CONVEX ANALYSIS AND FINANCIAL EQUILIBRIUM

A. Jofré
Center for Mathematical Modeling and DIM, Univ. of Chile
Casilla 170/3, Correo 3, Santiago, Chile (ajofre@dim.uchile.cl)

R. T. Rockafellar
Dept. of Mathematics, Univ. of Washington, Seattle, WA 98195-4350 (rtr@uw.edu)

R. J-B Wets
Dept. of Mathematics, Univ. of California, Davis, CA 95616 (rjbwets@ucdavis.edu)

Abstract

Convexity has long had an important role in economic theory, but some recent developments have featured it all the more in problems of equilibrium. Here the tools of convex analysis are applied to a basic model of incomplete financial markets in which assets are traded and money can be lent or borrowed between the present and future. The existence of an equilibrium is established with techniques that include bounds derived from the duals to problems of utility maximization. Composite variational inequalities furnish the modeling platform. Models with and without short-selling are handled, moreover in the absence of any requirement that agents must initially have a positive amount of every asset, as is typical in equilibrium work in economics.

Keywords: *financial equilibrium, incomplete markets, variational inequalities, convex analysis*

13 March 2013

1 Introduction

Microeconomics centers largely on models of production and consumption by "agents" who optimize in accordance with their preferences. It further puts such optimization into models of equilibrium in which groups of agents compete for resources in markets governed by prices. A key question then is whether prices exist that can bring the supply and demand for resources into balance.

Convex analysis, with its close ties to optimization, is a natural tool in this subject and has been employed since the early days of rigorous modern theory [2], [3]. In the landmark 1954 contribution of Arrow and Debreu [2], for example, consumers maximize quasi-concave utility functions over convex sets of goods vectors in a nonnegative orthant, subject to budget constraints dictated by their initial holdings of goods, while producers maximize the value of output minus the cost of input over convex technology sets of input-output pairs of goods vectors. Markets provide opportunities for the buying and selling of goods and establishing their value as part of the optimization. Conditions are furnished which ensure that an equilibrium can be reached in this setting. The equilibrium is not itself a direct result of optimization, though. Its existence requires a fixed-point argument.

Arrow-Debreu equilibrium takes hold in a single period of time, without past or future, and with full information. A much bigger challenge lies in properly formulating, and establishing, an equilibrium when decisions made in the present play out only in the future and require planning for uncertainties in that future. There may be no agreement on the probabilities of various future states, but an agent's preferences can nonetheless subjectively incorporate beliefs and appraisals of the risks in them.

In this article we develop an equilibrium model in which the agents are "financial entities" concerned only with the amounts of money they will be able to secure for use in the present and future. Utility functions support comparisons that can be the basis of optimization. In optimizing, agents compete in putting together portfolios comprised of assets that are available in fixed supply in financial markets. They furthermore can borrow or lend money, or for that matter, just set it aside in the present for use in the future. However, even with borrowing and lending, the portfolios that can be created may fall short of covering all possible patterns of payment in the future states, and in that sense the market would be incomplete. The goal is to demonstrate the existence of asset prices and a rate of return for lending (and borrowing as its reverse) under which the optimization problems solved independently by the agents achieve an overall equilibrium.

Arrow [1] in 1953 studied "optimality" in financial markets revolving around assets that pay money money amounts in various future states, as here, and with agents partaking of a fixed supply of money in the present. However, in the mainstream financial market literature steming from Radner [14] in 1972, the agents are consumers of goods who plan for the future by writing contacts with each other about the future delivery of such goods, with payment in so-called units of account. Trading in existing stocks and bonds is not covered, and even money as such, is absent (not being admitted as a "good"). There is no way then to compare the value of goods in the present with the value of goods in the future.

Furthermore, instead of convex analysis, the methodology of differential topology was employed in that economics work, as inspired by the tactics of Debreu [4]. Such methodology leads largely to generic results and could not cope with the ability of an agent to hold nonnegative amounts of various assets in a portfolio, rather than strictly positive amounts. Specifically, the boundaries of nonnegative orthants, familiar and welcome to analysis in optimization, are troublesome for the kind of mathematics that relies always on smooothness. For expositions of this line of research, see Geanakoplos [7], Hens [8], and Magill and Shafer [13]. Here, our approach to financial equilibrium draws instead on ideas from a broader effort we have recently made in [11] in order to address these shortcomings, but it focuses on investment and exhibits novel features.

On a different track, earlier but likewise with reliance on convex analysis, financial equilibrium in markets for trading assets was studied in 2007 by Rockafellar, Uryasev and Zabarankin [16]. Nonnegativity of holdings was handled, but probabilities of future states were assumed known, and the preferences of investors were derived from the means and deviations (possibly nonstandard) of random pay-offs.

Before our involvement with financial markets, we emphasized convex analysis also in our work on single-stage models of equilibrium in consumption and production [9], [10]. Those models, as here and in [11], made strong use of something further which is very close to convex analysis, yet unknown in most of economics, namely variational inequalities.

Along with contributing new results and insights in the modeling of financial markets, one of our major aims here, on the side, is explaining to mathematicians this wide, and potentially much larger, role of convex analysis and its variational analysis extensions in applications to economics.

2 The Equilibrium Model

The future is represented by a finite set of states $s = 1, \ldots, S$. The present is represented by the state $s = 0$. Agent i for $i = 1, \ldots, I$ has utility $u_i(m_{i0}, m_{i1}, \ldots, m_{iS})$ with respect to having money amounts m_{is} available in the states $s = 0, 1, \ldots, S$. What will be done with that money is not our concern. Perhaps it will be spent on consumption in goods markets that are not part of this model. The agent might, for example, be arranging for retirement in an uncertain future while maintaining adequate funds for the present. Another interpretation could be that the agent is a financial firm making plans for pay-outs to its owners.

