A TIME-EMBEDDED APPROACH TO ECONOMIC EQUILIBRIUM WITH INCOMPLETE FINANCIAL MARKETS

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Abstract

In the models of multi-stage equilibrium with uncertain financial markets that have so far been formulated in extension of the classical Walrasian model with only a single stage, each state is completely isolated in its activity. If there is production, it ends in the state in which it begins. Goods that are not consumed within a state merely perish. Nothing can carry over from one period to the next like money might, and comparisons between the units of account in different states may be problematical.

This paper furnishes a two-stage model in which money assists as a special good which agents like to retain instead of consume. Holding money influences utility through pleasures of wealth and in safeguarding against completely unforseen events. The agents' planning is able in this way also to reflect continuing expectations of the future beyond the second stage.

The ability of agents to retain money furthermore has the technical benefit that the survivability conditions needed to establish the existence of equilibrium can reduced to a very simple and yet much more appealing form than has been discerned until now. Endogenous transaction costs on the sellers of financial contracts help in this as well.

Keywords: general economic equilibrium, incomplete financial markets, time embedding, retention of money, transaction costs, ample survivability.

1 Introduction

Many conceptual hurdles must be faced in the modeling of economic equilibrium. Everyone understands that the very idea of equilibrium is an artifical abstraction in the ongoing world of economic activity. Nonetheless, this idea has the potential to help greatly in identifying sources of stability and instability along with the main features of market behavior and the conditions on which they depend.

The basic goal in formulating and analyzing a mathematical model of equilibrium is to determine the extent to which *market prices* exist which will *balance supplies with demands* when the agents in an economy *optimize their activities according to their preferences*. However, models can differ in the kinds of markets and the ranges of activites that they try to encompass, and in their degree of success in capturing economic realities.

The celebrated Walrasian model of equilibrium, placed on a modern mathematical footing by Arrow and Debreu [2], has yielded valuable insights but is very limited in its scope. It is *isolated in time* without any past or future, and is therefore unable deal with the large component of economic behavior concerned with making arrangements beyond the needs and incentives of the immediate present. However, any extension to future planning must confront *uncertainty*, and that raises another set of touchy issues.

Financial markets can assist by furnishing agents with possibilities to buy and sell contracts in the present which affect their circumstances in the future. Exchanges of value between different time periods are then enabled, but it is not realistic to expect such markets to be able to hedge or ensure precisely against *all* conceivable uncertainties. Thus, financial markets must be modeled as *incomplete*. That leads to mathematical obstacles which preclude the straightforward approach of Arrow and Debreu. Moreover, some of the central conclusions drawn from the classical one-stage model, such as the capablility of market prices to achieve Pareto optimality in the distribution of resources, are then called into question.

One of the daunting challenges in financial markets is that the promises incorporated in the contracts to be bought and sold need to *enforced* somehow when they come due. But that entails, in the first place, being sure that the promises are realistic. Future supplies and demands must be taken into account, yet those supplies and demands will depend on the present and future actions of the various agents. How then can there be any knowledge of that on which an individual agent might plan? Indeed, how can prices for the contracts in a financial market be brought into line without some grasp of future market prices for goods, which in turn ought to reflect balances in future supply and demand?

The availability of information about the future is thus a critical consideration. But it cannot be supposed that the agents have a common view even of likelihoods. Indeed, no representation of the future can ever deserve full trust. Too much will inevitably be unforseen. Moreover, this deeper level uncertainty is bound to affect attitudes of the agents in a Keynesian manner. As explained by Skidelsky [**?**, Chapter 4]: "for Keynes, money was a 'store of value' as well as a means of transactions; it was 'above all a subtle device for linking the present and the future'." The model presented here tries for the first time to account for this in the modeling of equilibrium by allowing utility of an agent to benefit from the retention of money. By treating money as a special kind of good which can store value, it skirts the customary view of the worth of goods being tied solely to their consumption.

2 Background

To set the stage for the new developments that are the focus of this paper, we first recall the one-stage Arrow-Debreu model of economic equilibrium. We concentrate on a basic version in which agents can exchange goods but firms and production activities are left out. In two-stage models with uncertainty, such as we will come to later, the modeling of intertemporal production carried out by "firms" is an especially difficult matter. Progress is being made, cf. [**?**], but production will not be considered in this paper.

