APPLICATIONS OF CONVEX VARIATIONAL ANALYSIS TO NASH EQUILIBRIUM

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Abstract. Problems of Nash equilibrium, in original or generalized form, have mainly been studied for the existence of a solution, but much more is possible with additional structure involving convexity. An equilibrium can be characterized then as a solution to a variational inequality problem, and this enables the application of a wide range of results in convex analysis and modern variational analysis. In particular, an equilibrium problem can be formulated with a parameter vector, and the question of expressing it as a well behaved function of that parameter vector can be given a good answer.

Key Words. Nash equilibrium, variational inequalities, parametric stability, convex analysis, variational analysis, maximal monotonicity, strong monotonicity

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1 Equilibrium modeling with variational inequalities

The notion of an "equilibrium" gained its prominence in physics. The word itself derives from Latin roots meaning "equal weight," as in the traditional device for measuring the unknown weight of an object by balancing it with known weights. Equilibrium in physics is typically a balance of forces, and the standard mathematical model for it is an equation $f(x) = 0$ for a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, or perhaps a mapping from some infinite-dimensional function space into itself, instead of $\mathbb{R}^n$.

Besides looking for the existence of a solution to such an equation and studying its properties, there is interest in understanding dependence of the solution on parameters. The formulation then is $f(p, x) = 0$ for a parameter vector p, and the issue is whether this can be "solved for x in terms of p" to get a function $x = s(p)$. The main tool for that is the *implicit function theorem*, one of the most celebrated results in classical analysis.

Beyond physics, there are many situations where equilibrium notions are important but an equation model is inadequate. In economics, for instance, equilibrium may be associated with a balance between supply and demand for various goods in a market. The balance is supposed to emerge from the actions of a collection of "agents" who optimize their buying and selling according to their preferences. In that setting, inequalities have to be respected as well as equations, because the quantity of a good acquired by an agent can be ≥ 0 but not < 0. Indeed, inequalities are characteristic in general of optimization, where expressions are usually constrained by upper bounds or lower bounds rather than being required to hold at a fixed level.

Equilibrium is a key concept also in game theory. The balance then must be found between the competing interests of different "players" as agents, and optimization likewise has a very essential role.

What kind of mathematical model can serve in those optimization-oriented settings in place of $f(x) = 0$? An attractive answer has slowly emerged after decades of work in optimization theory and variational analysis. The name in use for it (one might wish for something better) is a *generalized equation*, taking the following form:

$$-f(x) \in F(x) \text{ for } f : \mathbb{R}^n \to \mathbb{R}^n \text{ and } F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n. \tag{1.1}$$

The notation $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ signals that F is, in general, a *set-valued mapping* (multi-valued mapping) with a subset $\operatorname{gph} F$ of $\mathbb{R}^n \times \mathbb{R}^n$ serving its graph (instead of a subset of $\mathbb{R}^n \times 2^{\mathbb{R}^n}$); $y \in F(x)$ means $(x, y) \in \operatorname{gph} F$. The effective domain of F is $\operatorname{dom} F = \{\, x \mid F(x) \neq \emptyset \,\}$. In the special case of the *zero mapping*, where $y \in F(x)$ if and only if $y = 0$, which is expressed as $F \equiv 0$, the generalized equation $-f(x) \in F(x)$ reduces to the equation $f(x) = 0$.

The theory of generalized equations is by now very well developed. The book [1] offers an introduction along with many powerful results. In particular there are extensions of the implicit function theorem to parameterized generalized equations,

$$-f(p, x) \in F(x) \text{ with parameter vector } p. \tag{1.2}$$

The issue once more is "solving" this to get a function $x = s(p)$, at least in some localized sense. In contrast to the classical situation, a solution function can't be expected to be differentiable, because of "kinks" introduced by underlying one-sided constraints, however Lipschitz continuity is often with in grasp. The first to produce such an implicit function theorem was Robinson [8] in 1980, and he was also the originator of the term "generalized equation." Since then, the subject has gone much farther.