Agent i has an initial money amount $m^0_{i0} > 0$ and will get inputs $m^0_{is} > 0$ also in the future, but can also take actions to supplement or redistribute these funds. One kind of action involves portfolios and another is borrowing or lending. There is a financial market in the buying and selling of assets, which can be held in fractional amounts. The assets could be stocks and bonds or just shares in a future cash stream coming from a project. Asset j for $j = 1, \ldots, J$ yields the money amounts $a_{js} \geq 0$ in the future states $s = 1, \ldots, S$, with $a_{js} > 0$ for at least one s.

Initially, agent i possesses the amount $x^0_{ij} \geq 0$ of asset j but can trade it for a different amount $x_{ij} \geq 0$ subject to budget constraints formulated below. The price p_j at which asset j is traded will be determined by equilibrium considerations. An agent thus puts together a portfolio that costs $\sum_{j=1}^J x_{ij} p_j$ and yields the money amounts $\sum_{j=1}^J x_{ij} a_{js}$ in the future states s. The value of the initial holdings, namely $\sum_{j=1}^J x^0_{ij} p_j$, can help along with the initial money amount m^0_{i0} in financing this.

In addition, money can be transfered between present and future through borrowing and lending. Money lent earns a rate of return $r \geq 0$, so that in lending an amount y_i at time 0 an agent gets back $(1 + r) y_i$ at time 1. Lending an amount of money is tantamount to investing it in a money market. Borrowing corresponds to $y_i < 0$. Like asset prices, the rate of return r is to be determined through market equilibrium.

The utility function u_i is assumed to be concave and differentiable[1] on the positive orthant $I\!R^{1+S}_{++}$ and upper semicontinuous on $I\!R^{1+S}$, with its value being $-\infty$ outside of $I\!R^{1+S}_+$ (and perhaps at places on the boundary). It increases on $I\!R^{1+S}_{++}$ with respect to increases in any m_{is} with the others fixed; all components of the gradient $\nabla u_i(m_{i0}, m_{i1}, \ldots, m_{iS})$ are assumed to be positive, The max of those components tending to ∞ as the boundary of $I\!R^{1+S}_+$ is approached. Since the components of ∇u_i give

[1] Differentiability could be dropped, with gradients replaced by subgradients, but for that we would need to work in Section 3 with a more complicated type of variational inequality, as we did in [11].

the marginal utility (in the present) of money in the various states, this boundary assumption means that agent i cannot tolerate the vanishing of funds in any state.

Agent optimization problem. *Agent i chooses a money vector $m_i = (m_{i0}, m_{i1}, \ldots, m_{iS})$ with $m_{is} > 0$, a portfolio vector $x_i = (x_{i1}, \ldots, x_{iJ})$ with $x_{ij} \geq 0$, and a lending amount y_i (possibly negative), so as to maximize utility $u_i(m_i)$ subject to the budget constraints:*

$$\begin{array}{l} m_{i0} + \sum_{j=1}^{J} x_{ij} p_j + y_i \;\leq\; m_{i0}^0 + \sum_{j=1}^{J} x_{ij}^0 p_j, \\ m_{is} \;\leq\; m_{is}^0 + \sum_{j=1}^{J} x_{ij} a_{js} + (1+r) y_i \;\; \text{for } s = 1, \ldots, S. \end{array} \tag{1}$$

An important consideration is "conservation of assets" in the sense that in solving their optimization problems the agents end up just redistributing their holdings. The total $\sum_{i=1}^{I} x_{ij}$ should agree with that initial total $\sum_{i=1}^{I} x_{ij}^0$, which is assumed to be positive for each asset j. Similarly, the borrowed amounts of money should be covered by the amounts lent, as in the following definition.

Equilibrium definition. *Asset prices $p_j \geq 0$ for $j = 1, \ldots, J$ and a rate of return $r \geq 0$ furnish an equilibrium, along with the decision elements m_i, x_i and y_i of the agents $i = 1, \ldots, S$, if*

(a) *those decision elements solve the agents' optimization problems,*
(b) $\sum_{i=1}^{I} x_{ij} = \sum_{i=1}^{I} x_{ij}^0$ *for* $j = 1, \ldots, J$,
(c) $\sum_{i=1}^{I} y_i = 0$ *if* $r > 0$, *whereas* $\sum_{i=1}^{I} y_i \geq 0$ *if* $r = 0$.

The provision in (c) means that if no interest is earned on lending—so there is no need for anyone to pay interest—an agent can essentially lend to itself merely by setting an amount of money aside in the present for use in the future. Conditions (b) and (c) constitute "clearing" in the asset market and the money market.

Theorem 1 (existence)**.** *Under the stated assumptions, an equilibrium is sure to exist. The asset prices p_j in it must all be positive, and the budget constraints must all hold as equations.*

The proof of existence will be carried out in Section 3 by way of a variational inequality representation of equilibrium and a series of harmless truncations which make it possible to apply an existence theorem for solutions to "bounded" variational inequalities. Optimality conditions for the agents' problems have a key role in that, as does concavity of the utility functions instead of just quasiconcavity. The assertions of Theorem 1 beyond existence are immediate from those conditions, according to the result stated next.

Theorem 2 (optimality)**.** *With respect to given asset prices $p_j \geq 0$ and a return rate $r \geq 0$, the money vector $m_i = (m_{i0}, m_{i1}, \ldots, m_{iS})$ with $m_{is} > 0$, portfolio vector $x_i = (x_{i1}, \ldots, x_{iJ})$ with $x_{ij} \geq 0$ and lending amount y_i solve the optimization problem for agent i if and only if they satisfy the budget constraints (1) as equations and, in terms of the gradient vector*

$$\lambda_i = (\lambda_{i0}, \lambda_{i1}, \ldots, \lambda_{iS}) = \nabla u_i(m_{i0}, m_{i1}, \ldots, m_{iS}) \;\; \text{with } \lambda_{is} > 0, \tag{2}$$

the following conditions are fulfilled, which in particular necessitate $p_j > 0$ for $j = 1, \ldots, J$:

$$\begin{array}{l} \lambda_{i0} = (1+r)(\lambda_{i1} + \cdots + \lambda_{iS}), \\ \lambda_{i0} p_j = \sum_{s=1}^{S} \lambda_{is} a_{js} \;\; \text{for assets } j \text{ with } x_{ij} > 0, \\ \lambda_{i0} p_j \geq \sum_{s=1}^{S} \lambda_{is} a_{js} \;\; \text{for assets } j \text{ with } x_{ij} = 0. \end{array} \tag{3}$$

Proof. Because the given money amounts m_{is}^0 are all positive, there is a feasible solution to the optimization problem in which m_i lies in the interior of the effective domain of u_i, namely $m_i = m_i^0$,

$x_i = x_i^0$ and $y_i = 0$. Since the budget constraints and the nonnegativity of x_i are linear constraints, this ensures that an optimal solution can be characterized in terms of a saddle point of the corresponding Lagrangian function, cf. [15, Section 28].

