_{xx}^2 f(\bar{p}, \bar{x})w > 0 \quad \forall \text{ nonzero } w \in \bar{K}(\bar{x}, \bar{z}) = K(\bar{x}, \bar{z}) - K(\bar{x}, \bar{z}) \text{ for } \bar{z} = -\nabla_x f(\bar{p}, \bar{x}), \quad (2.7)$$

where $K(\bar{x}, \bar{z})$ is the critical cone to C at \bar{x} defined by

$$K(\bar{x}, \bar{z}) = \{w \in T_C(\bar{x}) \mid w \perp \bar{z}\}. \quad (2.8)$$

The cone $K(\bar{x}, \bar{z})$ is polyhedral when C is polyhedral, in particular convex, and $\bar{K}(\bar{x}, \bar{z})$ is the smallest subspace containing $K(\bar{x}, \bar{z})$, the *critical subspace* to C at \bar{x} with respect to the normal \bar{z} .

For nonpolyhedral convex sets second-order optimality must take into account not only the hessian $\nabla_{xx}^2 f(\bar{p}, \bar{x})$ but also the ‘‘curvature’’ of the boundary of C . That is why a more complicated condition as in (2.6) has to be expected. Other reductions of (2.6) to something simpler ought to be possible in making use of a constraint structure for C , but this is a challenge not on the agenda here. It's worth mentioning anyway that

$$w \cdot u \geq 0 \text{ when } u \in D^*N_C(\bar{x}|\bar{z})(w). \quad (2.9)$$

Indeed, this property of nonnegativity holds always for coderivatives of maximal monotone mappings [5, Theorem 2.1], and N_C is maximal monotone because it's the subdifferential of the convex function δ_C (see for instance [8, 12.17]). We are tempted by this to define the *curvature indicator* of C at a point $x \in C$ with respect to a normal $z \in N_C(x)$ by

$$\omega_C(x, z)(w) = \inf\{w \cdot u \mid u \in D^*N_C(x|z)(w)\} \quad (2.10)$$

(with the understanding that this is ∞ when $D^*N_C(x|z)(w) = \emptyset$) and consider writing the second-order condition in the more appealing form

$$w \cdot \nabla_{xx}^2 f(\bar{p}, \bar{x})w + \omega_C(\bar{x}|\bar{z})(w) > 0 \text{ when } w \neq 0, \text{ where } \bar{z} = -\nabla_x f(\bar{p}, \bar{x}). \quad (2.11)$$

Certainly this property implies the one in Definition 2.3, but equivalence is unclear because the attainment of the infimum in (2.10) is elusive.²

The compelling reason for bringing the strong second-order condition for local optimality into the picture here is that it turns out to furnish an exact criterion for the availability of the stability properties in Theorem 2.2.

²The author thanks a referee for indicating that the equivalence is supported by [4, Corollary 4.5].

Theorem 2.5 (second-order criterion for strongly stable local optimality). *The strong second-order condition is both necessary and sufficient for the solution mapping \bar{S}_0 to provide strongly stable local optimality as in Theorem 2.2, and the same then for the solution mappings S^* and \bar{S}^* , thereby implying that property also for the mappings S and \bar{S} .*

Joint proof of Theorems 2.2 and 2.5. The argument will be based on results in the paper [5] and its follow-up [2]. The prime virtue of the strong second-order condition for local optimality is its role, revealed in [5], in characterizing tilt stability of the minimization problem behind Definition 2.3, as follows. Consider yet another solution mapping:

$$\tilde{S}_0(v) = \{x \mid v - \nabla f(\bar{p}, x) \in N_C(x)\}, \quad \text{with } (0, \bar{x}) \in \text{gph } \tilde{S}_0, \quad (2.12)$$

which associates with $v \in \mathbb{R}^n$ the points $x \in C$ that satisfy the first-order condition for local optimality in the minimization of $f(\bar{p}, x)$ over $x \in C$. (This differs from \bar{S}_0 by having the gradient of $f(\bar{p}, \cdot)$ in place of the gradient of the function \bar{f} in (2.2).) The concept of tilt stability in this situation, first explored in [5], is kin to the ideas already treated. It refers (in our notation that maintains \bar{p}) to \tilde{S}_0 being strongly metrically regular at $(0, \bar{x})$ and such that, with respect to the neighborhoods V of 0 and X of \bar{x} in that property, and the corresponding implicit function \tilde{s}_0 , one has

$$\tilde{s}_0(v) = \operatorname{argmin}_{x \in C \cap X} \{f(\bar{p}, x) - v \cdot x\} \quad \text{for every } v \in V. \quad (2.13)$$

According to [5, Theorem 4.2], the strong second-order condition is both necessary and sufficient for this. Obviously it is also then both necessary and sufficient for the corresponding property of \bar{S}_0 , since the hessian appearing in it is the same for the functions $f(\bar{p}, \cdot)$ and $\bar{f}(\bar{p}, \cdot)$. This falls short, though, of guaranteeing its necessity and sufficiency for the corresponding property of S^* , which is what we need in order to establish the two theorems. (The implication from S^* to S in Theorem 2.2 is elementary.)