An especially rich and well appreciated category of generalized equations is associated with the set-valued mappings F coming from convex analysis as *normal cone mappings*. For a nonempty, closed, convex set $C \subset I\!R^n$, and a point $x \in C$, normality is defined by

$$v \text{ is a normal vector to } C \text{ at } x \quad \Longleftrightarrow \quad v{\cdot}(x' - x) \leq 0 \text{ for all } x' \in C. \tag{1.3}$$

This is written as $v \in N_C(x)$; thus, N_C is the set-valued mapping $I\!R^n \rightrightarrows I\!R^n$ described by

$$\operatorname{gph} N_C = \{\, (x, v) \mid x \in C,\ v \text{ normal to } C \text{ at } x \,\}. \tag{1.4}$$

The corresponding generalized equation

$$-f(x) \in N_C(x), \quad \text{i.e., } x \in C \text{ and } f(x){\cdot}(x' - x) \geq 0 \text{ for all } x' \in C \tag{1.5}$$

is called the *variational inequality for f and C*. When $x \in \operatorname{int} C$ one has $N_C(x) = \{0\}$ (and conversely), so that $-f(x) \in N_C(x)$ becomes $f(x) = 0$. Indeed the variational inequality for f and C reduces entirely to that equation when C is all of $I\!R^n$.

An immediate example in optimization comes from the minimization of a differentiable function g over a nonempty, closed, convex set C. The first-order necessary condition for that is $-f(x) \in N_C(x)$ for $f = \nabla g$ (gradient mapping). When g is convex, this necessary condition is also sufficient. Note that when the minimizing point x is not on the boundary of C, this comes out as $\nabla g(x) = 0$, the classical first-order condition for unconstrained minimization. When C is specified by a system of constraints, the elements of $N_C(x)$ can be expressed by linear combinations of the gradients of those constraints. The coefficients are Lagrange multipliers, and the variational inequality then provides a Lagrange multiplier rule.

Extensions of all this can be made to nonconvex sets C through a broader definition of N_C, but convex analysis [10] will dominate here. For the extended theory, we refer to the variational analysis book [11].

With enough convexity, Nash equilibrium can be expressed by a variational inequality as well. Consider players $i = 1, \ldots m$ with strategy vectors x_i taken from nonempty, closed, convex sets $C_i \subset I\!R^{n_i}$. Player i has a cost function $g_i(x_i, x_{-i})$, where the notation x_{-i}, standard in game theory, refers to the part of the "supervector" $(x_1, \ldots, x_m)$ obtained by deleting x_i. We speak of a cost function instead of a pay-off function in order to keep with minimization instead of switching to maximization. A Nash equilibrium, by definition then, is an $x = (x_1, \ldots, x_m)$ such that

$$x_i \text{ minimizes } g_i(\cdot, x_{-i}) \text{ over } C_i \text{ for } i = 1, \ldots, m. \tag{1.6}$$

Under the assumption that $g_i(x_i, x_{-i})$ is differentiable with respect to x_i, this relates to the first-order conditions

$$-\nabla_{x_i} g_i(x_i, x_{-i}) \in N_{C_i}(x_i) \text{ for } i = 1, \ldots, m, \tag{1.7}$$

which moreover are sufficient for (1.6) when $g_i(x_i, x_{-i})$ is convex with respect to x_i. The crucial observation is that the optimality conditions can be put together as the variational inequality

$$-f(x_1, \ldots, x_m) \in N_C(x_1, \ldots, x_m) \tag{1.8}$$

for the function

$$f(x_1, \ldots, x_m) = (\nabla_{x_1} g_1(x_1, x_{-1}), \cdots, \nabla_{x_m} g_m(x_m, x_{-m})) \tag{1.9}$$

and the closed, convex set

$$C = C_1 \times \cdots \times C_m. \tag{1.10}$$

This due to the fact in convex analysis that

$$N_{C_1 \times \cdots \times C_m}(x_1, \ldots, x_m) = N_{C_1}(x_1) \times \cdots \times N_{C_m}(x_m). \tag{1.11}$$

Thus, as long as the sets C_i are nonempty, closed and convex, and the functions $g_i(x_i, x_{-i})$ are, in the x_i argument, convex and differentiable, a Nash equilibrium is *equivalent* to the variational inequality (1.8) under (1.9) and (1.10).