Disregarding (2) temporarily, we can introduce λ_{is} as the nonnegative Lagrange multiplier for the budget inequality constraint in state s, so that the Lagrangian is

$$\begin{aligned} L_i(m_i, x_i, y_i; \lambda_i) = u_i(m_i) + \lambda_{i0}[m_{i0}^0 - m_{i0} + \textstyle\sum_{j=1}^J (x_{ij}^0 - x_{ij})p_j - y_i] \\ + \textstyle\sum_{s=1}^S \lambda_{is}[m_{is}^0 - m_{is} + \sum_{j=1}^J x_{ij}a_{js} + (1+r)y_i]. \end{aligned} \tag{4}$$

We are concerned with a saddle point in which maximization occurs in the decision variables and minimization occurs in the multipliers. The minimization simply requires the budget constraints to hold. The maximization with respect to m_i yields (2) (plus the positivity of each m_{is}) and implies $\lambda_{is} > 0$. In the maximization with respect to y_i (unconstrained), the first equation in (3) is revealed. In the maximization with respect to $x_{ij} \geq 0$, the rest of (3) comes out. Because $\lambda_{is} > 0$, and each asset j has at least one payment $a_{js} > 0$, we see that (3) forces $p_j > 0$. □

Interpretation of optimality. The multiplier $\lambda_{is} > 0$ gives, through (2), the marginal utility to agent i of money in state s (as seen from the present). The first condition in (3) tells us that $\lambda_{is} < \lambda_{i0}$ for $s > 0$, and moreover that by taking

$$\pi_{is} = (1+r)\lambda_{is}/\lambda_{i0} \ \text{ for } \ s = 1, \dots, S \tag{5}$$

we uncover "imputed probabilities" in the decision making of agent i:

$$\pi_{is} > 0, \qquad \pi_{i1} + \cdots + \pi_{iS} = 1. \tag{6}$$

The portfolio conditions in (3) then involve the corresponding "expected" future payouts $\sum_{s=1}^S \pi_{is}a_{js}$ of asset j in the stochastic assessment of agent i. None of asset j will be purchased by agent i if this imputed valuation is exceeded by $(1+r)p_j$. That is because $(1+r)p_j$ is the assured future return obtained by lending out the price amount p_j instead of devoting it to asset purchase.

Leeway in stochastic assessments. The imputed probabilities (5)–(6) that the agents i come up with can differ, but by how much? As just explained, the portfolio conditions in (3) can be described in "subjective expectation" form as

$$\begin{aligned} (1+r)p_j = \textstyle\sum_{s=1}^S \pi_{is}a_{js} \ \text{ for assets } j \text{ with } x_{ij} > 0, \\ (1+r)p_j \geq \textstyle\sum_{s=1}^S \pi_{is}a_{js} \ \text{ for assets } j \text{ with } x_{ij} = 0. \end{aligned} \tag{7}$$

These conditions, relative to an optimal portfolio vector x_i, specify a polyhedral subset of the probability simplex to which the vector $\pi_i = (\pi_{i1}, \dots, \pi_{iS})$ must belong. The dependence on a particular x_i (perhaps not unique in optimality) is illusory, because the same optimality conditions have to work for all such x_i, and by taking convex combinations one can get an x_i with the maximal set of positive components. Anyway, it is clear that with more and more assets j in the market the possibilities for π_i generally get narrower and narrower. Each asset has can reveal more about an agent's beliefs in the probabilities of the future states.

This analysis is based on requiring $x_{ij} \geq 0$ in the agents' porfolio choices, but it is possible to relax this to allowing negative amounts x_{ij}. This extension, which in financial markets corresponds to "short-selling," will be taken up in Section 4.

3 Variational Inequality Representation and Truncation

Our strategy for establishing the existence of an equilibrium in the circumstances described is to recast the equilibrium conditions as a finite-dimensional variational inequality, i.e., a condition of the form

$$-f(w) \in \partial g(w)$$

for a continuous mapping $f : I\!R^N \to I\!R^N$ and a closed proper convex function $g : I\!R^N \to (-\infty, \infty]$. The special case of an indicator $g = \delta_C$ for a nonempty closed convex set $C \subset I\!R^N$, in which $\operatorname{dom} g = C$, turns into the normal cone condition

$$-f(w) \in N_C(w).$$

The elementary criterion that can be used for the existence of a solution is the boundedness of the effective domain of g, or in the geometric case, the boundedness of C; cf. [10], [11]. For variational inequalities in which the mapping f is *monotone* (in the sense of Minty), additional criteria can be brought in, cf. [17, Chapter 12], but such monotonicity will not be available in the application we are engaged in here.

An important bit of methodology at our disposal is *composite structure* for a variational inequality. This refers to the fact that a collection of conditions

$$-f_k(w_1, \ldots, w_M) \in \partial g_k(w_k) \quad \text{for } k = 1, \ldots, M, \tag{8}$$

with each g_k closed proper convex on $I\!R^{N_k}$ and f_k a continuous mapping, together comprise a single variational inequality, namely

$$\begin{array}{l} -f(w_1, \ldots, w_M) \in \partial g(w_1, \ldots, w_M) \;\text{ for }\; (w_1, \ldots, w_M) \in I\!R^{N_1} \times \cdots \times I\!R^{N_M} \\ \quad\text{with } \begin{cases} g(w_1, \ldots, w_M) = g_1(w_1) + \cdots + g_M(w_M) \text{ on } I\!R^N \text{ with } N = N_1 + \cdots + N_M, \\ f(w_1, \ldots, w_M) = (f_1(w_1, \ldots, w_M), \ldots, f_M(w_1, \ldots, w_M)) \in I\!R^N. \end{cases} \end{array} \tag{9}$$

Here $\operatorname{dom} g = \operatorname{dom} g_1 \times \cdots \times \operatorname{dom} g_M$, and its boundedness corresponds to that of every $\operatorname{dom} g_k$.

