

Calibrating Risk Preferences with Generalized CAPM Based on Mixed CVaR Deviation

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The generalized Capital Asset Pricing Model based on mixed CVaR deviation is used for calibrating risk preferences of investors protecting investments in S&P500 by means of options. The corresponding new generalized beta is designed to capture tail performance of S&P500 returns. Calibration is done by extracting information about risk preferences from option prices on S&P500. Actual market option prices are matched with the estimated prices from the pricing equation based on the generalized beta. In addition to the risk preferences, an optimal allocation to a portfolio of options for the considered group of investors is calculated.

Key words: Capital Asset Pricing Model (CAPM), master fund, risk preferences, value-at-risk (VaR), conditional value-at-risk (CVaR), mixed CVaR deviation

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1 INTRODUCTION

The Capital Asset Pricing Model (CAPM, see Sharpe (1964), Linther (1965), Mossin (1966), Treynor (1961, 1962)) after its foundation in the 1960's became one of the most popular methodologies for estimation of returns of securities and explanation of their combined behavior. This model assumes that all investors want to minimize risk of their investments, and all investors measure risk by the standard deviation of return. The model implies that all optimal portfolios are mixtures of the Market Fund and risk free instrument. The Market Fund is commonly approximated by some stock market index, such as S&P500.

An important practical application of the CAPM model is the possibility to calculate hedged portfolios uncorrelated with the market. To reduce the risk of a portfolio, an investor can include additional securities and hedge market risk. The risk of the portfolio in terms of CAPM model is measured by "beta". The value of beta for every security or portfolio is proportional to the correlation between its return and market return. This follows from the assumption that investors have risk attitudes expressed with the standard deviation (volatility). The hedging is designed to reduce portfolio beta with the idea to protect the portfolio in case of a market downturn. However, beta is just a scaled correlation with the market and there is no guarantee that hedges will cover losses during sharp downturns, because the protection works only on average for the frequently observed market movements. Recent credit crises have shown that hedges have tendencies to perform very poorly when they are most needed in extreme market conditions. The classical hedging procedures based on standard beta set up a defence around the mean of the loss distribution, but fail in the tails. This deficiency has led to multiple attempts to improve the CAPM.

One approach to CAPM improvement is to include additional factors in the model. For example, Fama and French (1996) added to the asset return linear regression model two additional terms: the difference between the return on a portfolio of small stocks and the return on a portfolio of large stocks, and the difference between the return on a portfolio of high-book-to-market stocks and the return on a portfolio of low-book-to-market stocks.

The second approach is to find alternative risk measures, which may more precisely represent risk preferences of investors. For instance, Konno and Yamazaki (1991) applied an \mathcal{L}^1 risk model (based on mean absolute deviation) to the portfolio optimization problem with NIKKEI 225 stocks. Their approach led to linear programming instead of quadratic programming in the classical Markowitz's model, but computational results weren't significantly better. Further re-

search has been focused on risk measures more correctly accounting for losses. For example, Estrada (2004) applied downside semideviation-based CAPM for estimating returns of Internet company stocks during the Internet bubble crisis. Downside semideviation calculates only for the losses underperforming the mean of returns. Nevertheless, semideviation, similarly to standard deviation, doesn't pay special attention to extreme losses, associated with heavy tails. Sortino and Forsey (1996) also point out that downside deviation does not provide complete information needed to manage risk.

A much more advanced line of research is considered in papers of Rockafellar et al. (2006a, 2006b, 2007). The assumption here is that there are different groups of investors having different risk preferences. The generalized Capital Asset Pricing Model (GCAPM), see Rockafellar et al. (2006a, 2006b, 2007), proposes that there is a collection of deviation measures, representing risk preferences of the corresponding groups of investors. These deviation measures substitute for the standard deviation of the classical theory. With the generalized pricing formula following from GCAPM one can estimate the deviation measure for a specific group of investors from market prices. This is done by considering parametric classes of deviation measures and calibrating parameters of these measure. The GCAPM provides an alternative to the classical CAPM measure of systematic risk, so-called "generalized beta". Similarly to classical beta, the generalized beta can be used in portfolio optimization for hedging purposes.

This paper considers the class of so-called mixed CVaR deviations, having several attractive properties. First, different terms in the mixed CVaR deviation give credit to different parts of the distribution. Therefore, by varying parameters (coefficients), one can approximate various structures of risk preferences. In particular, so-called tail-beta can be built which accounts for heavy tail losses (e.g., losses in the top 5% of the tail distribution). Second, mixed CVaR deviation is a "coherent" deviation measure, and it therefore satisfies a number of desired mathematical properties. Third, optimization of problems with mixed CVaR deviation can be done very efficiently. For instance, for discrete distributions, the optimization problems can be reduced to linear programming.

This paper considers a group of investors in the S&P500 Index fund, who manage risk by purchasing European put options. We assume that these investors estimate risks with the mixed CVaR deviations having fixed quantile levels: 50, 75, 85, 95 and 99 percent of the loss distribution. By definition, this mixed CVaR deviation is a weighted combination of average losses exceeding these quantile levels. The weights for CVaRs with the different quantile levels determine a specific instance of the risk measure. The generalized pricing formula and generalized beta for this class

of deviation measures are used in this approach. With market option prices the parameters of the deviation measure are calibrated, thus estimating risk preferences of investors. An optimal allocation of funds between the Index and the options is also calculated. To solve both problems simultaneously, an iterative process with converging approximations of the deviation measure and the optimal portfolio is built.

Several numerical experiments calibrating risk preferences of investors at different time moments were conducted. We have found that the deviation measure, representing investors' risk preferences, has the biggest weight on the $\text{CVaR}_{50\%}$ term, which equals the average loss below median return. On average, about 40% of the weight is assigned to $\text{CVaR}_{85\%}$, $\text{CVaR}_{95\%}$ and $\text{CVaR}_{99\%}$ evaluating heavy-loss scenarios. Experiments also showed that risk preferences tend to change over time reflecting investors' opinions about the state of market.

We are not the first who attempted to extract risk preferences from option prices. It is a common knowlengne that option prices convey risk neutral probability distribution. Some studies, such as Ait-Sahalia and Lo (2000), Jackwerth (2000), Bliss and Panigirtzoglou (2004), contain various approaches to extracting risk preferences in the form of utility function by comparing objective (or statistical) probability density function with risk neutral probability density function, estimated from option prices. In this paper risk preferences are expressed in the form of deviation measure, thus making it impossible to compare results with previous studies. We believe, however, that a wide range of applicability of generalized CAPM framework make our results being useful in a greater variety of applications in practical finance.

The paper is structured as follows. Section 2 recalls the necessary background, describes the assumptions of the model, provides the main definitions and statements, and presents the derivation of the generalized pricing formula. Section 3 contains description of the case study. Section 4 presents the results of the case study. The conclusion section provides several ideas for further research that can be performed in this area.

2 DESCRIPTION OF THE APPROACH

2.1 Generalized CAPM Background

In the classical Markovitz portfolio theory (see Markovitz (1952)) all investors are mean-variance optimizers. Contrary to the classical approach, consider now a group of investors who form their

portfolios by solving optimization problems of the following type:

$$P(\Delta) \quad \min_{\substack{x_0 r_0 + x^T E r \geq r_0 + \Delta \\ x_0 + x^T e = 1}} \mathcal{D}(x_0 r_0 + x^T r) ,$$

where \mathcal{D} is some measure of deviation (not necessarily standard deviation), r_0 denotes the risk-free rate of return, r is a column vector of (uncertain) rates of return on available securities, and e is a column-vector of ones. Problem $P(\Delta)$ minimizes deviation of the portfolio return subject to a constraint on its expected return and the budget constraint. Different investors within the considered group may demand different excess return Δ . Unlike classical theory, instead of variance or standard deviation, investors measure risk with their generalized deviation measure \mathcal{D} . According to the definition in Rockafellar et al. (2006a), a functional $\mathcal{D} : \mathcal{L}^2 \rightarrow [0, \infty]$ is a deviation measure if it satisfies the following axioms:

- (D1) $\mathcal{D}(X + C) = \mathcal{D}(X)$ for all X and constants C .
- (D2) $\mathcal{D}(0) = 0$ and $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for all X and all $\lambda > 0$.
- (D3) $\mathcal{D}(X + Y) \leq \mathcal{D}(X) + \mathcal{D}(Y)$ for all X and Y .
- (D4) $\mathcal{D}(X) \geq 0$ for all X with $\mathcal{D} > 0$ for nonconstant X .