We will reach our goal by specializing the main result in [2] to the situation at hand. That result is stated far more broadly in terms of minimizing (in adapted notation) an expression $\varphi(p, x) - v \cdot x$ over $x \in \mathbb{R}^n$, where φ is an extended-real-valued function on $\mathbb{R}^d \times \mathbb{R}^n$ that is lower semicontinuous and proper. The first-order condition for local optimality in that framework is $v \in \partial_x \varphi(p, x)$, with the subgradients being the general ones of variational analysis [8]. The solution mapping to look at then is the mapping M defined by

$$M(v, p) = \{x \mid v \in \partial_x \varphi(p, x)\}. \quad (2.14)$$

The issue at a given $(0, \bar{p}, \bar{x}) \in \text{gph } M$ is whether M will be strongly metrically regular and such that the associated implicit function furnishes locally optimal solutions in the manner of Definition 2.1. Conditions both necessary and sufficient for that are given in [2, Theorem 2.3] in terms of the coderivatives of M under some restrictions on φ . Here we are dealing with

$$\varphi(p, x) = f(p, x) + \delta_C(x), \quad \partial_x \varphi(p, x) = \nabla_x f(p, x) + N_C(x), \quad \text{hence } M(v, p) = S^*(v, p), \quad (2.15)$$

and our task is to work out the applicability and reduced formulation of the conditions in question for this setting.

To start with, we have to verify that the function φ in (2.15) for a closed convex set C and a \mathcal{C}^2 function f fulfills the requirement in [2] of being ‘‘continuously prox-regular.’’ That can be confirmed with the help of [2, Proposition 2.2] by demonstrating that φ is ‘‘strongly amenable in x at \bar{x} with compatible parameterization in p at \bar{p} .’’ This means that, in some neighborhood of (\bar{p}, \bar{x}) , there is a composite representation $\varphi(p, x) = \psi(G(p, x))$ in which $G : \mathbb{R}^d \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a \mathcal{C}^2 mapping and $\psi : \mathbb{R}^m \rightarrow (-\infty, \infty]$ is a convex, proper, lsc function such that $G(\bar{p}, \bar{x}) \in \text{dom } \psi$ and

$$z \in N_{\text{dom } \psi}(G(\bar{p}, \bar{x})), \quad \nabla_p G(\bar{p}, \bar{x})^* z = 0, \quad \nabla_x G(\bar{p}, \bar{x})^* z = 0 \implies z = 0. \quad (2.16)$$

We can satisfy these requirements by taking

$$G(p, x) = (f(p, x), x) \in \mathbb{R} \times \mathbb{R}^n, \quad \psi(t, x) = t + \delta_C(x),$$

so that $\text{dom } \psi = \mathbb{R} \times C$ and $N_{\text{dom } \psi}(t, x) = \{0\} \times N_C(x)$. Then having $z \in N_{\text{dom } \psi}(G(\bar{p}, \bar{x}))$ means $z = (0, v)$ for some $v \in N_C(\bar{x})$. Moreover $\nabla_p G(\bar{p}, \bar{x})^* = (\nabla_p f(\bar{p}, \bar{x}), 0)$ while $\nabla_x G(\bar{p}, \bar{x})^* = (\nabla_x f(\bar{p}, \bar{x}), I)$. The condition $\nabla_x G(\bar{p}, \bar{x})^* z = 0$ for $z = (0, v)$ already by itself implies $v = 0$ and makes z vanish.