We will take advantage of this structure by interpreting the conditions for equilibrium one by one as having the form in (8). In a first pass, this is readily accomplished but with sets $\operatorname{dom} g_k$ that are unbounded. In a second pass, truncations are introduced to create boundedness. Of course, this has to be done in such a manner that solutions to the ultimate variational inequality (9) with truncation are the same as solutions to the original version without truncation. An interesting aspect, promoted by the utiliy functions being concave instead of merely quasi-concave, is that analysis of the optimization problem *dual* to the utility maximization of agent i will yield crucial insights.

A small change of variables will be convenient. Instead of working with y_i and r, we can pass to z_i and q, where q is the *market discount rate* corresponding to the rate of return r:

$$q = (1 + r)^{-1}, \qquad y_i = q z_i, \tag{10}$$

with $q \in (0, 1]$ in place of $r \in [0, \infty)$; $q = 0$ would stand for a "free lunch" in which all agents can increase future funds at no cost, and (with utility always increasing) their optimization problems would be unsolvable (and equilibrium could not exist).

Theorem 3 (basic representation). *With respect to (10), a price vector $p = (p_1, \ldots, p_J)$ and a market discount rate q, together with decision elements m_i, x_i, z_i and multiplier vectors λ_i for all the agents i, furnish an equilibrium if and only if*

(a) $-\lambda_i = \nabla[-u_i](m_i)$,

(b) $m_{i0} - m^0_{i0} + \sum_{j=1}^J (x_{ij} - x^0_{ij})p_j + qz_i \in N_{[0,\infty)}(\lambda_{i0})$,
(c) $m_{is} - m^0_{is} - \sum_{j=1}^J x_{ij}a_{js} - z_i \in N_{[0,\infty)}(\lambda_{is})$ *for* $s > 0$,
(d) $\sum_{s=1}^S \lambda_{is}a_{js} - \lambda_{i0}p_j \in N_{[0,\infty)}(x_{ij})$,
(e) $(\lambda_{i1} + \cdots + \lambda_{iS}) - q\lambda_{i0} \in N_{(-\infty,\infty)}(z_i)$,
(f) $\sum_{i=1}^I x_{ij} - \sum_{i=1}^I x^0_{ij} \in N_{[0,\infty)}(p_j)$,
(g) $\sum_{i=1}^I z_i \in N_{[0,1]}(q)$.

Proof. Obviously (a) corresponds to (2) and is equivalent to $-\lambda_i \in \partial(-u_i)$ for the closed proper convex function $-u_i$ on $\mathbb{R}^{1+S}$. Conditions (b) and (c) express complementary slackness with respect to Lagrange multipliers in the budget constraints (1). In (d) we have the complementary slackness of the portfolio conditions in (3), and in (e) a way of writing the initial equation in (3). Altogether, (a)–(e) express the optimality conditions in Theorem 2, or in other words, part (a) in the definition of equilibrium. This has the positivity of p_j as a consequence, and that allows us to write (b) in the definition of equilibrium as (f) here. Finally, (c) in the definition of equilibrium is exactly the complementary slackness condition in (g) here. □

The conditions in Theorem 3 add up to a composite variational inequality, as promised, but the domain is unbounded in every variable except q. Except for z_i, we do at least have lower bounds for all the variables, but that is not enough to enable us to apply the elementary criterion for existence of a solution. To proceed with that it is crucial to determine upper bounds which can act in tandem with the lower bounds. We will be helped by fixing some

$$a > \max_{s=1,\ldots,S} \sum_{j=1}^J a_{js} \text{ and } b > \max_{j=1,\ldots,J} \sum_{i=1}^I x^0_{ij}. \tag{11}$$

Simple observations will move us toward the more crucial issues in taking advantage of the fact that (a)–(e), which correspond to the optimality in Theorem 2 for each agent, necessitate $p_j > 0$ and $\lambda_{is} > 0$. From $p_j > 0$ we can be sure in (f) that $\sum_{i=1}^I x_{ij} = \sum_{i=1}^I x^0_{ij}$ (> 0), so that $x_{ij} < b$ for all i and $x_{i_0 j} > 0$ for at least one i_0, in which case $\sum_{s=1}^S \lambda_{i_0 s}a_{js} - \lambda_{i0}p_j = 0$ in (d). Recalling the interpretation of that aspect of optimality in terms of imputed probabilities as $(1 + r)p_j = \sum_{s=1}^S \pi_{i_0 s}a_{js}$ in (7), we see that the optimality conditions (a)–(e) holding for all agents i imply

$$x_{ij} \in [0, b), \qquad \sum_{j=1}^J x_{ij}a_{js} < abJ, \qquad p_j < a. \tag{12}$$

Thus, any elements solving the variational inequality comprised of (a)–(g) in Theorem 3 must also solve the variational inequality comprised of the following list with truncated versions of (d) and (f):

(a′) $-\lambda_i = \nabla[-u_i](m_i)$,
(b′) $m_{i0} - m^0_{i0} + \sum_{j=1}^J (x_{ij} - x^0_{ij})p_j + qz_i \in N_{[0,\infty)}(\lambda_{i0})$,
(c′) $m_{is} - m^0_{is} - \sum_{j=1}^J x_{ij}a_{js} - z_i \in N_{[0,\infty)}(\lambda_{is})$ for $s > 0$,
(d′) $\sum_{s=1}^S \lambda_{is}a_{js} - \lambda_{i0}p_j \in N_{[0,b]}(x_{ij})$,
(e′) $\lambda_{i1} + \cdots + \lambda_{iS} - q\lambda_{i0} \in N_{(-\infty,\infty)}(z_i)$,
(f′) $\sum_{i=1}^I x_{ij} - \sum_{i=1}^I x^0_{ij} \in N_{[0,a]}(p_j)$,
(g′) $\sum_{i=1}^I z_i \in N_{[0,1]}(q)$.