Similarly to Rockafellar et al. (2006b), we can eliminate x_0 , which is equal to $1 - x^T e$:

$$P^0(\Delta) \quad \min_{x^T(Er - r_0 e) \geq \Delta} \mathcal{D}(x^T r) .$$

A pair (x_0, x) is an optimal solution to $P(\Delta)$ if and only if x is an optimal solution to $P^0(\Delta)$ and $x_0 = 1 - x^T e$. Theorem 1 in Rockafellar et al. (2006c) shows that an optimal solution to $P^0(\Delta)$ exists, if deviation measure \mathcal{D} satisfies the property (D5):

- (D5) $\{X \mid \mathcal{D}(X) \leq C\}$ is closed for every constant C .

A deviation measure \mathcal{D} satisfying this property is called *lower semicontinuous*. For further results we will also require an additional property, called *lower range dominance*:

- (D6) $\mathcal{D}(X) \leq EX - \inf X$ for all X .

In this paper we consider only lower semicontinuous, lower range-dominated deviation measures.

Rockafellar et al. (2006b) show that if a group of investors solves problems $P(\Delta)$, the optimal investment policy is characterized by the Generalized One-Fund Theorem (Theorem 2 in that paper). According to the result, the optimal portfolios have the following general structure:

$$x^\Delta = \Delta x^1 , \quad x_0^\Delta = 1 - \Delta(x^1)^T e ,$$

where x_0^Δ is the investment in risk free instrument, x^Δ is a vector of positions in risky instruments, and (x_0^1, x^1) is an optimal solution to $P(\Delta)$, with $\Delta = 1$. Portfolio (x_0^1, x^1) is called a *basic fund*. It is important to note that, in full generality, $(x^1)^T E r$ could be positive, negative, or equal 0 (threshold case), although for most situations the positive case should prevail.

According to the same paper, a portfolio $x^\mathcal{D}$ is called a *master fund* of positive (negative) type if $(x^\mathcal{D})^T e = 1$ ($(x^\mathcal{D})^T e = -1$), and $x^\mathcal{D}$ is a solution to $P^0(\Delta^*)$ for some $\Delta^* > 0$. From the definition follows that master fund contains only risky securities, with no investment in risk free security. With this definition, the generalized One-Fund Theorem can be reformulated in terms of the master fund. Below we present its formulation as it was given in Rockafellar et al. (2006b).

Theorem 1 (One-Fund Theorem in Master Fund Form). *Suppose a master fund of positive (negative) type exists, furnished by an $x^\mathcal{D}$ -portfolio that yields an expected return $r_0 + \Delta^*$ for some $\Delta^* > 0$. Then, for any $\Delta > 0$, the solution for the portfolio problem $P(\Delta)$ is obtained by investing the positive amount Δ^*/Δ (negative amount $-\Delta^*/\Delta$) in the master fund, and the amount $1 - (\Delta^*/\Delta)$ (amount $1 + (\Delta^*/\Delta) > 1$) in the risk free instrument.*

From Theorem 1 follows that for every investor in the considered group, the optimal portfolio can be expressed as a combination of investment in the master fund, and investment in the risk free security.

Rockafellar et al. (2007) extends the framework to the case with multiple groups of investors. Every group of investors i , where $i = 1, \dots, I$, solves the problem $P(\Delta)$ with their own deviation measure \mathcal{D}_i . It was shown that there exists a market equilibrium, and optimal policy for every group of investors is defined by the Generalized One-Fund Theorem. In this framework investors from different groups may have different master funds. From now on we assume that a generalized deviation measure represents risk preferences of a given group of investors.

Consider a particular group of investors with risk preferences defined by a generalized deviation \mathcal{D} . If their master fund is known, the corresponding Generalized CAPM relations can be formulated. The exact relation depends on the type of the master fund. Let r_M denote the rate of return of the master fund. Then

$$r_M = (x^\mathcal{D})^T r = \sum_{j=1}^n x_j^\mathcal{D} r_j ,$$

where the random variables r_j stand for rates of return on the securities in the considered economy, $x_j^\mathcal{D}$ are the corresponding weights of these securities in the master fund, and $\sum_{j=1}^n x_j^\mathcal{D} = 1$.

The generalized beta of a security j , replacing the classical beta, is defined as follows:

$$\beta_j = \frac{\text{cov}(-r_j, Q_M^D)}{\mathcal{D}(r_M)} . \quad (1)$$

In this formula Q_M^D denotes the *risk identifier* corresponding to the master fund, taken from the risk envelope corresponding to the deviation measure \mathcal{D} . Examples of risk identifiers for specific deviation measures will be presented in the next subsection.

Rockafellar et al. (2006c) derives optimality conditions for problems of minimizing a generalized deviation of the return on a portfolio. The optimality conditions are applied to characterize three types of master funds. Theorem 5 in that paper, presented below, formulates the optimality conditions in the form of CAPM-like relations.

Theorem 2. *Let the deviation \mathcal{D} be finite and continuous.*

Case 1. *An x^D -portfolio with $x_1^D + \dots + x_n^D = 1$ is a master fund of positive type, if and only if $Er_M > r_0$ and $Er_j - r_0 = \beta_j (Er_M - r_0)$ for all j .*

Case 2. *An x^D -portfolio with $x_1^D + \dots + x_n^D = -1$ is a master fund of negative type, if and only if $Er_M > -r_0$ and $Er_j - r_0 = \beta_j (Er_M + r_0)$ for all j .*

Case 3. *An x^D -portfolio with $x_1^D + \dots + x_n^D = 0$ is a master fund of threshold type, if and only if $Er_M > 0$ and $Er_j - r_0 = \beta_j Er_M$ for all j .*

From now on we call the conditions specified in the Theorem 2 the Generalized CAPM (GCAPM) relations.

2.2 Pricing Formulas in GCAPM

Let $r_j = \zeta_j/\pi_j - 1$, where ζ_j is the payoff or the future price of security j , and π_j is the price of this security today.

Similarly to classical theory, pricing formulas can be derived from the Generalized CAPM relations, as it was done in Sarykalin (2008). The following Lemma presents these pricing formulas both in *certainty equivalent* form, and *risk adjusted* form.

Lemma 1.

Case 1. *If the master fund is of positive type, then*

$$\pi_j = \frac{E\zeta_j}{1 + r_0 + \beta_j (Er_M^D - r_0)} = \frac{1}{1 + r_0} \left(E\zeta_j + \frac{\text{cov}(\zeta_j, Q_M^D)}{\mathcal{D}(r_M^D)} (Er_M^D - r_0) \right) .$$

Case 2. *If the master fund is of negative type, then*

$$\pi_j = \frac{E\zeta_j}{1 + r_0 + \beta_j (Er_M^D + r_0)} = \frac{1}{1 + r_0} \left(E\zeta_j + \frac{\text{cov}(\zeta_j, Q_M^D)}{\mathcal{D}(r_M^D)} (Er_M^D + r_0) \right) .$$

Case 3. If the master fund is of threshold type, then

$$\pi_j = \frac{E\zeta_j}{1 + r_0 + \beta_j Er_M^{\mathcal{D}}} = \frac{1}{1 + r_0} \left(E\zeta_j + \frac{\text{cov}(\zeta_j, Q_M^{\mathcal{D}})}{\mathcal{D}(r_M^{\mathcal{D}})} Er_M^{\mathcal{D}} \right) .$$

See proof in Appendix.

Rockafellar et al. (2007) proved the existence of equilibrium for multiple groups of investors optimizing their portfolios according to their individual risk preferences, and therefore the pricing formulas in Lemma 1 hold true for all groups of investors.

2.3 Mixed CVaR Deviation and Betas

Conditional Value-at-Risk has been studied by various researchers, sometimes under different names (expected shortfall, Tail-VaR). We will use notations from Rockafellar and Uryasev (2002). For more details on stochastic optimization with CVaR-type functions see Uryasev (2000), Rockafellar and Uryasev (2000, 2002), Krockmal et al. (2002), Krockmal et al. (2006), Sarykalin et al (2008).

Suppose random variable X determines some financial outcome, future wealth or return on investment. By definition, Value-at-Risk at level α is the α -quantile of the distribution of $(-X)$:

$$\text{VaR}_\alpha(X) = q_\alpha(-X) = -q_{1-\alpha}(X) = -\inf\{z \mid F_X(z) > 1 - \alpha\} ,$$

where F_X denotes the probability distribution function of random variable X .