With that confirmation out of the way, we turn to the key conditions of [2, Theorem 2.3] in terms of the coderivatives of M in (2.14)–(2.15), namely

- (a) $(y, u) \in D^*M(\bar{p}, \bar{x}|0)(w), \quad w \neq 0 \implies w \cdot u > 0,$
- (b) $(y, 0) \in D^*M(\bar{p}, \bar{x}|0)(0) \implies y = 0.$

Here we have because of (2.15), in terms of $\bar{z} = -\nabla_x f(\bar{p}, \bar{x})$, that

$$D^*M(\bar{p}, \bar{x}|0)(w) = (\nabla_{px}^2 f(\bar{p}, \bar{x})w, \nabla_{xx}^2 f(\bar{p}, \bar{x})w) + (0, D^*N_C(\bar{x}|\bar{z})(w)) \quad (2.17)$$

in accordance with the rule for calculating coderivatives of the sum of a smooth mapping and another mapping [8, 10.43]. From this it's evident that (b) holds. As for (a), we are looking at whether having

$$y = \nabla_{px}^2 f(\bar{p}, \bar{x})w, \quad z = \nabla_{xx}^2 f(\bar{p}, \bar{x})w + u \text{ for some } u \in D^*N_C(\bar{x}|\bar{z})(w),$$

with $w \neq 0$ implies that $w \cdot z > 0$. This comes down to the condition that $w \cdot \nabla_{xx}^2 f(\bar{p}, \bar{x})w + w \cdot u > 0$ for all $u \in D^*N_C(\bar{x}|\bar{z})(w)$, which is the second-order condition we were aiming at. \square

3 Application to stability of Nash equilibrium

The results in Theorems 2.2 and 2.5 have immediate consequences for the agents' minimization problems in the multi-optimization framework in Section 1. The expression to be minimized by agent k over $x_k \in C_k$ depends then on the actions also of the other agents, and we can imagine it further to depend on a parameter vector $p \in \mathbb{R}^d$. It then takes the form $f_k(p, x_1, \dots, x_N)$ in which the parameterization is not only by p but by x_{-k} , i.e., the vector obtained from (x_1, \dots, x_N) by deleting x_k . A variational Nash equilibrium for a given \bar{p} corresponds then to

$$\begin{aligned} -F(\bar{p}, \bar{x}) \in N_C(\bar{x}) \text{ for } \bar{x} = (\bar{x}_1, \dots, \bar{x}_N), \quad C = C_1 \times \dots \times C_N, \\ \text{where } F(\bar{p}, \bar{x}) = (\nabla_{x_1} f_1(\bar{p}, \bar{x}), \dots, \nabla_{x_N} f_N(\bar{p}, \bar{x})). \end{aligned} \quad (3.1)$$

The ‘‘linearized’’ version of this, which will also be important, has \bar{F} instead of F through the replacement of each f_k by its quadratic expansion \bar{f}_k :

$$\begin{aligned} \bar{f}_k(p, x) = f_k(\bar{p}, \bar{x}) + \sum_{j=1}^N \nabla_{x_j} f_k(\bar{p}, \bar{x})[x_j - \bar{x}_j] + \frac{1}{2} \sum_{k=1, j=1}^{N, N} [x_k - \bar{x}_k] \cdot \nabla_{x_k x_j}^2 f(\bar{p}, \bar{x})[x_j - \bar{x}_j] \\ + \nabla_p f_k(\bar{p}, \bar{x})[p - \bar{p}] + \sum_{j=1}^N [x_j - \bar{x}_j] \cdot \nabla_{x_k p}^2 f(\bar{p}, \bar{x})[p - \bar{p}] + \frac{1}{2} [p - \bar{p}] \nabla_{pp}^2 f(\bar{p}, \bar{x})[p - \bar{p}] \end{aligned} \quad (3.2)$$

and therefore

$$\bar{F}(p, x) = (\dots, \nabla_{x_k} f_k(\bar{p}, \bar{x}) + \nabla_{x_k p}^2 f_k(\bar{p}, \bar{x})[p - \bar{p}] + \sum_{j=1}^N \nabla_{x_k x_j}^2 f_k(\bar{p}, \bar{x})[x_j - \bar{x}_j], \dots). \quad (3.3)$$

On the way to analyzing the effects of perturbations to $p \neq \bar{p}$, we next introduce solution mappings for agent subproblems and summarize the facts available about them. The subproblem for agent k

depends parametrically not only on p but also on x_{-k} , so the basic solution mapping for that agent with respect to first-order optimality is given by

$$S_k(p, x_{-k}) = \{x_k \in C_k \mid -\nabla_{x_k} f_k(p, x_k, x_{-k}) \in N_{C_k}(x_k)\} \text{ with } (\bar{p}, \bar{x}_{-k}, \bar{x}_k) \in \text{gph } S_k \quad (3.4)$$

and the canonical solution mapping is given by

$$S_k^*(v_k, p, x_{-k}) = \{x_k \in C_k \mid v_k - \nabla_{x_k} f_k(p, x_k, x_{-k}) \in N_{C_k}(x_k)\} \text{ with } (0, \bar{p}, \bar{x}_{-k}, \bar{x}_k) \in \text{gph } S_k^*. \quad (3.5)$$

