Importance Sampling in the Evaluation and Optimization of Buffered Failure Probability

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ABSTRACT: Engineering design is a process in which a system's parameters are selected such that the system meets certain criteria. These criteria vary in nature and may involve such matters as structural strength, implementation cost, architectural considerations, etc. When random variables are part of a system model, an added criterion is usually the failure probability. In this paper, we examine the buffered failure probability as an attractive alternative to the failure probability in design optimization problems. The buffered failure probability is more conservative and possesses properties that make it more convenient to compute and optimize. Since a failure event usually occurs with small probability in structural systems, Monte-Carlo sampling methods require large sample sizes for high accuracy estimates of failure and buffered failure probabilities. We examine importance sampling techniques for efficient evaluation of buffered failure probabilities, and illustrate their use in structural design of two multi-story frames subject to ground motion. We formulate a problem of design optimization as a cost minimization problem subject to buffered failure probability constraints. The problem is solved using importance sampling and a nonlinear optimization algorithm.

Uncertainty in loads, material properties and other parameters need to be accounted for in designing modern engineering systems. For that purpose, the choice of how reliability is quantified plays an important role when assessing the feasibility of a certain choice of design parameters. One such quantification is the *failure probability*, see, e.g. Ditlevsen and Madsen (1996). The *buffered failure probability* is an alternative presented in Rockafellar and Royset (2010); see also Rockafellar and Royset (2015), which offers computational and practical benefits over the former choice. Design optimization problems with buffered failure probability constraints are in some sense no harder to solve than the underlying deterministic design optimization problems. In particular, if a deterministic design optimization problem is convex, the corresponding stochastic one with buffered failure probability constraints is also convex. This situation is dramatically different than that for failure probability constraints, which involves significant added complications when passing from the deterministic to the stochastic problem. Moreover, the buffered failure probability captures tail behavior more comprehensively than the failure probability and, in fact, incorporates the degree of failure to some extent; see Rockafellar and Royset (2010) for a discussion.

Analytical computation of the buffered failure probability is usually not possible. Consequently, numerical sampling techniques are usually used; namely Monte-Carlo Sampling (MCS) methods. However, since failure events may occur with low probability, MCS methods require large sample sizes to estimate the value of both failure and buffered failure probabilities, and therefore also long computing times.

Importance Sampling (IS) is a method that can improve accuracy and require fewer sample points to estimate (buffered) failure probabilities. This paper uses buffered failure probability as a quantification of reliability, describes a method of evaluating buffered failure probability constraints by IS, and incorporates the method into the design of a structural system. An example involving two multi-story structural frames subject to ground motion is studied in detail.

1. IMPORTANCE SAMPLING

In this section we explain the concept of IS.

1.1. Definition

Given a random variable Y with probability density function f, the expected value of Y is defined as

$$E[Y] = \int_{-\infty}^{\infty} y f(y) dy.$$

Let $y_1, y_2, \dots, y_n$ be realizations of Y, independently sampled according to *f*. Then, the expectation of the random variable Y can be approximated by

$$E[Y] \approx \frac{1}{n} \sum_{i=1}^{n} y_i$$

Multiplying Eq. (1) by $\frac{h(y)}{h(y)}$, where h is a probability density that is zero only when f is zero, we obtain that

$$E[Y] = \int_{-\infty}^{\infty} \frac{y f(y)}{h(y)} h(y) dy = E\left[\frac{V f(V)}{h(V)}\right]$$

where now V is distributed according to h. Thus with $v_1, v_2, \dots, v_n$ sampled from h we find that

$$E[Y] \approx \frac{1}{n} \sum_{i=1}^{n} \frac{v_i f(v_i)}{h(v_i)}$$

Appropriate choices of sampling density h may result in an increased efficiency in estimating the expectation; see for example Asmussen and Glynn (2007).

2. BUFFERED FAILURE PROBABILITY

Failure of a structural component is defined by means of a *limit-state function* $g(\mathbf{x}, \mathbf{v})$**,** where $\mathbf{x}$ is a vector of design variables, and $\mathbf{v}$ is a vector of quantities that represent uncertain parameters of the system (e.g. loads and material strength), see Rockafellar and Royset (2010). It is common that the vector of uncertain parameters is modeled as a vector of random variables $\mathbf{V} = (V_1, V_2, \dots, V_m)$ described by a given or estimated joint probability density function. We henceforth denote all random variables using upper case letters and their realizations with lower case ones. For a realization $\mathbf{v}$ of $\mathbf{V}$, and a choice of design variables $\mathbf{x}$, the system is assumed to be in an unsatisfactory state, i.e., in failure, when $g(\mathbf{x}, \mathbf{v})$ is strictly positive. The event of failure is thus described by $\{g(\mathbf{x}, \mathbf{V}) > 0\}$. We note that $g(\mathbf{x}, \mathbf{V})$ is a random variable for any fixed design $\mathbf{x}$.