Yet another form of equilibrium, sometimes called *generalized* Nash equilibrium arises when the strategies available to player i are affected by the choices made by the other players. The simplest case is that of a joint constraint $(x_1, \ldots, x_m) \in C$ for a set C that isn't just a product of separate sets in the spaces $I\!R^{n_i}$ as in (1.10). An equilibrium corresponds then, at least from an initial perspective, to modifying the minimization requirement in (1.6) to

$$x_i \text{ minimizes } g_i(\cdot, x_{-i}) \text{ over } C_i(x_{-i}), \tag{1.12}$$

where

$$C_i(x_{-i}) = \{\, x_i \,|\, (x_i, x_{-i}) \in C \,\}. \tag{1.13}$$

When C is closed and *convex*, the sets $C_i(x_{-i})$ are closed and convex as well. The corresponding first-order conditions for optimality in (1.12), utilizing differentiability of $g_i(x_i, x_{-i})$ in x_i, then take the form

$$-\nabla_{x_i} g_i(x_i, x_{-i}) \in N_{C_i(x_{-i})}(x_i) \text{ for } i = 1 \ldots, m. \tag{1.14}$$

Again, these relations are sufficient as well as necessary for the equilibrium if $g_i(x_i, x_{-i})$ is also convex in x_i. We are then in the picture of a generalized equation

$$-f(x_1, \ldots, x_m) \in F(x_1, \ldots, x_m) \text{ for } f \text{ in (1.9) and } F(x_1, \ldots, x_m) = \Pi_{i=1}^m N_{C_i(x_{-i})}(x_i), \tag{1.15}$$

but this is not a variational inequality where $F = N_D$ for some D.

The generalized Nash equilibrium (1.12)–(1.13) can actually be posed as an "ungeneralized" Nash equilibrium for a different choice of sets $\tilde{C}_i$ and functions $\tilde{g}_i$. This is accomplished by taking $\tilde{C}_i = I\!R^{n_i}$ and

$$\tilde{g}_i(x_i, x_{-i}) = \begin{cases} g_i(x_i, x_{-i}) & \text{when } (x_i, x_{-i}) \in C, \\ \infty & \text{when } (x_i, x_{-i}) \notin C. \end{cases} \tag{1.16}$$

In other words, there isn't anything "generalized" about such an equilibrium notion except for the introduction of ∞ "costs." Nonetheless, there are technical obstacles in working with that ∞ format, and a variational inequality representation is anyway still out of reach.

An equilibrium in the sense of (1.12)–(1.13) may also be too weak an idea, not paying enough attention to capturing the effects of strategy interactions induced by the joint constraint $(x_1, \ldots, x_m) \in C$. As an improvement, in the case where C is a closed, convex set and the gradients in (1.9) exist, we propose turning directly to the variational inequality (1.8)–(1.9) for C itself. This is motivated by a further rule of convex analysis, beyond (1.11), namely that

$$(v_1, \ldots, v_m) \in N_C(x_1, \ldots, x_m) \implies v_i \in N_{C_i(x_{-i})}(x_i) \text{ for } i = 1 \ldots, m \tag{1.17}$$

under (1.13). On this basis, *any solution $x = (x_1, \ldots, x_m)$ to the variational inequality (1.8)–(1.9) is in particular a solution to the generalized equation (1.15), which is equivalent to the earlier version of equilibrium in (1.12)–(1.13) when $g_i(x_i, x_{-i})$ is convex in x_i.* Such a solution x will be called a *strong* generalized Nash equilibrium

Note also that, in passing this way to $N_C(x_1, \ldots, x_m)$ itself, the door is opened also to Lagrange multiplier rules associated with a specification of C by a system of equation or inequality constraints. As noted earlier, such rules enter through formulas for the normal vectors $v \in N_C(x)$ in that set-up.

The virtue of these observations is that a variational inequality model for equilibrium fits into a framework of analysis where perturbations can be studied and computational methods are available. Approaches to equilibrium through fixed-point theory alone fall short of providing the problem structure needed for that.

2 Results on Existence

In possession now of equilibrium examples enjoying convexity, we can proceed to look at what the theory of variational inequalities has to offer for them.

The first question to ask about a variational inequality (1.5) is whether it has a solution. This is much easier to answer than the same question for a generalized equation (1.1) with less structure, and there are two distinct approaches. The first makes use of the characterization of normal vectors v to a closed convex set C at a point $x \in C$ by means of the

$$\text{projection mapping } P_C : z \to P_C(z) = \text{ nearest point of } C \text{ to } z. \tag{2.1}$$

The mapping P_C is nonexpansive, i.e., globally Lipschitz continuous with Lipschitz constant 1, and one has

$$v \in N_C(x) \iff P_C(x + v) = x. \tag{2.2}$$

According to that, the problem of solving a variational is equivalent to a special fixed-point problem:

$$-f(x) \in N_C(x) \iff M(x) = x \text{ for } M(x) = P_C(x - f(x)). \tag{2.3}$$

This yields an immediate result through the fact that a continuous mapping from a compact convex set into itself is sure to have a fixed-point:

Background Theorem 1 (existence via compactness)**.** *If the function f is continuous from the nonempty, closed, convex set $C \in I\!R^n$ into $I\!R^n$, and C is bounded, then the variational inequality $-f(x) \in N_C(x)$ has at least one solution.*

We can readily apply this to the Nash equilibrium models above.