In terms of the quadratic expansions there are the associated mappings

$$\bar{S}_k(p, x_{-k}) = \{x_k \in C_k \mid -\nabla_{x_k} \bar{f}_k(p, x_k, x_{-k}) \in N_{C_k}(x_k)\} \text{ with } (\bar{p}, \bar{x}_{-k}, \bar{x}_k) \in \text{gph } \bar{S}_k \quad (3.6)$$

and

$$\bar{S}_k^*(v_k, p, x_{-k}) = \{x_k \in C_k \mid v_k - \nabla_{x_k} \bar{f}_k(p, x_k, x_{-k}) \in N_{C_k}(x_k)\} \text{ with } (0, \bar{p}, \bar{x}_{-k}, \bar{x}_k) \in \text{gph } \bar{S}_k^* \quad (3.7)$$

as well as the auxiliary solution mapping

$$\bar{S}_{0k}(v_k, x_{-k}) = \{x_k \in C_k \mid v_k - \nabla_{x_k} \bar{f}_k(\bar{p}, x_k, x_{-k}) \in N_{C_k}(x_k)\} \text{ with } (0, \bar{x}_k) \in \text{gph } \bar{S}_{0k}, \quad (3.8)$$

where in the last, as seen from (3.3), the gradient is simply

$$\nabla_{x_k} \bar{f}_k(\bar{p}, x_k, x_{-k}) = \nabla_{x_k} f_k(\bar{p}, \bar{x}) + \sum_{j=1}^N \nabla_{x_k x_j}^2 f_k(\bar{p}, \bar{x}) [x_j - \bar{x}_j]. \quad (3.9)$$

For each of these we can also raise the issue of strongly stable local optimality in the associated minimization problems and have at our disposal the facts in Theorems 1.3, 2.2 and 2.5.

Forgoing the exercise of laying out those details, we proceed toward articulating similar facts for analogous solution mappings associated with parameterized Nash equilibrium. The *basic equilibrium mapping* E in this setting is

$$E(p) = \{x = (x_1, \dots, x_N) \mid x_k \in S_k(p, x_{-k}), \forall k\}, \quad (3.10)$$

and the *canonical equilibrium mapping* E^* involving $v = (v_1, \dots, v_N)$ is

$$E^*(v, p) = \{x = (x_1, \dots, x_N) \mid x_k \in S_k^*(v_k, p, x_{-k}), \forall k\}. \quad (3.11)$$

The corresponding “linearized” versions are given by

$$\bar{E}(p) = \{x = (x_1, \dots, x_N) \mid x_k \in \bar{S}_k(p, x_{-k}), \forall k\} \quad (3.12)$$

and

$$\bar{E}^*(v, p) = \{x = (x_1, \dots, x_N) \mid x_k \in \bar{S}_k^*(v_k, p, x_{-k}), \forall k\}, \quad (3.13)$$

while the *auxiliary equilibrium mapping* only has

$$\bar{E}_0(v) = \{x = (x_1, \dots, x_N) \mid x_k \in \bar{S}_{0k}(v_k, x_{-k}), \forall k\}. \quad (3.14)$$

Although these definitions relate so far only to *variational* Nash equilibrium, our results on local optimality will be able to take us beyond that to classical Nash equilibrium in a local sense. First, though, we record the facts about strong metric regularity in which the equilibrium mappings just defined yield implicit functions to be denoted respectively by e , e^* , \bar{e} , \bar{e}^* , and \bar{e}_0 .

Theorem 3.1 (implicit functions for variational Nash equilibrium). *The equilibrium mapping E^* is strongly metrically regular at $(0, \bar{p}, \bar{x})$ if and only if the equilibrium mapping \bar{E}^* is strongly metrically regular at $(0, \bar{p}, \bar{x})$, in which case the implicit function \bar{e}^* serves as a first-order approximation to the implicit function e^* :*

$$e^*(v, p) = \bar{e}^*(v, p) + o(v, p - \bar{p}). \quad (3.15)$$

Moreover the strong metric regularity of the auxiliary equilibrium mapping \bar{E}_0 at $(0, \bar{x})$ is necessary and sufficient for this equivalence, in consequence of which the equilibrium mappings E and \bar{E} are strongly metrically regular at (\bar{p}, \bar{x}) with the implicit function \bar{e} serving as a first-order approximation to the implicit function e :

$$e(p) = \bar{e}(p) + o(p - \bar{p}), \quad \text{where furthermore } \bar{e}(p) = \bar{e}_0(-\nabla_p F(\bar{p}, \bar{x})[p - \bar{p}]). \quad (3.16)$$

Proof. This comes directly out of Theorem 1.3 as specialized to the notation of the variational inequality (3.1) and its linearization. \square

In elevating these properties to encompass local optimality in the agents' subproblems, we will not be dealing with classical Nash equilibrium but a local form of it.