In this section we define the buffered failure probability and describe methods for assessing whether it is sufficiently small.

2.1. Definition

For a fixed $\mathbf{x}$ and a given probability level α, we recall that the α-quantile of $g(\mathbf{x}, \mathbf{V})$, denoted by $q_\alpha(\mathbf{x})$, is the smallest scalar q such that

$$P(g(\boldsymbol{x}, \boldsymbol{V}) \leq q) \geq \alpha$$

where P is the probability measure. We define the α-superquantile as

$$\bar{q}_\alpha(\mathbf{x}) = \frac{1}{1-\alpha} \int_{\alpha}^{1} q_\beta(\mathbf{x}) d\beta$$

It is clear that a superquantile is a normalized average of quantiles for a range of probability levels**.**

If the cumulative distribution function of $g(\mathbf{x},\mathbf{V})$ is continuous at $q_\alpha(\mathbf{x})$, then a superquantile is equivalently expressed by

$$\bar{q}_\alpha(\mathbf{x}) = E[g(\mathbf{x},\mathbf{V}) | g(\mathbf{x},\mathbf{V}) \geq q_\alpha(\mathbf{x})\,].$$

It is clear that in this case, a superquantile is the expected value of $g(\mathbf{x},\mathbf{V})$, conditioned on $g(\mathbf{x},\mathbf{V})$ being greater or equal to the α-quantile. Regardless of the cumulative distribution function of $g(\mathbf{x},\mathbf{V})$, a superquantile can also be expressed as the optimal value of a minimization problem (see Rockafellar and Royset, 2010), i.e.,

$$\bar{q}_\alpha(x) = \min_y \{y + E[\max\{g(\mathbf{x},\mathbf{V}) - y, 0\}]\} \tag{1}$$

Computing a superquantile thus involves finding a scalar *y* that minimizes a convex function given in terms of an expectation, which is straightforward when the distribution of **V** is known and the limit-state function can be evaluated relatively easily.

If $g(\mathbf{x},\mathbf{V})$ follows a discrete distribution with realizations $y_1 < y_2 < \cdots < y_n$ and corresponding probabilities $p_1, p_2, \ldots, p_n$, a superquantile obeys the expressions

$$\bar{q}_\alpha(\mathbf{x}) = \sum_{j=1}^{n} p_j y_j \quad \text{if } \alpha = 0$$

$$\bar{q}_\alpha(\boldsymbol{x}) = \frac{1}{1-\alpha}\left[\left(\sum_{j=1}^{i} p_j - \alpha\right) y_i + \sum_{j=i+1}^{n} p_j y_j\right]$$
$$\text{if } \sum_{j=1}^{i-1} p_j < \alpha \leq \sum_{j=1}^{i} p_j < 1$$

and

$$\bar{q}_\alpha(\mathbf{x}) = y_n \text{ if } \alpha > 1 - p_n$$

The buffered failure probability $\bar{p}(\mathbf{x})$ is defined in terms of a superquantile. Specifically,

$$\bar{p}(\mathbf{x}) = 1 - \alpha$$

where α is the probability level such that $\bar{q}_\alpha(\mathbf{x}) = 0$. Consequently, the buffered failure probability constraint

$$\bar{p}(\mathbf{x}) \leq 1 - \alpha$$

is satisfied if and only if

$$\bar{q}_\alpha(\mathbf{x}) \leq 0.$$

This equivalence will be utilized when formulating design optimization problems below.

2.2. Superquantile estimates using IS

In this subsection we discuss sampling techniques for estimating superquantiles for a given probability level. Moreover, we present a method to compute an estimate of an upper bound of a superquantile and build a confidence interval around it. For probability levels close to 1, the event $\{g(\boldsymbol{x},\boldsymbol{V}) > y\}$ appearing in Eq. (1) occurs with small probabilities for typical values of *y*, and the use of IS may become beneficial.

After sampling n' realizations $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n'}$ from a suitable density h, an estimate for the superquantile becomes

$$\bar{q}_\alpha(\mathbf{x}) \approx \min_y \; y + \frac{1}{n'}\frac{1}{1-\alpha}\sum_{i=1}^{n'} \frac{\max(g(\mathbf{x},\mathbf{v}_i) - y, 0)\, f(\mathbf{v}_i)}{h(\mathbf{v}_i)} \tag{2}$$

which is easily computed using a linear programming algorithm or specialized procedures.