Moreover the converse holds as well. If these modified conditions are satisfied, with $x_{ij} < b$ always, there is no difference between the effect of (d′) and (d), so that the bound on p_j derived in (12) is still in place and (f′) reduces to (f). On the other hand, if somehow $x_{i_0 j_0} = b$ for a particular agent i_0 and asset j_0, the corresponding requirement of (d′), expressed in the "expectation" context of (7), is $(1+r)p_{j_0} \leq \sum_{s=1}^S \pi_{i_0 s} a_{j_0 s}$. Then again $p_{j_0} < a$, with (f′) again reverting to (f) and implying $\sum_{i=1}^I x_{ij_0} \leq \sum_{i=1}^I x_{ij_0}^0$. The latter is incompatible $x_{i_0 j_0} = b$.

Therefore the variational inequality specified by (a′)–(g′) has the same solutions (if any) as the one specified by (a)–(g), entailing (12).

Aiming for additional truncation, we look now at the budget condition (c′), where the expression on the left has to equal 0 because of $\lambda_{is} > 0$ in (a′). Since $m_{is} > 0$, while the portfolio sum is bounded above by a through (12), we see that

$$z_i > m_{is} - m_{is}^0 - abJ. \tag{13}$$

The budget condition (b′), with the expression on the left side likewise having to equal to 0, can be added over all agents i to get

$$\sum\nolimits_{i=1}^I m_{i0} + q \sum\nolimits_{i=1}^I z_i = \sum\nolimits_{i=1}^I m_{i0}^0, \tag{14}$$

where (g′) implies

$$q \sum\nolimits_{i=1}^I z_i = \max\left\{0, \sum\nolimits_{i=1}^I z_i\right\} \tag{15}$$

Since all money amounts are positive, we deduce from (13), (14) and (15) the existence of $c > 0$ such that, with the other conditions in place, necessarily $-c < z_i < c$ for all i. We furthermore see then the existence of an upper bound M on money such that (13) and (14), together now with $|z_i| < c$, ensure that $0 < m_{is} < M$ for all i and s. This upper bound can of course be taken high enough that

$$m_{is}^0 < M \text{ for all } i \text{ and } s. \tag{16}$$

This tells us that any solution to the variational inequality for (a′)–(g′) must satisfy the following conditions in which (a′) and (e′) have been truncated to (a″) and (e″):

(a″) $-\lambda_i - \nabla[-u_i](m_i) \in \Pi_{s=0}^S N_{[0,M]}(m_{is})$,
(b″) $m_{i0} - m_{i0}^0 + \sum_{j=1}^J (x_{ij} - x_{ij}^0) p_j + q z_i \in N_{[0,\infty)}(\lambda_{i0})$,
(c″) $m_{is} - m_{is}^0 - \sum_{j=1}^J x_{ij} a_{js} - z_i \in N_{[0,\infty)}(\lambda_{is})$ for $s > 0$,
(d″) $\sum_{s=1}^S \lambda_{is} a_{js} - \lambda_{i0} p_j \in N_{[0,b]}(x_{ij})$,
(e″) $\lambda_{i1} + \cdots + \lambda_{iS} - q\lambda_{i0} \in N_{[-c,c]}(z_i)$,
(f″) $\sum_{i=1}^I x_{ij} - 1 \in N_{[0,a]}(p_j)$,
(g″) $\sum_{i=1}^I z_i \in N_{[0,1]}(q)$,

The additional terms in (a″) vanish under (16). Moreover (a″) is identical to $-\lambda_i \in \partial g_i(m_i)$ for the closed proper convex function g_i that agrees with $-u_i$ when $m_{is} \leq M$ for all s but equals ∞ otherwise. Hence (a″)–(g″) do specify a composite variational inequality.

Because the truncations utilized in (a″) and (e″) were developed in the background of the other conditions, which remain unchanged, they must be inactive for solutions to (a″)–(g″), and that variational inequality therefore has the same solution set as the one in Theorem 3.

The important thing to note, in line with our goal of reducing to a variational inequality with bounded domain in order to obtain the existence of a solution, is that only upper bounds on the multipliers λ_{is} are still missing. To get a handle on that, we argue next that conditions (a″)–(e″) are necessary and sufficient for m_i, x_i and z_i to solve the following modification of the optimization problem of agent i under the change of variables in (10):

$$\begin{array}{l} \text{choose } m_i,\ x_i,\ z_i \text{ to maximize } u_i(m_i) \text{ subject to} \\ m_{is} \in [0, M],\ x_{ij} \in [0, b],\ z_i \in [-c, c],\ \text{ and the constraints} \\ m_{i0} + \sum_{j=1}^J x_{ij} p_j + q z_i \ \leq\ m_{i0}^0 + \sum_{j=1}^J x_{ij}^0 p_j, \\ m_{is} \ \leq\ m_{is}^0 + \sum_{j=1}^J x_{ij} a_{js} + z_i \ \text{ for } s = 1, \ldots, S. \end{array} \tag{17}$$

Indeed, (a″)–(e″) describe a saddle point of the modified Lagrangian function

$$\begin{aligned} \widetilde{L}_i(m_i, x_i, z_i; \lambda_i) = u_i(m_i) &+ \lambda_{i0}[m_{i0}^0 - m_{i0} + \textstyle\sum_{j=1}^J (x_{ij}^0 - x_{ij}) p_j - q z_i] \\ &+ \textstyle\sum_{s=1}^S \lambda_{is}[m_{is}^0 - m_{is} + \sum_{j=1}^J x_{ij} a_{js} + z_i]. \end{aligned} \tag{18}$$

with respect to maximizing over $m_{is} \in [0, M]$, $x_{ij} \in [0, b]$ and $z_i \in [-c, c]$, while minimizing over $\lambda_{is} \geq 0$. This Lagrangian is concave in the maximization variables and convex (actually affine) in the minimization variables.