Application Theorem 1. *Let C be a nonempty, closed, convex set in $I\!R^n = \Pi_{i=1}^m I\!R^{n_i}$, and let the functions $g_i(x_1, \ldots, x_m)$ on C be convex and differentiable in x_i with their gradients in x_i being continuous with respect to $x = (x_1, \ldots, x_m)$. If C is also bounded, then the variational inequality (1.8)–(1.9) has at least one solution $x = (x_1, \ldots, x_m)$.*

(a) *In the Nash equilibrium model (1.6), with $C = C_1 \times \cdots \times C_m$, this is equivalent to x being an equilibrium.*

(b) *In the generalized Nash equilibrium model (1.12)–(1.13), this means that x furnishes a strong equilibrium.*

Of course, the existence of a Nash equilibrium (1.6) can be obtained via compactness and continuity of the functions g_i themselves without having to look at any gradients. The importance of (a) in this theorem lies not simply in such existence but rather in tying it to a variational inequality, with the potential advantages mentioned at the end of the preceding section. In (b), there is something more: the strong equilibrium in question even depends on a variational inequality formulation for its definition. Existence of a generalized Nash equilibrium is obtained in manner that discriminates in favor of stronger properties than would follow from just the existence of a solution to (1.12)–(1.13), which anyway would be difficult to guarantee in the absence of sharper assumptions.

Although compactness is common in connection with Nash equilibrium, situations can certainly be contemplated in which the set C might be unbounded. This can often be handled through truncation arguments, which systematically cut parts away from C while demonstrating that equilibrium is not thereby affected. However, another approach to existence of solutions to variational inequalities can get around boundedness by appealing instead to "monotonicity."

Monotonicity is a property deeply embedded in convex analysis, and it is best explained by starting from a general perspective. Consider a set-valued mapping $T : I\!R^n \rightrightarrows I\!R^n$; monotonicity of T isn't related to an "ordering," but refers rather to having

$$y \in T(x),\ y' \in T(x') \quad \Longrightarrow \quad (x' - x)\cdot(y' - y) \geq 0. \tag{2.4}$$

If $T : I\!R^n \rightrightarrows I\!R^n$ is monotone and there is no $T' : I\!R^n \rightrightarrows I\!R^n$ such that $\operatorname{gph} T' \supset \operatorname{gph} T$, $\operatorname{gph} T' \neq \operatorname{gph} T$, then T is *maximal* monotone. Monotonicity and maximal monotonicity are obviously preserved in passing from T to its (set-valued) inverse T^{-1}, where $x \in T^{-1}(y)$ if and only if $y \in T(x)$. When T is maximal monotone, its effective domain $\operatorname{dom} T = \{\, x \,|\, T(x) \neq \emptyset \,\}$ is almost convex in the sense of lying between some convex set D and its closure. That holds also then for the effective range of T, which is $\operatorname{dom} T^{-1}$.

A special case of monotonicity is that of T being single-valued, i.e., just a function f. The condition in (2.4) reduces then to $(x' - x)\cdot(f(x') - f(x)) \geq 0$. If f is monotone with all of $I\!R^n$ as its domain, and also continuous, then f is maximal monotone. In the case of f being differentiable with Jacobian matrices $\nabla f(x) \in I\!R^{n\times n}$, monotonicity comes down to the property that

$$w{\cdot}\nabla f(x)w \geq 0 \quad \text{for all } x \text{ and } w, \tag{2.5}$$

or in other words, that the Jacobians are alway positive semidefinite. Beware, however, that this is positive semidefiniteness of a matrix J which need not be symmetric; it depends only in the symmetric part $\frac{1}{2}[J + J^T]$ of J. These facts and many others are laid out in [11, Chapter 12].

Along with monotonicity there is the property of *strong* monotonicity, where the inequality in (2.4) is strengthened to $(x' - x){\cdot}(y' - y) \geq \mu||x' - x||^2$ for some $\mu > 0$, with $||\cdot||$ denoting the Euclidean norm. In the case of $T = f$, this would be $(x' - x){\cdot}(f(x') - f(x)) \geq \mu||x' - x||^2$, and it would be guaranteed for differentiable f by having $w{\cdot}\nabla f(x)w \geq \mu||w||^2$ in (2.5). The advantage of T being both maximal monotone and strongly monotone with constant $\mu > 0$ is that T^{-1} is then a single-valued Lipschitz continuous mapping from all of $I\!R^n$ into itself with Lipschitz constant μ^{-1}.