Agents, indexed by $i = 1, \ldots, m$, will exchange goods indexed by $l = 1, \ldots, L$ at market prices $p_l \geq 0$. Agent i starts with a goods vector $x_i^0 \in I\!R^L_+$ with components x_{il}^0 and trades it for a goods vector $x_i \in I\!R^L_+$ with components x_{il} subject to the budget constraint $p{\cdot}x_i \leq p{\cdot}x_i^0$, where $p = (p_1, \ldots, p_L)$. In doing so, agent i maximizes the utility $u_i(x_i)$ of x_i. The function u_i is taken here to be defined on all of $I\!R^L$ with values in $[-\infty, \infty)$ and nonempty effective domain

$$U_i = \operatorname{dom} u_i = \{\, x_i \,|\, u_i(x_i) > -\infty \} \subset I\!R^L_+.$$

The constraint $x_i \in U_i$ is implicit then in the maximization; we speak of U_i as the *survival* set for agent i. It is assumed that u_i is nondecreasing and upper semicontinuous (usc); that requires sets of the form $\{\, x_i \,|\, u_i(x_i) \geq \alpha \}$ to be closed for all $\alpha \in I\!R$, but does not necessitate U_i itself being closed.

Definition: classical equilibrium. *A equilibrium in this setting of pure exchange is comprised of a price vector $\bar{p}$ and goods vectors $\bar{x}_i$ for $i = 1, \ldots, m$ such that*

(a) *$\bar{x}_i$ maximizes u_i under the budget dictated by $\bar{p}$, and*
(b) *the markets clear: $\sum_{i=1}^m x_i^0 - \sum_{i=1}^m \bar{x}_i \geq 0,\ \bar{p}{\cdot}\Big[\sum_{i=1}^m x_i^0 - \sum_{i=1}^m \bar{x}_i\Big] = 0$*

The market clearing conditions are written in a convenient vector form. They reflect free disposal and amount to requiring for each good l that $\sum_{i=1}^m x_{il}^0 = \sum_{i=1}^m \bar{x}_{il}$ unless $\bar{p}_l = 0$, in which case $\sum_{i=1}^m x_{il}^0 \geq \sum_{i=1}^m \bar{x}_{il}$ is allowed.

An issue that comes up in connection with establishing the existence of such an equilibrium is the extent to which the agents can, without trading anything, at least *survive*. Plain survival only refers to having $x_i^0 \in U_i$, but existence arguments generally require more than just that. The following standard result, in which U_i is replaced by its interior, $\operatorname{int} U_i$, in the survivability condition, provides a basis for comparisons. It also draws on the concept of *insatiability* of utility in the sense of there being no goods vector at which u_i attains a maximum value over U_i.

Theorem of Arrow and Debreu [2]**.** *For utility functions u_i that are insatiable, quasi-concave and continuous relative to the sets U_i, a classical equilbrium exists under the strong survivability assumption that*

$$x_i^0 \in \operatorname{int} U_i \ \text{ for all agents } i.$$

Although strong survivability is a simple condition that suffices to obtain the existence of equilibrium, it is distressingly restrictive. Insistence on having $x_i^0 \in \operatorname{int} U_i \subset I\!R^L_+$ entails that *each agent must start with a positive quantity of every good.* Some technical ways around this were developed in [2] and elaborated later for instance by Florig [13, 14], but they are quite complicated and their general economic implications are not easy to fathom.

The mathematical reason why strong survivability comes into play is usually understood by economists in the context of properties needed in fixed-point arguments, but it is equally understandable from the perspective of optimization theory. In problems of optimization with constraints, such as the one faced by agent i, some "constraint qualification" is typically required. In this setting of convexity, that would most naturally be the *Slater condition*: there should be an $\hat{x}_i \in U_i$ such that $p{\cdot}x_i < p{\cdot}x_i^0$. Then too a Lagrange multiplier for the budget constraint will be available.

The trouble with this constraint qualification, though, is that the price vector p is not fixed in advance and will only be determined by the equilibrium. The Slater condition might seemingly have to be guaranteed for *every* possible price vector $p \geq 0$, $p \neq 0$. The only way to do that is through strong survivability.

If some of the price components of p may be 0, and agent i happens only to start with goods that are in that case worthless, then the Slater condition would definitely fail. But even with positive prices, it could fail if survivability can only effected with the budget constraint tight.

Questions of about the prices that may prevail in equilibrium are significant from other angles as well. If the price of a particular good comes out as 0, does that mean no agent has any desire for it, or at least no marginal desire for more of it in the equilibrium state? The answer is yes on the basis of Lagrange multiplier analysis of optimality in an agent's utility problem, but that leads back to the need for a constraint qualification supporting the existence of a Lagrange multiplier for the budget constraint. In that case anyway, the positivity of a good's price can be ensured by an assumption on the utility function that prevents marginal utility of that good from ever being 0. This would only have to be true for at least one agent i. Of course, it is desirable to keep in a mode where an agent may only be attracted to a subset of the goods.