Conditional Value-at-Risk for continuous distributions equals the expected loss exceeding VaR:

$$\text{CVaR}_\alpha(X) = -E[X \mid X \leq -\text{VaR}_\alpha(X)] .$$

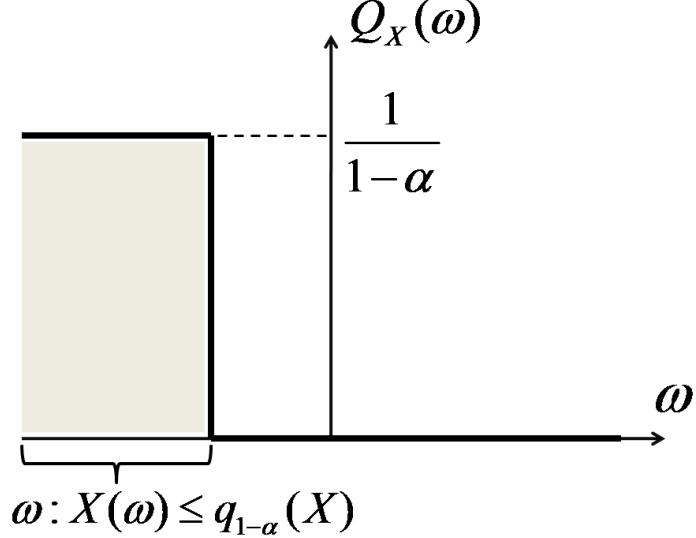
This formula underlies the name of CVaR as conditional expectation. For the general case the definition is more complicated, and can be found, for example, in Rockafellar, Uryasev (2000). Conditional Value-at-Risk deviation is defined as follows:

$$\text{CVaR}_\alpha^\Delta(X) = \text{CVaR}_\alpha(X - EX) .$$

As follows from Theorem 1 in Rockafellar, et al. (2006a), there exists a one-to-one correspondence between lower-semicontinuous, lower range-dominated deviation measures \mathcal{D} and convex positive risk envelopes \mathcal{Q} :

$$\mathcal{Q} = \left\{ Q \mid Q \geq 0, EQ = 1, EXQ \geq EX - \mathcal{D}(X) \text{ for all } X \right\} ,$$

Figure 1: CVaR-type Risk Identifier for a Given Outcome Variable X



$$\mathcal{D}(X) = EX - \inf_{Q \in \mathcal{Q}} EXQ . \quad (2)$$

The random variable $Q_X \in \mathcal{Q}$, for which $\mathcal{D}(X) = EX - EXQ_X$, is called the *risk identifier*, associated by \mathcal{D} with X .

For a given X and CVaR deviation, the risk identifier can be viewed as a step function, with a jump at the quantile point:

$$Q_X(\omega) = \frac{1}{1-\alpha} \mathbb{1}\{X(\omega) \leq q_{1-\alpha}(X)\} , \quad (3)$$

where ω denotes an elementary event on the probability space, and $\mathbb{1}\{\text{condition}\}$ is an indicator function, defined on the same probability space, which equals 1 if *condition* is true, and 0 otherwise. Figure 1 illustrates the structure of the CVaR risk identifier, corresponding to some random outcome X . For simplicity, the probability space, assumed in the figure, is the space of values of the random variable X .

If the group of investors constructs its master fund by minimizing CVaR deviation, and all r_j are continuously distributed, beta for security j has the following expression, derived in Rockafellar, et al. (2006c):

$$\beta_j = \frac{\text{cov}(-r_j, Q_M)}{\text{CVaR}_\alpha^\Delta(r_M)} = \frac{E[Er_j - r_j | r_M \leq -\text{VaR}_\alpha(r_M)]}{E[Er_M - r_M | r_M \leq -\text{VaR}_\alpha(r_M)]} . \quad (4)$$

Classical beta is a scaled covariance between the security and the market. The new beta focuses on events corresponding to big losses in the master fund. For big α ($\alpha > 0.8$), this expression can be called tail-beta.

The following two theorems lead to the definition of mixed CVaR deviation, which is used for the purpose of this paper.

Theorem 3. *Let deviation measure \mathcal{D}_l correspond to risk envelope \mathcal{Q}_l for $l = 1, \dots, L$. If deviation measure \mathcal{D} is a convex combination of the deviation measures \mathcal{D}_l :*

$$\mathcal{D} = \sum_{l=1}^L \lambda_l \mathcal{D}_l, \text{ with } \lambda_l \geq 0, \sum_{l=1}^L \lambda_l = 1,$$

then \mathcal{D} corresponds to risk envelope $\mathcal{Q} = \sum_{l=1}^L \lambda_l \mathcal{Q}_l$.

See proof in Appendix.

The following theorem presents a formula for the beta corresponding to a deviation measure that is a convex combination of a finite number of deviation measures.

Theorem 4. *If the master fund M , corresponding to the deviation measure \mathcal{D} , is known, and \mathcal{D} is a convex combination of a finite number of deviation measures \mathcal{D}_l , $l = 1, \dots, L$:*

$$\mathcal{D} = \lambda_1 \mathcal{D}_1 + \dots + \lambda_L \mathcal{D}_L,$$

then

$$\beta_j = \frac{\lambda_1 \text{cov}(-r_j, Q_M^{\mathcal{D}_1}) + \dots + \lambda_L \text{cov}(-r_j, Q_M^{\mathcal{D}_L})}{\lambda_1 \mathcal{D}_1(r_M) + \dots + \lambda_L \mathcal{D}_L(r_M)},$$

where $Q_M^{\mathcal{D}_l}$ is a risk identifier of master fund return corresponding to deviation measure \mathcal{D}_l .

See proof in Appendix.

For a given set of confidence levels $\alpha = (\alpha_1, \dots, \alpha_L)$ and coefficients $\lambda = (\lambda_1, \dots, \lambda_L)$ such that $\lambda_l \geq 0$ for all $l = 1, \dots, L$, and $\sum_{l=1}^L \lambda_l = 1$, *mixed CVaR deviation* $\text{CVaR}_{\alpha; \lambda}^{\Delta}$ is defined in the following way:

$$\text{CVaR}_{\alpha; \lambda}^{\Delta}(X) = \lambda_1 \text{CVaR}_{\alpha_1}^{\Delta}(X) + \dots + \lambda_L \text{CVaR}_{\alpha_L}^{\Delta}(X). \quad (5)$$

Corollary 1. *If $\mathcal{D} = \text{CVaR}_{\alpha; \lambda}^{\Delta}$, where $\alpha = (\alpha_1, \dots, \alpha_L)$ and $\lambda = (\lambda_1, \dots, \lambda_L)$, and distribution of r_M is continuous, then*

$$\beta_j = \frac{\lambda_1 E[Er_j - r_j | r_M \leq -\text{VaR}_{\alpha_1}(r_M)] + \dots + \lambda_L E[Er_j - r_j | r_M \leq -\text{VaR}_{\alpha_L}(r_M)]}{\lambda_1 \text{CVaR}_{\alpha_1}^{\Delta}(r_M) + \dots + \lambda_L \text{CVaR}_{\alpha_L}^{\Delta}(r_M)}. \quad (6)$$

See proof in Appendix.

2.4 Risk Preferences of Investors Protecting Index Positions by Options

How can risk preferences of a particular group of investors be extracted from market prices?

According to GCAPM, risk preferences of a group of investors are represented by a deviation measure. This deviation measure determines the structure of a master fund. For a known deviation measure and a master fund, a risk identifier for the master fund can be specified. If a joint distribution of payoffs for securities is also known, then one can calculate the betas for securities, and then calculate GCAPM prices for these securities. Therefore, according to GCAPM, the deviation measure and the distribution of payoff determine the price for each security. To estimate the deviation measure, having expected returns on securities and market prices, one can find a candidate deviation measure \mathcal{D} for which the GCAPM prices are equal to the market prices.

Consider the group of investors who buy European put options on the S&P500 Index with time to maturity 1 month to hedge their investments in the Index. Put options provide insurance against the fall of the index below strike prices K_j , where $j = 1, \dots, J$ is index of strike price. Alternatively to standard deviation, which measures the magnitude of possible price changes in both directions, Conditional Value-at-Risk deviation measures the average loss for the α worst-case scenarios. We consider a mixed CVaR deviation, which is a weighted combination of several CVaR deviations with appropriate weights, to capture different parts of the tail of the distribution.

Suppose that the master fund for this group of investors is already known, and their risk preferences can be expressed by a mixed CVaR deviation defined by formula (5). Extraction of risk preferences is then reduced to estimation of the coefficients $\lambda_1, \dots, \lambda_L$ in the mixed CVaR deviation.