Let y^* denote a scalar that minimizes the expression in Eq. (2). Next we obtain n'' sample point from h**,** namely $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n''}$, and define

$$u_i^{n'} = y^* + \frac{1}{1-\alpha}\frac{\max(g(\mathbf{x},\mathbf{v}_i) - y^*, 0)\, f(\mathbf{v}_i)}{h(\mathbf{v}_i)}$$

Since the expectation of this expression is an upper bound on $\bar{q}_\alpha(\mathbf{x})$, due to the fact that y^* might not minimize the right-hand side in Eq. (1), it can be seen that

$$U^{n'} = \frac{1}{n''}\sum_{i=1}^{n''} u_i^{n'}$$

is an approximate upper bound on $\bar{q}_\alpha(\mathbf{x})$. For large n'' and an independent sample $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n''}$, it is clear that $U^{n'}$ is approximately normal, making the computation of confidence interval straightforward.

With sample variance

$$\sigma^2 \approx \frac{1}{n''-1}\sum_{i=1}^{n''} ({u_i}^{n'} - U^{n'})^2$$

we obtain, for example, an approximate 95% confidence interval for an upper bound on the α-superquantile as $[U^{n'} - 1.96\sigma, U^{n'} + 1.96\sigma]$.

3. SYSTEM DESIGN

In this section we present a mathematical formulation of a design problem incorporating uncertainty and also give an algorithm for designing a system as well as assessing the resulting design.

Given a system described by a limit-state function g, we seek a design $\mathbf{x}$ that minimize a cost function C, which typically depends on the design. Moreover, we would like the buffered failure probability of the design to be no larger than $1-\alpha$. Consequently, we seek to solve the optimization problem

$$\min_{x\in X} C(\boldsymbol{x})\ \ s.t.$$
$$\bar{p}(\mathbf{x}) \le 1-\alpha$$

where X is subset of the space of design parameters considered admissible. For example, X could incorporate bounds on the design parameters that ensure a practical design. In view of the equivalence between buffered failure probability constraints and superquantile constraints, we find that this problem is identical to

$$\min_{x\in X} C(\boldsymbol{x})\ \ s.t.$$
$$\bar{q}_\alpha(\mathbf{x}) \le 0$$

Moreover, from Eq. (1), this simplifies to the problem

$$\min_{x\in X} C(\boldsymbol{x})\ \ s.t.$$
$$z_0 + \frac{1}{1-\alpha}E[\max\{g(\boldsymbol{x},\boldsymbol{V}) - z_0, 0\}] \le 0$$

where z_0 is an auxiliary variable corresponding to y in Eq. (1). Relying on IS and the sample $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ from h, an approximation of this problem takes the form

$$\min_{x\in X} C(\boldsymbol{x})\ \ s.t.$$
$$z_0 + \frac{1}{n(1-\alpha)}\sum_{j=1}^{n} \max\{g(\mathbf{x},\mathbf{v}_j) - z_0, 0\}\frac{f(\mathbf{v}_j)}{h(\mathbf{v}_j)} \le 0$$

This problem can be rewritten as

$$\min_{x\in X} C(\boldsymbol{x})\ \ s.t.$$
$$z_0 + \frac{1}{n(1-\alpha)}\sum_{j=1}^{n} \frac{z_j f(\mathbf{v}_j)}{h(\mathbf{v}_j)} \le 0 \qquad (3)$$
$$g(\mathbf{x},\mathbf{v}_j) - z_0 \le z_j \quad j = 1,2,\dots.,n$$
$$z_j \ge 0\ \ j = 1,2,\dots,n\ .$$

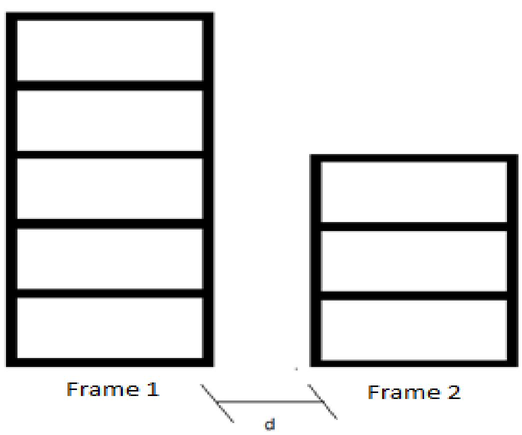
Here, $z_1, \dots, z_n$ are auxiliary optimization variables; see Rockafellar and Royset (2010).

The design and assessment process can be summarized as follows:

- Select a suitable sampling density h.
- Sample independently from h and obtain the realizations $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.
- Solve Eq. (3) using an optimization solver to obtain a solution $\hat{\boldsymbol{x}}$.
- Sample independently from h and obtain realizations $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n'}$, where n' does not need to be particularly large.