Marginal utility may seem to involve derivatives, but they might just be one-sided directional derivatives, which would be available without any differentiability assumption if the functions u_i are concave.

Another feature of the classical model is that the price vector p only provides *relative* prices, which are unscaled, instead of *money* prices. If the price of a particular good is positive in equilibrium, then that good can serve as a *numéraire* having unit price scaled to 1. However, that does not lead to money prices unless "money" can be viewed somehow as a "good." Indeed, money generally fits the picture of being limited in supply and freely tradable in markets, but if included as a "good," would it be able to serve as a numéraire? Not unless its price relative to other goods turned out to be positive — that returns us to the preceding discussion of utility.

The fundamental difficulty is that the classical framework ties utility to consumption *only*, and the marginal utility of "consuming money" in competition with other consuming goods is doubtful, although perhaps salvageable through some interpretation. *We see this largely as an artifact of the peculiar nature of a one-period model of equilibrium without past or future, since much of the economic importance of money revolves around its role in connecting past and future with the present.*

In what follows, we present a time-embedded two-period model of equilibrium which does give value to money. It is a realization of a broader model we have put together in [**?**] with many additional features. By getting to the heart of the main issues, we hope it will help in making the innovations clearer.

3 Incorporating an Uncertain Future

As a general principle, the future is uncertain to agents making decisions in the present, and any two-stage model of equilibrium must reflect that. Agents need to plan for the future despite its uncertainty. They ought to be able to do that through buying and selling *contracts* that promise deliveries or exchanges of goods in the future. However, for that to make sense there must be some way of assessing the present value of such contracts.

Especially, agents should be able to *borrow and lend money* in the present, settling accounts in the future. The interest rates in that operation ought to relate to future purchasing power. On the other hand, the future prices of goods ought to relate to future supply and demand. Thus, future markets in goods inevitably need to be contemplated.

But how should information about the future be generated for making decisions in the present? Can anything be learned from a model with just one stage of future? No matter how many stages, modeling issues will arise over having a fixed time horizon. Our idea is to mitigate the difficulties over this by an expanded view of an agent's wishes and actions. In fact, we see the incorporation of multiple future stages as perhaps detrimental to rather than beneficial to reality by overloading the structure of decisions and undermining the plausibility of the information behind them.

Although contracts for the delivery of all kinds of goods would be good to include in the model, we restrict ourselves here to money alone. That will make the pattern of our development easier to understand. A broader model with additional features is given in our paper [**?**].

The model here has states $s = 0, 1, \ldots, S$, with $s = 0$ standing for the present (at time 0) and $s = 1, \ldots, S$ standing for the different futures being taken into consideration (at time 1) because of uncertainty. The agents can buy and sell goods in all of these states, but they also have money and can pass it from the present to the future. Money is not "consumed"; instead, agents like to *retain* it.

Specifically, agent i, in each state s, consumes a goods vector $x_i(s) \geq 0$ and retains a money amount $m_i(s)$ in the environment of benefiting from incoming endowments $x_i^0(s) \geq 0$ and $m_i^0(s) \geq 0$. The agent's utility function has the form $u_i(x_i, m_i)$ for

$$x_i = (x_i(0), x_i(1), \ldots, x_i(S)) \in (I\!R^L_+)^{1+S}, \qquad m_i = (m_i(0), m_i(1), \ldots, m_i(S)) \in I\!R^{1+S}_+.$$

We allow u_i to take on $-\infty$ and in that way focus implicitly on a survival set

$$U_i = \{\, (x_i, m_i) \,|\, u_i(x_i, m_i) > -\infty\} \subset (I\!R^L_+)^{1+S} \times I\!R^{1+S}_+.$$

We assume that

> u_i is upper semicontinuous, concave and nondecreasing, moreover continuous relative to the sets $\{\, (x_i, m_i) \,|\, u_i(x_i, m_i) \geq c\}$ for $c \in I\!R$.

Those level sets are closed because of the upper semicontinuity, but U_i might not be closed. (Utility might tend to $-\infty$ as the boundary of U_i is approached.) Having u_i be nondecreasing means that $u_i(x_i, m_i) \leq u_i(x_i', m_i')$ when $(x_i, m_i) \leq (x_i', m_i')$. The concavity, in contrast to quasi-concavity, is a slight retreat from the classical setting but it simplifies and enhances the model in many respects. In particular it assists us in working with the following key assumption:

> $u_i(x_i, m_i)$ increases on U_i with respect to each of the components $m_i(s)$.