To estimate the coefficients $\lambda_1, \dots, \lambda_L$ we will use GCAPM formulas, presented in Theorem 2. Let P_K denote the market price of a put option with strike price K and 1 month to maturity, ζ_K denote its (random) monthly return, and $r_K = \frac{\zeta_K}{P_K} - 1$ denote its (random) return in one month. Let r_M be (random) return on the master fund, with its distribution at this moment assumed to be known; r_0 is the return on a risk free security. If market prices are exactly equal to GCAPM prices, and the deviation measure is a mixed CVaR deviation with fixed confidence levels $\alpha_1, \dots, \alpha_L$, then the set of coefficients $\lambda_1, \dots, \lambda_L$ is a solution to the following system of equations:

$$Er_K - r_0 = \beta_K(\lambda) (Er_M - r_0), \quad K = K_1, K_2, \dots, K_{J-1}, K_J, \quad (7)$$

where

$$\beta_K(\lambda) = \frac{\sum_{l=1}^L \lambda_l E[Er_K - r_K | r_M \leq -\text{VaR}_{\alpha_l}(r_M)]}{\sum_{l=1}^L \lambda_l \text{CVaR}_{\alpha_l}^\Delta(r_M)}, \quad (8)$$

$$\sum_{l=1}^L \lambda_l = 1, \quad (9)$$

and

$$\lambda_l \geq 0, \quad l = 1, \dots, L. \quad (10)$$

Equations (7) are GCAPM formulas from Theorem 2, applied to market prices P_K of put options with strike prices $K = K_1, \dots, K_J$, and random payoffs ζ_K . Systematic risk measure $\beta(\lambda)$ is expressed through the coefficients λ_l according to Corollary 1.

By multiplying both sides of equation (7) by $\sum_{l=1}^L \lambda_l \text{CVaR}_{\alpha_l}^\Delta(r_M)$ and taking into account (8), we get

$$(Er_K - r_0) \sum_{l=1}^L \lambda_l \text{CVaR}_{\alpha_l}^\Delta(r_M) = (Er_M - r_0) \sum_{l=1}^L \lambda_l E[Er_K - r_K | r_M \leq -\text{VaR}_{\alpha_l}(r_M)], \\ K = K_1, \dots, K_J,$$

or, equivalently,

$$\sum_{l=1}^L ((Er_K - r_0) \text{CVaR}_{\alpha_l}^\Delta(r_M) - (Er_M - r_0) E[Er_K - r_K | r_M \leq -\text{VaR}_{\alpha_l}(r_M)]) \lambda_l = 0, \\ K = K_1, \dots, K_J. \quad (11)$$

If the number of equations (options with different strike prices K) is greater than the number of variables, then system of equations (11) may not have a solution. For this reason we replace the equations (11) with alternative expressions with error terms e_K :

$$\sum_{l=1}^L ((Er_K - r_0) \text{CVaR}_{\alpha_l}^\Delta(r_M) - (Er_M - r_0) E[Er_K - r_K | r_M \leq -\text{VaR}_{\alpha_l}(r_M)]) \lambda_l = e_K, \\ K = K_1, \dots, K_J. \quad (12)$$

We estimate the coefficients $\lambda_1, \dots, \lambda_L$ as the optimal point to the following optimization problem minimizing a norm of vector $(e_{K_1}, \dots, e_{K_J})$:

$$\min_{\lambda_1, \dots, \lambda_L} \|(e_{K_1}, \dots, e_{K_J})\| \quad (13)$$

subject to

$$\sum_{l=1}^L ((Er_K - r_0) \text{CVaR}_{\alpha_l}^\Delta(r_M) - (Er_M - r_0) E[Er_K - r_K | r_M \leq -\text{VaR}_{\alpha_l}(r_M)]) \lambda_l = e_K, \\ K = K_1, \dots, K_J. \quad (14)$$

$$K = K_1, \dots, K_J,$$

$$\lambda_l \geq 0, \quad l = 1, \dots, L, \quad \sum_{l=1}^L \lambda_l = 1. \quad (15)$$

In the above formulation $\|\cdot\|$ is some norm. We consider two norms:

\mathcal{L}^1 -norm:

$$\|(e_{K_1}, \dots, e_{K_J})\|_1 = \frac{1}{J} \sum_{j=1}^J |e_{K_j}|, \quad (16)$$

and \mathcal{L}^2 -norm:

$$\|(e_{K_1}, \dots, e_{K_J})\|_2 = \sqrt{\frac{1}{J} \sum_{j=1}^J e_{K_j}^2}.$$

2.5 Master Fund of Investors Protecting Index Positions by Options

What is the master fund for investors protecting their positions in the S&P500 Index buying put options on S&P500 Index? We determine the set of securities which they invest in by choosing some range of S&P500 put options with 1 month to maturity, and the Index itself.

If the risk preferences of this group of investors were known, we would be able to determine their master fund by solving the problem $P^0(\Delta)$ for $x = (x_I, x_{K_1}, \dots, x_{K_J})$:

$$\text{minimize } \mathcal{D}(x^T r) \text{ subject to } x^T [Er - r_0 e] \geq \Delta, \quad (17)$$

and then normalizing the vector x so that sum of components equals 1.

But in this framework risk preferences are unknown, except that we assume they are expressed with a mixed CVaR deviation. In order to find both the coefficients $\lambda_1, \dots, \lambda_L$ in the mixed CVaR deviation measure and the master fund simultaneously, we build an iterative process with successive approximations of these objects. As an initial approximation of the master fund we selected the Index, because the investors certainly have the Index in their optimal portfolio.

Assuming initially that the master fund is the Index, we calibrated the deviation coefficients according to the algorithm described in the previous subsection. Then, having the measure, we calculated the optimal portfolio for investors with such risk preferences by solving (17). This was the second approximation of the master fund. After that, we recalibrated coefficients in the deviation measure, than found the next approximation of the master fund, and so on. This process may either converge, or go around a loop. To enforce algorithm convergence we set a friction to changes in λ and x . That is, if on iteration n the solution to the constrained regression (13) equals $\tilde{\lambda}^n = (\tilde{\lambda}_1^n, \dots, \tilde{\lambda}_L^n)$, then we set $\lambda^n = \lambda^{n-1} + f^n(\tilde{\lambda}^n - \lambda^{n-1})$, where $f^n = n^{-1}$. Similarly, $x^n = x^{n-1} + f^n(\tilde{x}^n - x^{n-1})$.

3 Case Study Data and Algorithm

We did 10 experiments of estimating risk preferences, each for a separate date (henceforth: date of experiment). Dates were chosen with intervals approximately 1/2 year in such a way that each date is 1 month prior to a January or July option expiration date. Thus, the first experiment was done for 12/22/2004, the second for 6/16/2005, etc. For every experiment we used historical data starting from year 1994, up to the date of experiment. For every date we used a set of S&P500 put options with strike prices K_1, \dots, K_J , where K_J is a strike price of the at-the-money option. We chose K_1 as a minimum strike price, for which the following two conditions are satisfied: 1) Starting with the option K_1 , ASK prices P_{ask,K_j} and BID prices P_{bid,K_j} are strictly increasing, i.e. $P_{\text{ask},K_{j+1}} > P_{\text{ask},K_j}$ and $P_{\text{bid},K_{j+1}} > P_{\text{bid},K_j}$; 2) BID price for option with strike K_1 is at least 1 dollar, i.e. $P_{\text{bid},K_1} \geq \1 . We further define option market price P_K as an average of BID and ASK prices:

$$P_K = \frac{1}{2} (P_{\text{ask},K} + P_{\text{bid},K}) .$$

For every experiment we designed a set of scenarios of monthly Index rates of return in the following way. For every trading day s from historical observations we recorded the value $\widetilde{r}_I^{(s)} = \frac{I_{s+21}}{I_s} - 1$, where I_s is the Index value on day s .

We further calculate implied volatility σ of the at-the-money option (the option with strike price K_J), and the value

$$\widehat{\sigma} = \text{standard deviation}(\widetilde{r}_I^{(s)}) .$$

Next, every scenario return was modified as follows:

$$r_I^{(s)} = \frac{\sigma}{\widehat{\sigma}} \left(\widetilde{r}_I^{(s)} - E\widetilde{r}_I \right) + r_0 + \xi\sigma , \quad (18)$$

where the value for the monthly risk-free rate of return r_0 was selected equal to 0.4125%, and $\xi > 0$ is some parameter. The new scenarios will have volatility equal to the volatility σ of the at-the-money options, and expected return $r_0 + \xi\sigma$. In formula (18) the value of ξ was chosen such that expected returns on options are negative. We selected $\xi = \frac{1}{3}$. Numerical experiments showed that results are not very sensitive to the selection of the parameter ξ .