As is well known about saddle points [15, Section 28], the multiplier vectors λ_i taking part in these conditions are the solutions to the corresponding Lagrangian dual problem,

$$\begin{array}{l} \text{minimize } \phi_i(\lambda_i) \text{ subject to } \lambda_i \geq 0, \text{ where} \\ \quad \phi_i(\lambda_i) = \max\Big\{\, \widetilde{L}_i(m_i, x_i, z_i; \lambda_i) \,\Big|\, m_{is} \in [0, M],\ x_{ij} \in [0, b],\ z_i \in [-c, c] \,\Big\}. \end{array} \tag{19}$$

The function ϕ_i is finite and convex on $I\!R^{1+S}$. Because the primal problem has feasible solutions (due to (16), e.g., $m_i = m_i^0$, $x_i = x_i^0$, $z_i = 0$), and its variables are bounded, a solution to (17) must exist and yield

$$[\text{maximum in (17)}] = [\text{infimum in (19)}]. \tag{20}$$

It is on this platform that an upper bound on the multipliers will be derived. Note first that

$$[\text{maximum in (17)}] \ \leq\ u_i(M, M, \ldots, M), \tag{21}$$

because utility increases with respect to all components. Choose $\epsilon > 0$ small enough that

$$m_{is}^0 > \epsilon, \qquad m_{is}^0 + \epsilon < M, \text{ for all } s, \tag{22}$$

through (16), and let

$$\mu_i = \text{minimum of } u_i(m_i) \text{ subject to } |m_{is} - m_{is}^0| \leq \epsilon, \tag{23}$$

which exists because of the continuity of u_i on $I\!R^{1+S}_{++}$. In restricting the maximization in the definition of $\phi_i(\lambda_i)$ to $m_{is} \in [m_{is}^0 - \epsilon, m_{is}^0 + \epsilon]$ along with $x_i = x_i^0$ and $z_i = 0$, we get

$$\phi_i(\lambda_i) \ \geq \max_{|m_{ij} - m_{ij}^0| \leq \epsilon} \widetilde{L}_i(m_i, x_i^0, 0; \lambda_i) \ \geq\ \mu_i + \epsilon \sum_{s=0}^S |\lambda_{is}|.$$

The combination of this with (20) and (21) yields for any solution λ_i to (19) that

$$\mu_i + \epsilon \sum_{s=0}^S |\lambda_{is}| \ \leq\ u_i(M, M, \ldots, M). \tag{24}$$

This therefore has to hold for any vector λ_i appearing together with a solution to (17) in the optimality conditions (a″)–(e″).

Completion of the proof of Theorem 1. There is available through (24) an upper bound d such that necessarily $\lambda_{is} < d$ for $s = 0, 1, \ldots, S$, and even for $i = 1, \ldots, I$, whenever (a$''$)–(e$''$) hold. This allows us to replace $[0, \infty)$ by $[0, d]$ in (b$''$) and (c$''$) without shrinking the possibilities for having a solution to these conditions. With that restriction, and with (f$''$) and (g$''$) thrown in, we have a composite variational inequality for which the set of solutions is the same as the set of solutions to the variational inequality in Theorem 3, which represents equilibrium. Since the modified conditions have bounded domains, the elementary existence criterion is applicable, and we conclude that the set of solutions is nonempty. Thus, an equilibrium does exist. □

4 Extension to Short-selling

Financial markets can allow the possibility of an agent obtaining a fraction of p_j in the present for promising to pay out the corresponding fraction of the returns a_{js} in the future. This is called short-selling and amounts in principle to a contract between two parties, the short-seller and another agent who advances the money in the present for receiving the pay-outs later. The quantity of asset j that is short-sold is thereby balanced on the other side, so that from the mathematical point of view, short-selling is simply the case of allowing x_{ij} to be negative with the total $\sum_{i=1}^{I} x_{ij}$ still maintained.

With this relaxation of nonnegativity in the portfolio constraints in the optimization problems of the agents, but with everything else the same, we speak of an *equilibrium with short-selling.* Our goal now is to prove the following extension of Theorem 3.

Theorem 4 (existence with short-selling)**.** *Under the stated assumptions, an equilibrium with short-selling is likewise sure to exist. Again, the asset prices p_j in it must all be positive, and the budget constraints must all hold as equations.*

In constructing the proof of this, the following restriction will help us without actually incurring any loss of generality:

Nonredundancy assumption. *The asset vectors $a_j = (a_{j1}, \ldots, a_{jS})$ for $j = 1, \ldots J$ along with the vector $(1, \ldots, 1)$ are linearly independent in $\mathbb{R}^S$.*

This assumption guarantees that the coefficients x_{ij} and z_i are uniquely determined in any financial arrangement that produces

$$(m_{i1}, \ldots, m_{iS}) - (m_{i1}^0, \ldots, m_{iS}^0) = \sum\nolimits_{j=1}^{J} x_{ij}(a_{j1}, \ldots, a_{jS}) + z_i(1, \ldots, 1) \tag{25}$$

There is no loss of generality because, if there were linear dependence, one or more of the assets j could be dropped as redundant until a linearly independent set of vectors remained for which the possibilities achieved on the left side of (25) are the unaffected. Of course, this "nonredundancy" argument would not be valid in the earlier framework with $x_{ij} \geq 0$.