In other words, retaining money is always attractive to every agent i, although it must nevertheless compete with consumption as represented by the components of the goods vectors $x_i(s)$.

The various goods $l = 1, \ldots, L$ can be bought and sold in markets in every state s. These markets are governed by a *money-denominated* price system

$$p = (p(0), p(1), \ldots, p(S)), \text{ where } p(s) = (p_1(s), \ldots, p_L(s)) \text{ gives the prices in state } s.$$

In state $s = 0$ there is also a financial market in which the agents can participate. The financial market revolves here around contracts in money only, as already mentioned. Such contracts are available in pattens $k = 1, \ldots, K$. A unit of contract k costs q_k in the present and pays $d_k(s)$ in future state s. The payout vectors

$$d_k = (d_k(1), \ldots, d_k(S)), \text{ assumed} \neq (0, \ldots, 0),$$

are known data, but the price vector q must be determined together with the price vectors $p(s)$ in the present and future markets of goods. Examples will be provided shortly.

Agent i buys a (generally fractional) amount $z_{ik}^+ \geq 0$ of contract k and sells an amount $z_{ik}^- \geq 0$, thereby putting together a *portfolio*

$$(z_i^+, z_i^-) \text{ with } z_i^+ = (z_{i1}^+, \ldots, z_{ik}^+) \in I\!R_+^K, \quad z_i^- = (z_{i1}^-, \ldots, z_{ik}^-) \in I\!R_+^K.$$

Simultaneously buying and selling a contract k is not prohibited but will be eliminated in optimality by a transaction cost. We introduce this *endogenously* by supposing that the *selling* of a unit of contract k uses up a goods vector $g_k \in I\!R_+^L$ which will end up costing costing $p(0){\cdot}g_k$. We assume that

g_k has a component > 0 in some good that is *universally attractive* in the present,

by which we mean that every agent's utility u_i increases on U_i with respect to that component of $x_i(0)$. Let

$$d(s) = (d_1(s), \ldots, d_K(s)), \qquad G = \text{the matrix with columns } g_k.$$

The portfolio (z_i^+, z_i^-) of agent i

costs $q{\cdot}[z_i^+ - z_i^-] + p(0){\cdot}Gz_i^-$ in the present, and then
pays $d(s){\cdot}[z_i^+ - z_i^-]$ in the future states $s = 1, \ldots, S$.

Precedent for such exogenous transaction costs can be seen in work of Arrow and Hahn [3] and especially Laitenberger [24], although articulated differently. Here, of course, our attention is limited to contracts that deliver money, but the costs enter more broadly in our paper [**?**] where the contracts can involve delivery of other goods as well.

The optimization problem for agent i takes the form of choosing choosing x_i, m_i and (z_i^+, z_i^-) subject to the budget constraints

$$\begin{aligned} &p(0){\cdot}x_i(0) + m_i(0) + q{\cdot}[z_i^+ - z_i^+] + p(0){\cdot}Gz_i^- \leq p(0){\cdot}x_i^0 + m_i^0(0), \\ &p(s){\cdot}x_i(s) + m_i(s) \leq p(0){\cdot}x_i^0(s) + m_i^0(s) + d(s){\cdot}[z_i^+ - z_i^-] + m_i(0) \text{ for } s > 0. \end{aligned} \tag{1}$$

Note the term $m_i(0)$ in the budget for the future states s. This conforms to the provision that money retained at time 0 is available with certainty at time 1.

Here are two elementary examples of contracts in our picture. For concreteness of language, they are explained with dollars as money. The transaction cost will be expressed by

$$\delta_k(p) = p(0)\cdot q.$$

Example 1: simple lending and borrowing. *As a particular case, contract k could have*

$$d_k(s) = 1 \text{ for all future states } s = 1, \ldots, S.$$

Buying a unit refers then to giving out q_k dollars in the present and always getting back 1 dollar in the future, with

$$\text{interest rate } = \frac{1}{q_k} - 1.$$

Selling refers instead to getting $q_k - \delta_k(p)$ dollars in the present in return for promising to pay back 1 dollar in the future, with

$$\text{interest rate } = \frac{1}{q_k - \delta_k(p)} - 1.$$

The transaction cost thus induces a natural difference in interest rates between lending and borrowing.