Suppose, for modeling purposes, that the investors' preferences are described by a mixed CVaR deviation with confidence levels 50%, 75%, 85%, 95% and 99%:

$$\mathcal{D}(\lambda) = \sum_{l=1}^L \lambda_l \text{CVaR}_{\alpha_l}^\Delta , \quad (19)$$

Table 1: Case Study Data

Description	Notation	Value				
Date of experiment		12/22/04	6/16/05	12/21/05	6/22/06	12/20/06
Current Index value	I_0	1207.62	1208.79	1264.50	1246.73	1426.28
Lowest option strike price	K_{min}	1135	1140	1180	1150	1325
Highest option strike price	K_{max}	1210	1210	1265	1250	1425
Number of scenarios (days)	S	2744	2865	2996	3121	3247
Description	Notation	Value				
Date of experiment		6/21/07	12/19/07	6/19/08	12/17/08	6/18/09
Current Index value	I_0	1513.83	1454.95	1339.08	907.40	914.94
Lowest option strike price	K_{min}	1355	1240	1190	700	800
Highest option strike price	K_{max}	1515	1455	1340	910	915
Number of scenarios (days)	S	3371	3497	3622	3748	3873
Description	Notation	Value				
Risk-free monthly interest rate	r_0	0.4125%				
Number of terms in mixed CVaR deviation	L	5				
Confidence level 1	α_1	99%				
Confidence level 2	α_2	95%				
Confidence level 3	α_3	85%				
Confidence level 4	α_4	75%				
Confidence level 5	α_5	50%				

where

$$L = 5, \quad \alpha_1 = 99\%, \quad \alpha_2 = 95\%, \quad \alpha_3 = 85\%, \quad \alpha_4 = 75\%, \quad \alpha_5 = 50\%,$$

and

$$\lambda_l \geq 0, \quad \sum_{l=1}^5 \lambda_l = 1. \quad (20)$$

The input data for the case study are listed in Table 1.

Multiple tests demonstrated that the results do not depend significantly on the choice of norm in the optimization problem (13). Further in this paper we present results obtained using \mathcal{L}^1 -norm.

The following algorithm was used to estimate risk preferences from the option prices.

Algorithm

Step 1. Assign initial iteration number $n := 0$. Calculate scenarios indexed by $s = 1, \dots, S$ for payoffs and net returns of put options according to the formula:

$$\zeta_K^{(s)} = \max(0, K - I_0(1 + r_I^{(s)})) , \quad r_K^{(s)} = \frac{\zeta_K^{(s)}}{P_K} - 1 ,$$

where $K = K_1, \dots, K_J$ are the strike prices, and I_0 is the Index value at time of the experiment.

Step 2. Let the vector $x = (x_I, x_{K_1}, x_{K_2}, \dots, x_{K_{J-1}}, x_{K_J})$ denote the weights in the master fund, where x_I is a fraction invested in index, and x_{K_1}, \dots, x_{K_J} are the weights invested in the options. Assign: $x := (1, 0, 0, \dots, 0, 0)$, so that the master fund equals the Index.

Step 3. Calculate scenarios indexed by $s = 1, \dots, S$ for the master fund return:

$$r_M^{(s)} = x_I r_I^{(s)} + \sum_{j=1}^J x_{K_j} r_{K_j}^{(s)},$$

and the following values:

$$E[Er_K - r_K | r_M \leq -\text{VaR}_{\alpha_l}(r_M)] \text{ for all } K = K_1, \dots, K_J \text{ and } l = 1, \dots, L,$$

and

$$\text{CVaR}_{\alpha_l}^\Delta(r_M) \text{ for all } l = 1, \dots, L.$$

Step 4. Build the design matrix for the constrained regression (13)-(15).

Step 5. Find a set of coefficients $\tilde{\lambda}_l$ by solving constrained regression (13)-(15) with \mathcal{L}^1 norm, given by equation (16). If $n > 0$ then assign $\mu := \lambda$ else assign $\mu := 0$. Then assign $\lambda :=$

Table 2: Deviation Measure Calibration Results

Results for 10 experiments. Each experiment gives mixed CVaR deviation expressing risk preferences of investors (investments S&P500 and put options on S&P500 Index with 1 month expiration). Coefficient λ_{α_l} is a weight for $\text{CVaR}_{\alpha_l}^\Delta(r_M)$ in mixed CVaR deviation.

Date of Experiment	$\lambda_{99\%}$	$\lambda_{95\%}$	$\lambda_{85\%}$	$\lambda_{75\%}$	$\lambda_{50\%}$
12/22/2004	0.051	0.000	0.000	0.000	0.949
6/16/2005	0.115	0.000	0.000	0.000	0.885
12/21/2005	0.054	0.000	0.000	0.000	0.946
6/22/2006	0.260	0.000	0.000	0.000	0.740
12/20/2006	0.127	0.000	0.000	0.000	0.873
6/21/2007	0.286	0.000	0.000	0.000	0.714
12/19/2007	0.000	0.011	0.370	0.000	0.619
6/19/2008	0.083	0.061	0.000	0.000	0.857
12/17/2008	0.000	0.000	0.000	0.000	1.000
6/18/2009	0.008	0.181	0.116	0.024	0.672
mean values	0.098	0.025	0.049	0.002	0.825
standard deviation	0.102	0.058	0.119	0.008	0.130

$$\mu + \frac{1}{n+1}(\tilde{\lambda} - \mu).$$

Step 6. Find the optimal portfolio by solving problem (17) and normalizing solution, and assign the optimal portfolio weights to the vector w . Assign $v := x$. Then assign $x := v + \frac{1}{n+1}(w - v)$.

Step 7. Assign $n := n + 1$.

Step 8. If $n = 10$ then Stop. Vector λ gives coefficients in mixed CVaR deviation, and vector x gives master fund.

Step 9. Goto Step 3.

As a stopping criteria we selected number of iterations, with a maximum number of iterations equal 10. Numerical experiments showed that algorithm converges quite fast, and 10 iterations is usually sufficient for the algorithm convergence.

4 Case Study Computational Results

Computations were performed on the laptop PC with Intel Core2 Duo CPU P8800 2.66GHz, 4GB RAM and Windows 7, 64-bit. Algorithm, described in previous section, was programmed in MATLAB. Both optimization problems, the constrained regression and CVaR portfolio optimization, on each iteration of the algorithm were solved with AORDA Portfolio Safeguard decision support tool (see, American Optimal Decisions (2009)). For one date the computational time is around 15 seconds.

The set of coefficients in the mixed CVaR deviation for every date is presented in Table 2. This table shows that in all experiments the obtained deviation measure has the biggest weight on $\text{CVaR}_{50\%}$, and smaller weights on either $\text{CVaR}_{85\%}$, $\text{CVaR}_{95\%}$, or $\text{CVaR}_{99\%}$. This can be interpreted as that investors are concerned both with the middle part of the loss distribution, expressed with $\text{CVaR}_{50\%}$, and extreme losses expressed with $\text{CVaR}_{85\%}$, $\text{CVaR}_{95\%}$, or $\text{CVaR}_{99\%}$.

Master fund converges to a portfolio consisting of Index and some options. The number of contracts of options in the final portfolio is always equal to the number of units of Index; this confirms that investors in S&P500 Index protect their full Index position with options.

Let us denote by π_K the GCAPM option prices, calculated with pricing formulas in Lemma 1, using calculated mixed CVaR deviation measure and the master fund. We mapped the obtained option prices π_K and the market prices P_K into the implied volatility scale. This mapping is defined by the Black-Scholes formula in implicit form. The graphs of π_K and P_K for 10 dates in the scale of monthly implied volatilities are presented in Figures 2, 3 and 4. All graphs show that the GCAPM prices are close to market prices, except for the graph for 6/21/2007.

Figure 2: Calculated Prices and Market Prices in the Scale of Implied Volatilities

The charts present results of 10 experiments for different dates. We assume that risk preferences of investors, who buy put options to protect their investments in S&P500 Index, are expressed by mixed CVaR deviation. In each experiment we use market option prices to calculate coefficients in the mixed CVaR deviation, and master fund, presenting the optimal portfolio for the considered group of investors. We then use generalized pricing formulas to calculate option prices. Each graph presents market prices and calculated prices, mapped to the scale of monthly implied volatilities. This mapping is the inverse to the Black-Scholes formula.