What is the effect of relaxing the nonnegativity of x_{ij} on the analysis already carred out above? As far as the optimality conditions for the agents are concerned, the only change is that, instead of (7), one has

$$(1 + r)p_j = \sum_{s=1}^{S} \pi_{is} a_{js} \quad \text{for all assets } j, \tag{26}$$

with $\pi_i = (\pi_{i1}, \ldots, \pi_{is})$ being a "subjective probability" vector having every $\pi_{is} > 0$. For the bound a in (11), this implies

$$0 < p_j < a \quad \text{for } j = 1, \ldots, J. \tag{27}$$

Hence, for a starter, we can take the conditions for a equilibrium with short-selling to be of the form

(aa) $-\lambda_i = \nabla[-u_i](m_i)$,
(bb) $m_{i0} - m_{i0}^0 + \sum_{j=1}^J (x_{ij} - x_{ij}^0)p_j + qz_i \in N_{[0,\infty)}(\lambda_{i0})$,
(cc) $m_{is} - m_{is}^0 - \sum_{j=1}^J x_{ij}a_{js} - z_i \in N_{[0,\infty)}(\lambda_{is})$ for $s > 0$,
(dd) $\sum_{s=1}^S \lambda_{is}a_{js} - \lambda_{i0}p_j \in N_{(-\infty,\infty)}(x_{ij})$,
(ee) $(\lambda_{i1} + \cdots + \lambda_{iS}) - q\lambda_{i0} \in N_{(-\infty,\infty)}(z_i)$,
(ff) $\sum_{i=1}^I x_{ij} - \sum_{i=1}^I x_{ij}^0 \in N_{[0,a]}(p_j)$,
(gg) $\sum_{i=1}^I z_i \in N_{[0,1]}(q)$.

The positivity of p_j and the equation form of the budget constraints are implied by this, as before, and our focus can again be in introducing truncations that leave the set of solutions (if any) unchanged.

Our first step is to argue that conditions (aa), (bb), (cc), (ff) and (gg) force a bound on the elements m_i, x_i and z_i. Instead of proceeding as before, we employ a recession cone technique. Specifically, we consider the nonempty closed subset $W \subset (\mathbb{R}^{1+S} \times \mathbb{R}^J \times \mathbb{R})^I$ consisting of all

$$\begin{aligned} &(m_i, x_i, z_i) \text{ for } i = 1, \ldots, I \text{ with } m_{is} \geq 0, \text{ such that (ff), (gg), hold for some } p,\ q, \text{ and} \\ &m_{i0} - m_{i0}^0 + \textstyle\sum_{j=1}^J (x_{ij} - x_{ij}^0)p_j + qz_i \leq 0, \\ &m_{is} - m_{is}^0 - \textstyle\sum_{j=1}^J x_{ij}a_{js} - z_i \leq 0 \text{ for } s > 0, \end{aligned} \tag{28}$$

and establish its boundedness by demonstrating that its horizon cone W^∞ (in the variational analysis sense) consists only of the zero vector; cf. [17, Theorem 3.5].

Suppose there is an unbounded sequence in W with components (m_i^k, x_i^k, z_i^k) with corresponding p^k and q^k, $k = 1, 2, \ldots$:

$$\begin{array}{ll} m_{i0}^k - m_{i0}^0 + \sum_{j=1}^J (x_{ij}^k - x_{ij}^{0k})p_j^k + q^k z_i^k \leq 0, & m_{i0}^k \geq 0, \\ m_{is}^k - m_{is}^0 - \sum_{j=1}^J x_{ij}^k a_{js} - z_i^k \leq 0, & m_{is}^k \geq 0, \text{ for } s > 0, \\ \sum_{i=1}^I x_{ij}^k - \sum_{i=1}^I x_{ij}^{0k} - \in N_{[0,a]}(p_j^k), & \sum_{i=1}^I z_i^k \in N_{[0,1]}(q^k). \end{array} \tag{29}$$

We will show that this leads to a contradiction to our redundancy assumption.

It is possible (by considering norms) to find a sequence of coefficients $\theta^k > 0$ decreasing to 0 such that the vector sequence with components $\theta^k(m_i^k, x_i^k, z_i^k)$ is bounded and yet stays outside some neighborhood of the origin. Invoking compactness, we can assume then that $\theta^k(m_i^k, x_i^k, z_i^k) \to (m_i^*, x_i^*, z_i^*)$ with at least one of these limits not being $(0,0,0)$, say for i_0. We also can assume $p^k \to p^* \in [0,a]$ and $q^k \to q^* \in [0,1]$. In multiplying the conditions in (29) by θ^k, we see in the limit that

$$\begin{array}{ll} m_{i0}^* + \sum_{j=1}^J x_{ij}^* p_j^* + q^* z_i^* \leq 0, & m_{i0}^* \geq 0, \\ m_{is}^* - \sum_{j=1}^J x_{ij}^* a_{js} - z_i^* \leq 0, & m_{is}^* \geq 0, \text{ for } s > 0, \\ \sum_{i=1}^I x_{ij}^* \in N_{[0,a]}(p_j^*), \quad \sum_{i=1}^I z_i^* \in N_{[0,1]}(q^*). & \end{array} \tag{30}$$

The third line of (30) entails

$$p_j^* \textstyle\sum_{i=1}^I x_{ij}^* = a \max\left\{0, \sum_{i=1}^I x_{ij}^*\right\}, \qquad q^* \displaystyle\sum_{i=1}^I z_i^* = \max\left\{0, \textstyle\sum_{i=1}^I z_i^*\right\}. \tag{31}$$

Adding the first inequality in (30) over i and utilizing (31), we get

$$\textstyle\sum_{i=1}^I m_{i0}^* + a \max\left\{0, \sum_{i=1}^I x_{ij}^*\right\} + \max\left\{0, \sum_{i=1}^I z_i^*\right\} \leq 0, \text{ with } m_{i0}^* \geq 0,$$

with the consequence that

$$m_{i0}^* = 0, \qquad \sum\nolimits_{i=1}^{I} x_{ij}^* \le 0, \qquad \sum\nolimits_{i=1}^{I} z_i^* \le 0. \tag{32}$$

Adding now the second inequality in (30) over i we get

$$\sum\nolimits_{i=1}^{I} m_{is}^* - \sum\nolimits_{j=1}^{J} \Big(\sum\nolimits_{i=1}^{I} x_{ij}^*\Big) a_{js} - \sum\nolimits_{i=1}^{I} z_i^* \le 0, \qquad m_{is}^* \ge 0,$$

which implies through (32) and the same line of (30) that

$$m_{is}^* = 0, \qquad -\sum\nolimits_{j=1}^{J} x_{ij}^* a_{js} - z_i^* = 0.$$

Then, however we have

$$(0, \dots, 0) = \sum\nolimits_{j=1}^{J} x_{ij}^* a_j + z_i^*(1, \dots, 1),$$

and through the nonredundancy assumption must have $x_{ij}^* = 0$ and $z_i^* = 0$. This contradicts having these elements not all being 0, at least for i_0.