Definition 3.2 (local Nash equilibrium). *A Nash equilibrium in the local sense is a Nash equilibrium $(\bar{x}_1, \dots, \bar{x}_N)$ with respect to sets $C_k \cap X_k$, where X_k is a neighborhood of \bar{x}_k (which without loss of generality can be taken to be closed and convex).*

A local Nash equilibrium can also be considered of course with respect to the replacement of the functions f_k by their quadratic expansions \bar{f}_k .

Definition 3.3 (strong stability of local Nash equilibrium). *The equilibrium mapping E provides at $(\bar{p}, \bar{x}) \in \text{gph } E$ a local Nash equilibrium that is strongly stable if strong metric regularity is present with the neighborhoods P and $X = X_1 \times \dots \times X_N$ in that property being such that local optimality holds in the agents' subproblems:*

$$x = e(p) \implies x_k = \underset{u_k \in C_k \cap X_k}{\operatorname{argmin}} f_k(p, u_k, x_{-k}) \quad \forall k \text{ when } p \in P. \quad (3.17)$$

For \bar{E} the property added to strong metric regularity is

$$x = \bar{e}(p) \implies x_k = \underset{u_k \in C_k \cap X_k}{\operatorname{argmin}} \{ \bar{f}_k(p, u_k, x_{-k}) \} \quad \forall k \text{ when } p \in P. \quad (3.18)$$

For E^* at $(0, \bar{p}, \bar{x})$ a neighborhood V of 0 also comes in and the extra property is

$$x = e^*(v, p) \implies x_k = \underset{u_k \in C_k \cap X_k}{\operatorname{argmin}} \{ f_k(p, u_k, x_{-k}) - v_k \cdot u_k \} \quad \forall k \text{ when } (v, p) \in V \times P, \quad (3.19)$$

and similarly for \bar{E}^* :

$$x = \bar{e}^*(v, p) \implies x_k = \underset{u_k \in C_k \cap X_k}{\operatorname{argmin}} \{ \bar{f}_k(p, u_k, x_{-k}) - v_k \cdot u_k \} \quad \forall k \text{ when } (v, p) \in V \times P. \quad (3.20)$$

For the auxiliary equilibrium mapping \bar{E}_0 at $(0, \bar{x})$ it takes the form that

$$x = \bar{e}_0(v) \implies x_k = \underset{u_k \in C_k \cap X_k}{\operatorname{argmin}} \{ \bar{f}_k(\bar{p}, u_k, x_{-k}) - v_k \cdot u_k \} \quad \forall k \text{ when } v \in V. \quad (3.21)$$

Theorem 3.4 (implicit functions for local Nash equilibrium). *The mapping E^* provides at $(0, \bar{p}, \bar{x})$ a local Nash equilibrium that is strongly stable if and only if the same holds for the mapping \bar{E}^* , which in turn holds if and only if the mapping \bar{E}_0 provides a local Nash equilibrium that is strongly stable. Then E and \bar{E} have this property at (\bar{p}, \bar{x}) as well.*

Proof. This proceeds from Theorem 3.1 by invoking the facts in Theorem 2.2 with respect to the solutions mappings in (3.4)–(3.8). \square

It would be nice to go on next to establishing a “pointwise” criterion for the properties in Theorem 3.4 that builds on Theorem 2.5, but the research in that direction is incomplete. What we can offer at least is a good conjecture in the case of polyhedral sets C_k .