The crucial set-valued example for our purposes here is that

$$\begin{array}{l}\text{if } T = f + N_C \text{ with } C \text{ nonempty closed convex and } f : C \to I\!R^n \\ \text{monotone and continuous, then } T = f + N_C \text{ is maximal monotone.} \\ \text{If } f \text{ is strongly monotone on } C\text{, then } T = f + N_C \text{ is strongly monotone.}\end{array} \tag{2.6}$$

In particular, for $f = 0$ we have the maximal monotonicity of N_C itself. On the other hand, for $f = I$ (identity mapping) we have the maximal monotonicity of $(I + N_C)^{-1}$, which is the projection mapping P_C. Because $f = I$ is strongly monotone with constant $\mu = 1$, P_C is Lipschitz continuous with Lipschitz constant 1, as already noted.

The connection with the existence of solutions to variational inequalities is that having $-f(x)$ in $N_C(x)$ corresponds to having 0 in $(f + N_C)(x)$, hence

$$-f(x) \in N_C(x) \iff x \in (f + N_C)^{-1}(0). \tag{2.7}$$

Under the maximal monotonicity in (2.6), this corresponds to 0 belonging to the almost convex set $\operatorname{dom}(f + N_C)^{-1}$, for which various special criteria can be devised. Here is one of them, coming in part from [11, 12.52].

Background Theorem 2 (existence via monotonicity)**.** *Let C be a nonempty, closed, convex set, and let $f : C \to I\!R^n$ be monotone and continuous. For any $a \in C$ and $r > 0$,*

$$\begin{array}{l}\textit{if } f(x){\cdot}(x - a) > 0 \textit{ for all } x \in C \textit{ with } ||x - a|| > r\textit{, then} \\ \textit{there exists } x \textit{ with } ||x - a|| \leq r \textit{ such that } -f(x) \in N_C(x).\end{array} \tag{2.8}$$

More particularly, for f strongly monotone on C, a unique solution exists to $-f(x) \in N_C(x)$.

Making use of this in the context of Nash equilibrium requires us to assess the monotonicity of the mapping f in (1.9). This is easiest when f is continuously differentiable, which corresponds to the functions g_i being twice continuously differentiable. The Jacobians of f then have the form

$$\nabla f(x) = \Big[J_{ij}(x)\Big]_{i=1,j=1}^{m,m} \quad \text{for} \quad J_{ij}(x) = \nabla^2_{x_i x_j} g_i(x_1, \ldots, x_m), \tag{2.9}$$

where $\nabla^2 g_i$ denotes the Hessian of g_i (the matrix of its second-partial derivatives). The monotonicity criterion in (2.5) comes out in terms of $w = (w_1, \ldots, w_m)$ as

$$\sum_{i=1^m} w_i \cdot \nabla^2_{x_i x_i} g_i(x) w_i \;\geq\; -\sum_{i=1,j=1,i\neq j}^{m,m} w_i \cdot \frac{\nabla^2_{x_i x_j} g_i(x) + \nabla^2_{x_i x_j} g_j(x)}{2} w_j, \tag{2.10}$$

with strong monotonicity having a term $\mu \sum_{i=1}^m ||w_i||^2$ added on the right. In the context of g_i being convex with respect to x_i, the matrices $\nabla^2 g_i(x)$ are of course positive semidefinite. Strong convexity over C would make them be positive definite. The right side of (2.10), on the other hand, represents interactions between the different players. This yields good insight:

> f in (1.9) will be strongly montone on C if the functions g_i are strongly convex in x_i and the effects of the cross term interactions between players are sufficiently small. (2.11)

Application Theorem 2. *If the functions g_i are strongly convex in x_i and the sum on the right of (2.10) remains sufficiently small with respect to the positive sum on the left, for $w \neq 0$, then regardless of whether C is bounded the variational inequality (1.8)–(1.9) for Nash equilibrium will have a unique solution $x = (x_1, \ldots, x_m)$.*