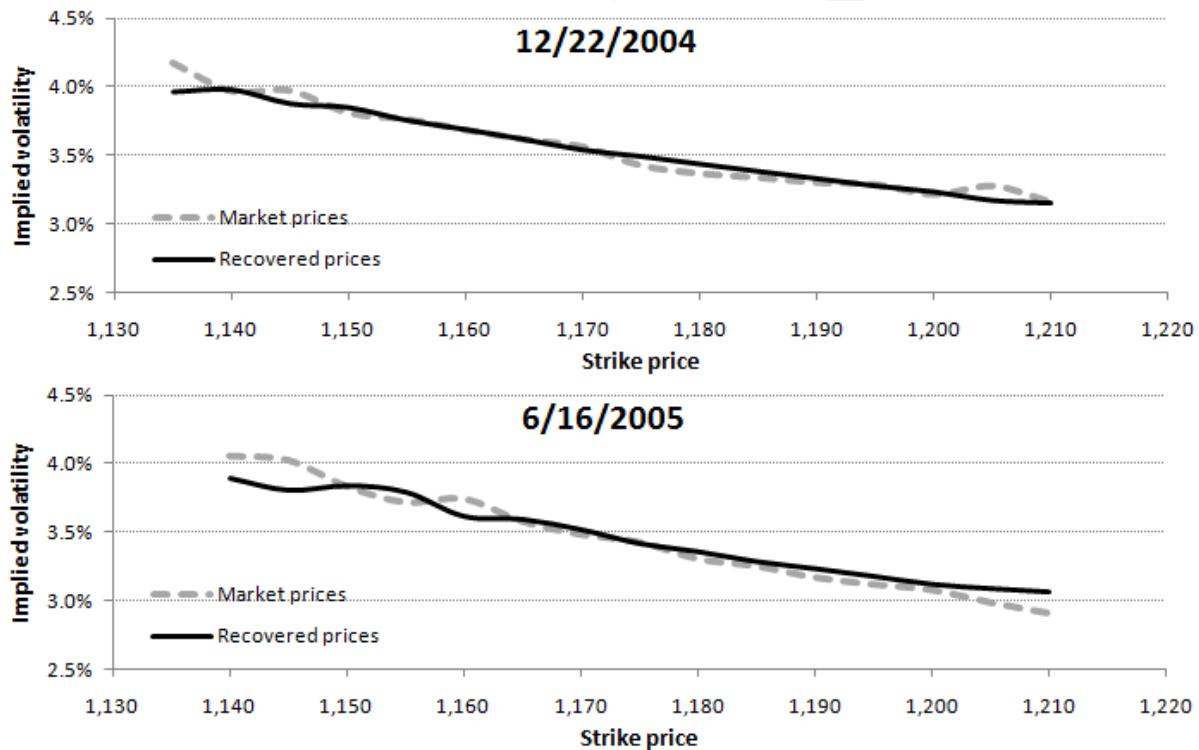


Figure 3: Calculated Prices and Market Prices in the Scale of Implied Volatilities (continued)

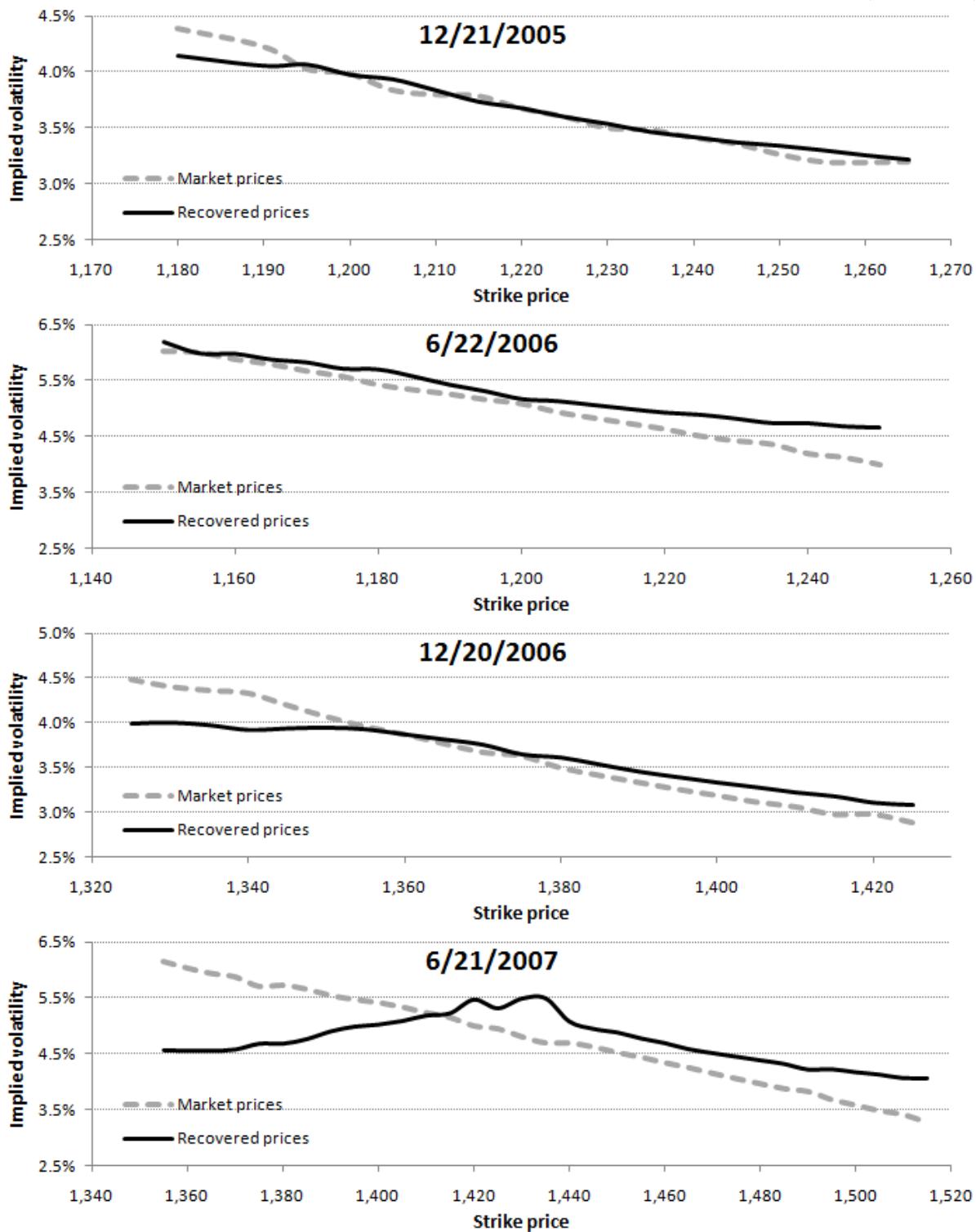


Figure 4: Calculated Prices and Market Prices in the Scale of Implied Volatilities (continued)