On the basis of having established the boundedness of W, we are able now to introduce an upper bound c on $|z_i|$, an upper bound M on m_{ij}, again satisfying (16), and an upper bound ξ on $|x_{ij}|$, such that all elements of W satisfy all these bounds with strict inequality. This permits us to identify the set of solutions to the variational inequality comprised of (aa)–(gg) with that of the variational inequality given by

(aa′) $-\lambda_i - \nabla[-u_i](m_i) \in \Pi_{s=0}^{S} N_{[0,M]}(m_{is})$,
(bb′) $m_{i0} - m_{i0}^0 + \sum_{j=1}^{J}(x_{ij} - x_{ij}^0)p_j + qz_i \in N_{[0,\infty)}(\lambda_{i0})$,
(cc′) $m_{is} - m_{is}^0 - \sum_{j=1}^{J} x_{ij}a_{js} - z_i \in N_{[0,\infty)}(\lambda_{is})$ for $s > 0$,
(dd′) $\sum_{s=1}^{S} \lambda_{is}a_{js} - \lambda_{i0}p_j \in N_{[-\xi,\xi]}(x_{ij})$,
(ee′) $\lambda_{i1} + \cdots + \lambda_{iS} - q\lambda_{i0} \in N_{[-c,c]}(z_i)$,
(ff′) $\sum_{i=1}^{I} x_{ij} - 1 \in N_{[0,a]}(p_j)$,
(gg′) $\sum_{i=1}^{I} z_i \in N_{[0,1]}(q)$,

From here on, the previous duality argument can be mimicked completely to obtain upper bounds on the multipliers λ_{is}. The additional truncation supported by those bounds brings us finally to a variational inequality to which the elementary criterion for existence of a solution can be applied.

Concluding remarks. An advantage of variational inequality modeling, as followed here are the enhanced possibilities for computation. In much of equilibrium theory in economics, computation is envisioned either in terms of an algorithm for finding a fixed point, or through reduction to a system of nonlinear equations which can be solved in a classical manner in Newton-like steps. However, the latter approach, in particular, is not well adapted to handling nonnegativity constraints.

It deserves emphasis also that our purpose here has been not only to contribute to financial theory, but to illustrate the important ways that convex analysis and its extensions in variational analysis enter the subject. On the side of financial equilibrium, much more generality is possible in the model. There can be markets for goods, which are influenced by consumption choices of the agents in the present and future. Financial instruments in the form of derivatives, involving options on future prices, can be incorporated as well. This is explained in detail in our paper [11].

As mentioned earlier, it is perfectly possible also to relax the smoothness assumption on the concave utility functions, but in that case the variational inequality model has to be modified to "functional" type. That too can be seen in [11].

References

[1] K. J. Arrow, Le rôle des valeurs boursiéres pour la répartition la meilleure des risques. *Econométrie Colloques Internationaux du Centre National de la Recherche Scientifique* 40 (1953), 41-47; English translation: The role of securities in the optimal allocation of risk-bearing. *The Review of Economic Studies* 31 (1964), 91–96.

[2] K. J. Arrow, G. Debreu, Existence of an equilibrium for a competitive economy. *Econometrica* 22 (1954), 265–290.

[3] G. Debreu, *Theory of Value,* (Wiley, 1959).

[4] G. Debreu, Economies with a finite set of equilibria. *Econometrica* 38 (1970), 387-392.

[5] A. D. Dontchev, R. T. Rockafellar, *Implicit Functions and Solution Mappings: A View From Variational Analysis*, Monographs in Mathematics, Springer-Verlag, Berlin, 2009.

[6] A. D. Dontchev, R. T. Rockafellar, Parametric stability of solutions in models of economic equilibrium. *Convex Analysis* 19 (2012), 975–993.

[7] J. Geanakoplos, An introduction to general equilibria with incomplete markets. *J. Mathematical Economics* 19 (1990), 1–38.

[8] T. Hens, Incomplete markets. Chapter 5 in *Elements of General Equilibrium Theory*, Festschrift in Honor of Gérard Debreu (A. Kirman, ed.), Blackwell Publishers, 1998.

[9] A. Jofre, R.T. Rockafellar, R. J-B Wets, A variational inequality model for determining an economic equilibrium of classical or extended type. *Variational Analysis and Applications* (F. Giannessi and A. Maugeri, eds.), Springer-Verlag, 2005, 553–578.

[10] A. Jofre, R.T. Rockafellar, R. J-B Wets, Variational inequalities and economic equilibrium. *Math. of Operations Research* 32 (2007), 32–50.

[11] A. Jofre, R.T. Rockafellar, R. J-B Wets, General equilibrium with incomplete markets and money. (Downloadable from www.math.washington.edu/∼rtr/mypage.html)

[12] A. Jofre, R.T. Rockafellar, R. J-B Wets, The robust stability of every equilibrium in economic models of exchange even under relaxed standard conditions. (Downloadable from www.math.washington.edu/∼rtr/mypage.html)

[13] M. Magill, W. Shafer, Incomplete markets. *Handbook of Mathematical Economics, Vol. IV* (W. Hildenbrand and H. Sonnenschein, eds.), Elsevier Science Publishers, 1991.

[14] R. Radner, The existence of equilibrium of plans, prices and price expectations. *Econometrica* 40 (1972), 289–303.

[15] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

[16] R. T. Rockafellar, S. Uryasev, M. Zabarankin, Equilibrium with investors using a diversity of deviation measures. *Journal of Banking and Finance* 31 (2007), 3251–3268.

[17] R. T. Rockafellar, R. J-B Wets, *Variational Analysis*, Grundlehren der Mathematischen Wissenschaften 317, Springer-Verlag, Berlin, 1997.