The mapping \bar{E}_0 in (3.8)–(3.9) for which the strong stability is the key in Theorem 3.4 assigns to $v = (v_1, \dots, v_N)$ the set of all $x = (x_1, \dots, x_N)$ such that

$$v_k - \left(\nabla_{x_k} f_k(\bar{p}, x) + \sum_{j=1}^N \nabla_{x_k x_j}^2 f_k(\bar{p}, \bar{x}) [x_j - \bar{x}_j] \right) \in N_{C_k}(x_k). \quad (3.22)$$

The strong second-order optimality conditions that pair up with these first-order conditions in the polyhedral case have (from Theorem 2.4) the form

$$\begin{aligned} w_k \cdot \nabla_{x_k x_k}^2 f_k(\bar{p}, \bar{x}) w_k > 0 \quad \forall \text{ nonzero } w_k \in \bar{K}_k(\bar{x}_k, \bar{z}_k), \text{ where } \bar{z}_k = -\nabla_{x_k} f_k(\bar{p}, \bar{x}) \text{ and} \\ \bar{K}_k(\bar{x}_k, \bar{z}_k) = K_k(\bar{x}_k, \bar{z}_k) - K_k(\bar{x}_k, \bar{z}_k) \text{ for } K_k(\bar{x}_k, \bar{z}_k) = \{ w_k \in T_{C_k}(\bar{x}_k) \mid w_k \perp \bar{z}_k \}. \end{aligned} \quad (3.23)$$

It might be hoped that the combination of these first- and second-order conditions would provide the desired criterion, but that’s not enough. The need from something more can be gleaned from the example of $C_k = \mathbb{R}^{n_k}$ for all k , which makes $N_{C_k}(x_k) = \{0\}$, thus reduces (3.22) to the equation

$$v_k = \nabla_{x_k} f_k(\bar{p}, x) + \sum_{j=1}^N \nabla_{x_k x_j}^2 f_k(\bar{p}, \bar{x}) [x_j - \bar{x}_j] \quad (3.24)$$

and (3.23) to the positive definiteness of the hessian $\nabla_{x_k x_k}^2 f_k(\bar{p}, \bar{x})$. That positive definiteness is inadequate to ensure that the system of equations (3.24) for $k = 1, \dots, N$ has a unique solution, i.e., the \bar{E}_0 mapping in this case might fail to yield single-valuedness. Some nonsingularity property of

$$H = \left[\nabla_{x_k x_j}^2 f_k(\bar{p}, \bar{x}) \right]_{k=1, j=1}^{N, N} \quad (3.25)$$

seems essential. Anyway, the right condition for the polyhedral case beyond this simple instance may very well be as follows.

Conjecture 3.5 (criterion for strongly stable equilibrium in the polyhedral case). *When the sets C_k are polyhedral, a condition both necessary and sufficient for the property in Theorem 3.4 to hold for \bar{E}_0 , with the indicated consequences for the other equilibrium mappings, may be the combination of the strong second-order optimality conditions (3.23) with the nonsingularity of H in (3.25) relative to the subspace*

$$\bar{K} = \{ w = (w_1, \dots, w_N) \mid w_k \in \bar{K}_k(\bar{x}_k, \bar{z}_k) \} \quad (3.26)$$

in the sense that

$$w \in \bar{K}, \quad Hw \perp \bar{K} \implies w = 0. \quad (3.27)$$

This is suggested by other known results, such as Theorem 2E.6 and Lemma 4H.3 of [1]. Further facts that could come into the picture have been developed in [3].

References

- [1] DONTCHEV, A. D., AND ROCKAFELLAR, R. T., *Implicit Functions and Solution Mappings*, second edition: Springer Verlag, 2014.
- [2] LEVY, A. B., POLIQUIN, R. A., AND ROCKAFELLAR, R. T., “Stability of locally optimal solutions,” *SIAM J. Optimization* **10** (2000), 580–604.
- [3] MORDUKHOVICH, B. M., AND NGHIA, T. T. A., “Local monotonicity and full stability for parametric variational systems,” *SIAM J. Optimization* **23** (2016), 1023–1059.
- [4] MORDUKHOVICH, B. M., NGHIA, T. T. A., AND ROCKAFELLAR, R. T., “Full stability in finite-dimensional optimization,” *Math. of Operations Research* **40** (2015), 226–252
- [5] POLIQUIN, R. A., AND ROCKAFELLAR, R. T., “Tilt stability of a local minimum,” *SIAM J. Optimization* **8** (1998), 287–299.
- [6] ROBINSON, S. M., “Strongly regular generalized equations,” *Math. of Operations Research* **5** (1980), 43–62.
- [7] ROCKAFELLAR, R. T., *Convex Analysis*, Princeton University Press, 1970.
- [8] ROCKAFELLAR, R. T., AND WETS, R. J-B, *Variational Analysis*, No. 317 in the series *Grundlehren der Mathematischen Wissenschaften*, Springer-Verlag, 1998.