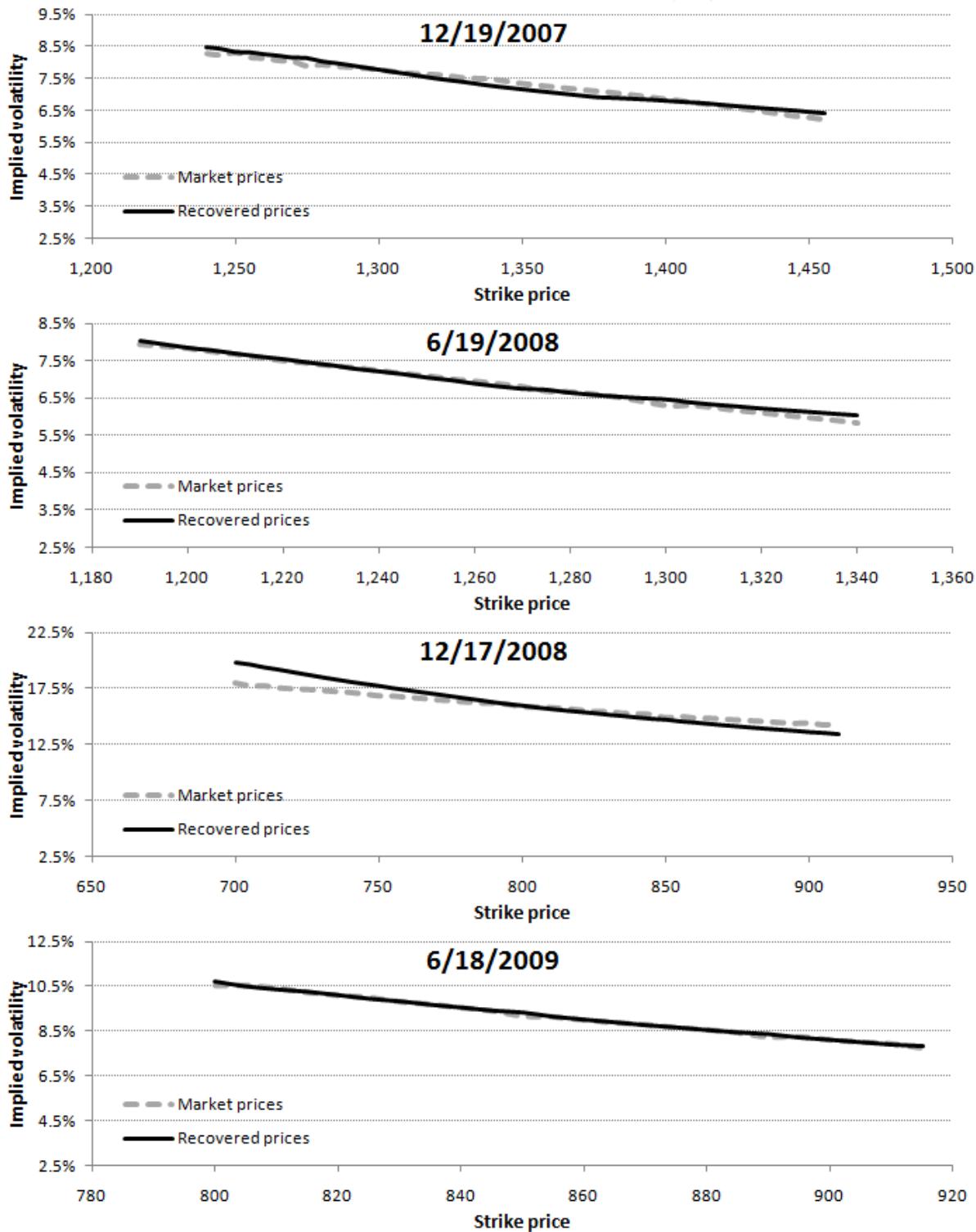


Figure 5: S&P500 Value and Risk Aversity Dynamics

The graphs compare S&P500 Index dynamics with the changes of risk preferences of investors. We assume that risk preferences of investors, who buy put options to protect their investments in S&P500 Index, are expressed by mixed CVaR deviation with confidence levels 50%, 75%, 85%, 95% and 99%. We conduct 10 experiments for different dates, and in each experiment we use market option prices to calculate coefficients in the mixed CVaR deviation, and master fund, presenting the optimal portfolio for the considered group of investors. Two curves: S&P500 index and $1 - \lambda_{50\%}$, or, equivalently, $\lambda_{75\%} + \lambda_{85\%} + \lambda_{95\%} + \lambda_{99\%}$. The curve $1 - \lambda_{50\%}$ reflects investors' apprehension about potential tail losses, and their tendency to hedge the risk of extreme losses. Investors concern about tail losses was increasing until the beginning of the market downturn, what demonstrates that the downturn was anticipated by the considered group of investors. After that, market participants poorly anticipated market trends. For example, in December 2008 the value of $1 - \lambda_{50\%}$ was 0, what can be interpreted as investor's belief that S&P500 reached its bottom, and there was no intention to hedge their investments in Index against losses. Nevertheless, in the beginning of 2009, market has fallen even further.

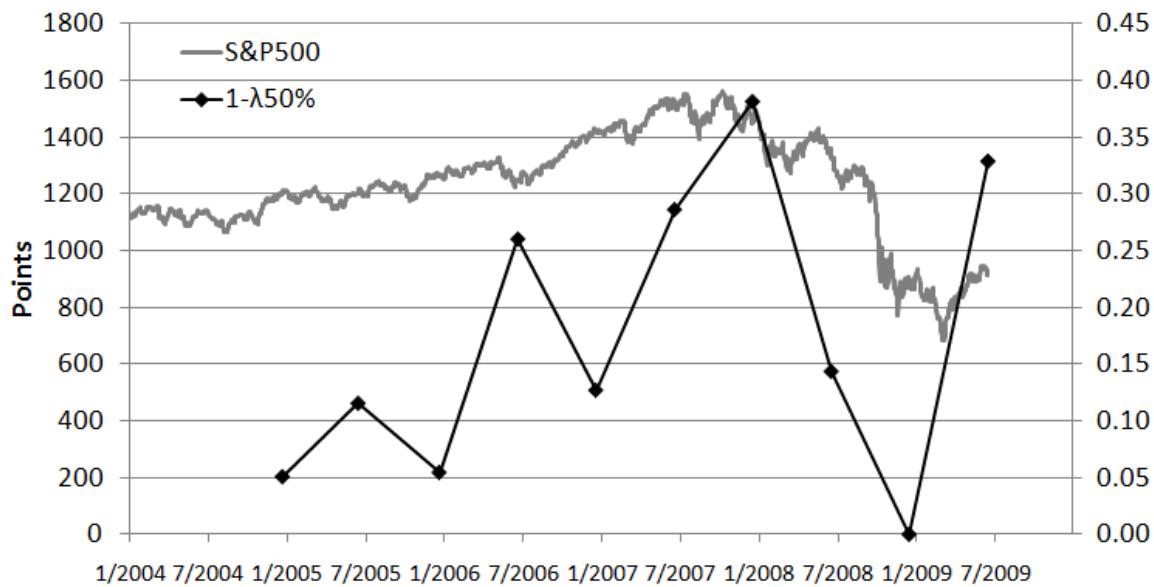


Figure 5 compares dynamics of the value $\eta = 1 - \lambda_{50\%}$ on 10 dates of experiment with S&P500 dynamics. High values of η indicate greater investors' apprehension about potential tail losses and greater inclination to hedge their investments in S&P500. It can be seen that risk preferences were relatively stable until 2008, when the distressed period began. It can also be seen that market participants didn't always properly anticipate future market trends. In particular, in December 2008 the value of η was 0, which indicated that market wrongly anticipated that Index reached its bottom and will go up. Nevertheless, 2009 started with further decline in the Index.

5 Conclusion

We have described a new technique of expressing risk preferences with generalized deviation measures. We have presented a method for extracting risk preferences from market option prices using these formulas. We have conducted a case study for extracting risk preferences of investors protecting their positions in S&P500 Index by buying put options on this Index.

We extracted risk preferences for 10 dates with 6 month intervals, and expressed them with mixed CVaR deviation. Results demonstrate that investors are concerned both with the middle part of the loss distribution, expressed with $\text{CVaR}_{50\%}$, and extreme losses expressed with $\text{CVaR}_{85\%}$, $\text{CVaR}_{95\%}$, or $\text{CVaR}_{99\%}$. Exact proportions vary, reflecting investors anticipation of high or low returns.

An important application of the theory is that it provides an alternative, more broad view on systematic risk, compared to the classical CAPM based on standard deviation. Similarly to the classical CAPM, we calculated new betas for securities, which measure systematic risk in a different way, capturing tail behavior of a master fund return. These betas can be used for hedging against tail losses, which occur in down market.

Potential applications go beyond identifying risk preferences of considered investors. An investor can express risk attitudes in the form of a deviation measure, and then recalculate betas for securities using this deviation measure. With these betas the investor can build a portfolio hedged according to his risk preferences.

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Appendix

Below we present proofs of statements formulated in the article. For the reader's convenience, we repeat formulations before every proof.

Lemma 1.

Case 1. If the master fund is of positive type, then

$$\pi_j = \frac{E\zeta_j}{1 + r_0 + \beta_j (Er_M^D - r_0)} = \frac{1}{1 + r_0} \left(E\zeta_j + \frac{\text{cov}(\zeta_j, Q_M^D)}{\mathcal{D}(r_M^D)} (Er_M^D - r_0) \right) .$$

Case 2. If the master fund is of negative type, then

$$\pi_j = \frac{E\zeta_j}{1 + r_0 + \beta_j (Er_M^D + r_0)} = \frac{1}{1 + r_0} \left(E\zeta_j + \frac{\text{cov}(\zeta_j, Q_M^D)}{\mathcal{D}(r_M^D)} (Er_M^D + r_0) \right) .$$

Case 3. If the master fund is of threshold type, then

$$\pi_j = \frac{E\zeta_j}{1 + r_0 + \beta_j Er_M^D} = \frac{1}{1 + r_0} \left(E\zeta_j + \frac{\text{cov}(\zeta_j, Q_M^D)}{\mathcal{D}(r_M^D)} Er_M^D \right) .$$