- Solve for y in Eq. (2) and obtain y^*.
- Sample independently from h and obtain $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n''}$. Since these realizations are only going to be used to obtain average values, and not to be used in any optimization formulation, n'' can be large.
- Obtain the upper bound estimate $U^{n'}$ and build a confidence interval around it as described in subsection 2.2.

Figure 1: *Structures subject to ground motion and consequently the risk of pounding.*



4. EXAMPLE DESIGN PROBLEM

In this section we present an example concerned with the design of two multi-story structures subject to random ground motion. The obtained design is assessed using IS.

4.1. Problem Definition

We consider two multi-story, single-bay frames with the same story height, separated by a distance d of 20 inches, which are both subject to an unknown dynamic load. The structures have five and three stories, and the lateral inertial of each story is 0.25 kip.s^2/in for both frames; see Figure 1. We assume that all columns in a frame have the same stiffness, and that all beams are rigid. The goal is to design the frames by choosing values for the column stiffness for each frame such that the buffered failure probability of pounding between the two frames is no more than 0.1.

4.1.1. Stochastic Load

The system is subject to a ground acceleration assumed to be white noise of duration 20s with uncertain stationary amplitude. The amplitude is modeled by a lognormal distribution with mean 0.4 G and standard deviation 0.05 G. We adopt a time discretization with timestep of 0.02 seconds. The random vector **V** therefore comprise 1000 independent lognormal random variables of this kind. The assumed load is a simple model of an earthquake ground motion here used for illustration.

4.1.2. Limit State Function

We let $\mathbf{x} = (k_1, k_2)$ be the vector of design parameters, with k_1 representing the stiffness of each column in each story of the leftmost (taller) frame in Figure 1 and k_2 representing those of the right frame.

With $u_i(t,m)$ being the displacement of the mth story of Frame i at time t (with displacements to the right taken as positive), we define the limit-state function

$$g(\boldsymbol{x}, \boldsymbol{v}) = -\min_{t,m=1,2,3}\{\, d + u_2(t,m) - u_1(t,m)\}$$

We use central differences to solve the dynamic system with time step 0.02 and ignore the inaccuracies this introduces in the evaluation of displacements. Likewise, the minimization over time is taken over the 1000 discretized time points.

4.1.3. Cost Function

Since it is reasonable to assume that construction cost is proportional to stiffness, we adopt the cost function

$$C(\boldsymbol{x}) = 5k_1 + 3k_2.$$

4.1.4. Computational Setup

As sampling distribution in IS, we use a lognormal distribution of mean 0.488 G, and standard deviation 0.05 G. The design process discussed in Section 3 is implemented in MATLAB, using “fmincon” to solve all optimization problems. Following the notation established in Section 3, we let n, n', and n'' be 40, 40, and 500 respectively. Computations are carried out on a laptop with 8.00 GB RAM and an Intel® Core™ i7-4700MQ CPU @2.40GHz.

4.2. Design Results

We solve Eq. (3) with IS and $n = 40$, which takes approximately 25 minutes of fmincon solver time, including repeated evaluations of the dynamical system. We obtain a column stiffness of 57.44 kips/in for Frame 1, and 105.93 kips/in for Frame 2.

4.3. Design Assessment Results

Although the obtained design satisfies all constraints in Eq. (3), the sample of only $n = 40$ results in an approximation of the buffered failure probability. We next make an assessment of the actual buffered failure probability. Using the explicit formula for the case of discrete distributions and a sample size of around 1.5 million, we obtain a 0.9-superquantile of -0.71, which is essentially exact. This shows that the 0.9-superquantile is below zero and, consequently, the buffered failure probability is below the required 0.1. However, such a large sample size is excessively costly in practice and we would like to reach the same conclusion using a much smaller sample size. Using a sample size $n' = 40$ and $n'' = 500$, we obtain an approximate 95% confidence interval of an upper bound of the 0.9-superquantile as

$$[-0.078, 0.039].$$

Ideally, we would have liked to have a design with a confidence interval that is entirely below zero. However, one cannot expect to achieve this for small sample sizes due to sampling error. In fact, if the sample sizes n' and n'' had been increased, then such a conclusion would be obtained.

5. CONCLUSIONS

We have presented an algorithm for the design and assessment of a system with stochastic parameters. The approach relies on the buffered failure probability to quantify reliability, which facilitates solution of the resulting design optimization problem using standard optimization solvers. Importance sampling provides hope that such design optimization can be carried with a relatively small sample size. In the design of two structural frames subject to ground motion, we find that a feasible design is obtained with as little as 40 samples. Subsequent assessment of that design using 540 samples provides near certainty that the design satisfies the required reliability level.

ACKNOWLEDGEMENT

This work is supported in part by AFOSR. Additional support for the third author is obtained from DARPA.

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