3 Results on Parametric Stability

Our attention focuses now on a parameterized variational inequality,

$$-f(p,x) \in N_C(x) \;\text{ for }\; x \in C \subset I\!R^n,\; p \in I\!R^d \tag{3.1}$$

and its set-valued solution mapping

$$S : p\{\, x \mid -f(p,x) \in N_C(x) \,\}, \tag{3.2}$$

having effective domain in $I\!R^d$ and effective range in C. The right way to think about extending the classical implicit function theorem from $f(p,x) = 0$ to the variational inequality framework is through the notion of a "localization" of S relative to a pair $(\bar{p}, \bar{x})$ with $\bar{x} \in S(\bar{p})$. Suppose there are neighborhoods P of $\bar{p}$ and X of $\bar{x}$ such that

$$\operatorname{gph} S \cap [P \times X] \;\text{ is the graph of a single-valued mapping }\; s : P \to C. \tag{3.3}$$

Then s is called a *single-valued localization* of S at $\bar{p}$ for $\bar{x}$. We will be interested not only in the existence of s on the basis of assumptions on f, C and $(\bar{p}, \bar{x})$, but also in whether s can be Lipschitz continuous. In the case of Lipschitz continuity, we speak of *parametric stability* of the variational inequality. Observe that parametric stability at $\bar{p}$ for $\bar{x}$ guarantees in particular the existence, for each p in the neighborhood P of $\bar{p}$, of a *unique* solution x belonging to the neighborhood X of $\bar{x}$ in C.

For Nash equilibrium this seems to be an entirely new topic. There, f has the form (1.9), so that in contemplating a parameterization we have in mind the pattern that

$$f(p, x_1, \dots, x_m) = (\nabla_{x_1} g_1(p, x_1, x_{-1}), \cdots, \nabla_{x_m} g_m(p, x_m, x_{-m})). \tag{3.4}$$

In other words, we are raising the question of how an equilibrium, or generalized equilibrium, might respond to shifts in parameters in the cost fnctions g_i. This widens the territory for equilibrium studies far beyond just existence and demonstrates at the same time the advantages of passing from fixed-point theory alone to approaches that utilize special structure in the context of variational analysis.

We present in this direction a result, which is far from the most general possible for extraction from the theory in [1, Chapter 2], but does serve to illustrate the ideas.

Background Theorem 3 (a criterion for parametric stability). *Let $\bar{x} \in S(\bar{p})$ for the variational inequality (3.1) and its solution mapping (3.2), with f continuously differentiable, and let l be the linearization of f with respect to x at $(\bar{p}, \bar{x})$:*

$$l(x) = f(\bar{p}, \bar{x}) + \nabla_x f(\bar{p}, \bar{x})(x - \bar{x}). \tag{3.5}$$

Suppose that $(l + N_C)^{-1}$ is single-valued and Lipschitz continuous. Then parametric stability holds for the parameterized variational inequality (3.1) at $\bar{p}$ for $\bar{x}$.

Application Theorem 3. *Consider the parameterized version of the variational inequality (1.8)–(1.9) indicated by (3.4) under the assumption that the functions $g_i(p, x_1, \dots, x_n)$ are twice continuously differentiable. Let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$ be a solution for $\bar{p}$, and suppose that the (nonsymmetric) matrix*

$$\nabla_x f(\bar{p}, \bar{x}) = \Big[J_{ij}(\bar{p}, \bar{x})\Big]_{i=1,j=1}^{m,m} \text{ with } J_{ij}(\bar{p}, \bar{x}) = \nabla^2_{x_i x_j} g_i(\bar{p}, \bar{x}_1, \dots, \bar{x}_m), \tag{3.6}$$

is positive definite. Then parametric stability holds: the solution mapping S has a single-valued Lipschitz continuous localization s at $\bar{p}$ for $\bar{x}$.

This result follows by invoking (2.6) for l to get the desired single-valuedness and Lipschitz continuity of $(l + N_C)^{-1}$ in Background Theorem 3.

Not only Lipschitz continuity but also semidifferentiability of the localization s can be investigated, i.e., the existence of directional derivatives

$$Ds(p; p') = \lim_{\varepsilon \searrow 0} \frac{s(p + \varepsilon p') - s(p)}{\varepsilon}. \tag{3.7}$$

Results in [1, Chapter 2] can assure this when C is *polyhedral*. (A closed, convex set C is polyhedral if and only if it is the intersection of a finite collection of hyperplanes and/or closed half-spaces, or in other words, when it can be specified by a system of finitely many linear equation or inequality constraints.)

Although Nash equilibrium has been the target of discussion here, models of *economic* equilibrium have likewise already been articulated in the variational inequality framework: see [4], [5], [6], and most recently [2] and [7]. The latter two draw heavily on the theory parametric stability.

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