Proof of Lemma 1. Proofs for all three cases are similar, so we present the proof only for a master fund of positive type. According to the GCAPM relation specified in Case 1,

$$Er_j - r_0 = \beta_j (Er_M^D - r_0) .$$

Since $r_j = \zeta_j/\pi_j - 1$, then $Er_j = E\zeta_j/\pi_j - 1$, from which we get

$$\frac{E\zeta_j}{\pi_j} - (1 + r_0) = \beta_j (Er_M^D - r_0) . \quad (21)$$

This yields the Generalized Capital Asset Pricing Formula in the certainty equivalent form:

$$\pi_j = \frac{E\zeta_j}{1 + r_0 + \beta_j (Er_M^D - r_0)} . \quad (22)$$

Using the expression for beta (1) we can also write

$$\pi_j = \frac{E\zeta_j}{1 + r_0 + \frac{\text{cov}(-r_j, Q_M^D)}{\mathcal{D}(r_M^D)} (Er_M^D - r_0)} . \quad (23)$$

By multiplying both sides of the equality (21) by π_j , we get

$$E\zeta_j - \pi_j(r_0 + 1) = \pi_j \beta_j (Er_M^D - r_0) . \quad (24)$$

With expression for beta (1) we get

$$\begin{aligned}\pi_j \beta_j &= \pi_j \frac{\text{cov}(-r_j, Q_M^D)}{\mathcal{D}(r_M^D)} = \frac{\text{cov}(-\pi_j r_j, Q_M^D)}{\mathcal{D}(r_M^D)} \\ &= \frac{\text{cov}(-\pi_j(1+r_j) + \pi_j, Q_M^D)}{\mathcal{D}(r_M^D)} = \frac{\text{cov}(-\pi_j(1+r_j), Q_M^D)}{\mathcal{D}(r_M^D)} + \frac{\text{cov}(\pi_j, Q_M^D)}{\mathcal{D}(r_M^D)}.\end{aligned}\quad (25)$$

Here π_j is a constant, consequently the second term in the last sum equals 0. Therefore,

$$\pi_j \beta_j = \frac{\text{cov}(-\pi_j(1+r_j), Q_M^D)}{\mathcal{D}(r_M^D)}.$$

Since $\pi_j(1+r_j) = \zeta_j$, then

$$\pi_j \beta_j = -\frac{\text{cov}(\zeta_j, Q_M^D)}{\mathcal{D}(r_M^D)}.$$

Substituting expression for $\pi_j \beta_j$ into (24) gives:

$$E\zeta_j - \pi_j(r_0 + 1) = -\frac{\text{cov}(\zeta_j, Q_M^D)}{\mathcal{D}(r_M^D)} (Er_M^D - r_0).$$

The last equation implies the risk-adjusted form of the pricing formula:

$$\pi_j = \frac{1}{1+r_0} \left(E\zeta_j + \frac{\text{cov}(\zeta_j, Q_M^D)}{\mathcal{D}(r_M^D)} (Er_M^D - r_0) \right). \quad (26)$$

◇

Theorem 3. Let deviation measure \mathcal{D}_l correspond to risk envelope \mathcal{Q}_l for $l = 1, \dots, L$. If deviation measure \mathcal{D} is a convex combination of the deviation measures \mathcal{D}_l :

$$\mathcal{D} = \sum_{l=1}^L \lambda_l \mathcal{D}_l, \text{ with } \lambda_l \geq 0, \sum_{l=1}^L \lambda_l = 1,$$

then \mathcal{D} corresponds to risk envelope $\mathcal{Q} = \sum_{l=1}^L \lambda_l \mathcal{Q}_l$.

Proof of Theorem 3. With formula (2) we get:

$$\begin{aligned}\mathcal{D}(X) &= \sum_{l=1}^L \lambda_l \mathcal{D}_l(X) = EX - \sum_{l=1}^L \lambda_l \inf_{Q \in \mathcal{Q}_l} EXQ = \\ &= EX - \inf_{(Q_1, \dots, Q_L) \in (\mathcal{Q}_1, \dots, \mathcal{Q}_L)} EX \left(\sum_{l=1}^L \lambda_l Q_l \right) = EX - \inf_{Q \in \sum_{l=1}^L \lambda_l \mathcal{Q}_l} EXQ.\end{aligned}\quad (27)$$

◇

Theorem 4. If the master fund M , corresponding to the deviation measure \mathcal{D} , is known, and \mathcal{D} is a convex combination of a finite number of deviation measures \mathcal{D}_l , $l = 1, \dots, L$:

$$\mathcal{D} = \lambda_1 \mathcal{D}_1 + \dots + \lambda_L \mathcal{D}_L,$$

then

$$\beta_j = \frac{\lambda_1 \text{cov}(-r_j, Q_M^{\mathcal{D}_1}) + \dots + \lambda_L \text{cov}(-r_j, Q_M^{\mathcal{D}_L})}{\lambda_1 \mathcal{D}_1(r_M) + \dots + \lambda_L \mathcal{D}_L(r_M)} ,$$

where $Q_M^{\mathcal{D}_l}$ is a risk identifier of master fund return, corresponding to deviation measure \mathcal{D}_l .

Proof of Theorem 4. From Theorem 3 follows:

$$\begin{aligned} \beta_j &= \frac{\text{cov}(-r_j, Q_M^{\mathcal{D}})}{\mathcal{D}} = \frac{\text{cov}(-r_j, \lambda_1 Q_M^{\mathcal{D}_1} + \dots + \lambda_L Q_M^{\mathcal{D}_L})}{\lambda_1 \mathcal{D}_1(r_M) + \dots + \lambda_L \mathcal{D}_L(r_M)} = \\ &= \frac{\lambda_1 \text{cov}(-r_j, Q_M^{\mathcal{D}_1}) + \dots + \lambda_L \text{cov}(-r_j, Q_M^{\mathcal{D}_L})}{\lambda_1 \mathcal{D}_1(r_M) + \dots + \lambda_L \mathcal{D}_L(r_M)} . \end{aligned} \quad (28)$$

Next,

$$\text{cov}(-r_j, Q_M^{\mathcal{D}_l}) = E(Er_j - r_j)(Q_M^{\mathcal{D}_l} - EQ_M^{\mathcal{D}_l}) . \quad (29)$$

According to the definition of risk envelope, $EQ_M^{\mathcal{D}_l} = 1$. Therefore, from (29) we have:

$$\text{cov}(-r_j, Q_M^{\mathcal{D}_l}) = E(Er_j - r_j)(Q_M^{\mathcal{D}_l} - 1) = E(Er_j - r_j)Q_M^{\mathcal{D}_l} - E(Er_j - r_j) = E(Er_j - r_j)Q_M^{\mathcal{D}_l} .$$

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Corollary 1. If $\mathcal{D} = \text{CVaR}_{\alpha, \lambda}^\Delta$, where $\alpha = (\alpha_1, \dots, \lambda_L)$ and $\lambda = (\lambda_1, \dots, \lambda_L)$, and distribution of r_M is continuous, then

$$\beta_j = \frac{\lambda_1 E[Er_j - r_j | r_M \leq -\text{VaR}_{\alpha_1}(r_M)] + \dots + \lambda_L E[Er_j - r_j | r_M \leq -\text{VaR}_{\alpha_L}(r_M)]}{\lambda_1 \text{CVaR}_{\alpha_1}^\Delta(r_M) + \dots + \lambda_L \text{CVaR}_{\alpha_L}^\Delta(r_M)} . \quad (30)$$

Proof of Corollary 1. For $\mathcal{D}_l = \text{CVaR}_{\alpha_l}^\Delta$, according to (3):

$$Q_M^{\mathcal{D}_l}(\omega) = \frac{1}{1 - \alpha_l} \mathbb{I}\left\{r_M(\omega) \leq -\text{VaR}_{\alpha_l}(r_M)\right\} .$$

Then,

$$\begin{aligned} \text{cov}(-r_j, Q_M^{\mathcal{D}_l}) &= E(Er_j - r_j) \frac{1}{1 - \alpha_l} \mathbb{I}\left\{r_M(\omega) \leq -\text{VaR}_{\alpha_l}(r_M)\right\} = \\ &= E[Er_j - r_j | r_M \leq -\text{VaR}_{\alpha_l}(r_M)] . \end{aligned} \quad (31)$$

Substituting expression for $\text{cov}(-r_j, Q_M^{\mathcal{D}_l})$ and expression for mixed CVaR deviation (5) into (28) gives:

$$\beta_j = \frac{\lambda_1 E[Er_j - r_j | r_M \leq -\text{VaR}_{\alpha_1}(r_M)] + \dots + \lambda_L E[Er_j - r_j | r_M \leq -\text{VaR}_{\alpha_L}(r_M)]}{\lambda_1 \text{CVaR}_{\alpha_1}^\Delta(r_M) + \dots + \lambda_L \text{CVaR}_{\alpha_L}^\Delta(r_M)} .$$

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