Example 2: simple insurance. *As a particular case, contract k could depend on a specific future state $\bar{s}$ and have*

$$d_k(\bar{s}) = 1, \text{ but } d_k(s) = 0 \text{ for all future states } s \neq \bar{s}.$$

Buying a unit of contract k refers then to giving out q_k dollars in the present but getting back 1 dollar only if the future state turns out to be $\bar{s}$, otherwise nothing. Selling refers to acquiring $q_k - \delta_k(p)$ dollars in the present, and only having to pay out 1 dollar if $\bar{s}$ occurs, otherwise nothing..

If simple insurance as in Example 2 is not in the market as one of the contracts k, there it the possibility that it can anyway be replicated by some linear combination of other contracts that are in the market. If this is true for *every* future state $\bar{s}$, the financial market is said to be *complete*. But in reality, financial markets are never complete. This observation has heavily influenced all extensions of equilibrium theory to financial markets after the early proposals of Arrow and Debreu, which did suppose completeness.

4 Existence of equilibrium

The main result about equilibrium in our model with its additional features about money will now be formulated.

Definition: money-supported equilibrium. *This is comprised of elements* $\bar{x}_i$, $\bar{m}_i$, $(\bar{z}_i^+, \bar{z}_i^-)$, *for the agents* $i = 1, \ldots, m$ *along with price vectors* $\bar{p}$ *and* $\bar{q}$ *such that*

(a) $(\bar{x}_i, \bar{m}_i)$ *maximizes* $u_i(x_i, m_i)$ *under the budget constraints (1) coming from* $\bar{p}$ *and* $\bar{q}$,

(b) *the goods markets clear in all states* $s = 0, 1, \ldots, S$:

$$\textstyle\sum_{i=1}^m x_i^0(s) - \sum_{i=1}^m \bar{x}_i(s) \geq 0, \qquad \bar{p}(s)\cdot\Big[\sum_{i=1}^m x_i^0(s) - \sum_{i=1}^m \bar{x}_i(s)\Big] = 0,$$

(c) *the financial markets clear:* $\sum_{i=1}^m \bar{z}_i^+ = \sum_{i=1}^m \bar{z}_i^-$,

(c) *the money supply is respected and money is conserved:*

$$\begin{aligned} &\textstyle\sum_{i=1}^m \bar{m}_i(0) = \sum_{i=1}^m m_i^0(0) \ \textit{ for } s = 0,\\ &\textstyle\sum_{i=1}^m \bar{m}_i(s) = \sum_{i=1}^m [m_i^0(s) + m_i(0)] \ \textit{ for } s > 0. \end{aligned}$$

A virtue of such a money-supported equilibrium is that existence can be ensured without resorting to strong survivability.

Assumption: ample survivability. *The agents* i *could, if they wished, choose elements* $(\hat{x}_i, \hat{m}_i) \in U_i$ *such that*

$$\begin{aligned} &\hat{x}_i(0) \leq x_i^0(0) \ \textit{ but } \hat{m}_i(0) < m_i^0(0) \ \textit{ for } s = 0,\\ &\hat{x}_i(s) \leq x_i^0(s) \ \textit{ and } \hat{m}_i(s) \leq m_i^0(s) \ \textit{ for } s = 1, \ldots, S,\\ &\textstyle\sum_{i=1}^m \hat{x}_i(s) < \sum_{i=1}^m x_i^0(s) \ \textit{ for } s = 0, 1, \ldots, S. \end{aligned} \tag{2}$$

The interpretation of this condition is that the agents would be able to survive, outside of any participation in the markets, *individually* without using all their money in the present, and *collectively* with a surplus remaining in every good. Thus, in contrast to the situation under strong survivability, there is no need for every agent to possess some of every good initially. Nevertheless, a way is opened up for invoking the Slater condition for an agent's budget constraints. This is evident from (2) for the intial budget, but it also follows then effectively for the subsequent budgets because the agent can save some of the initial surplus of money to furnish a buffer in the future.

Existence Theorem. *Under ample survivability and the assumptions on the financial contracts and the agent's utility functions, a money-supported equilibrium exists.*

Insatiability of utility is present here in the assumption that agents are always attracted to retaining money. That attraction is deemed to have a cultural origin based on a history of how money is valued in society and persists in usefulness because of its convenience and liquidity.

We proceed to discuss this new result, which stands as a specialization of the one in our paper [**?**]. The nature of the markets and how they are viewed is a key issue.

The markets here are modeled, according to custom in the economic literature, as brought into line by *Walrasian brokers* who match supply with demand. This avoids having to view transactions as one-on-one and goes back to Walras himself. Radner [29] in 1972 initiated having separate markets in the present and the future. His agents could promise delivery of goods (in so-called *real* contracts), but not "money" (unless that referred only to a numéraire good). An arbitrary number of uncertain future stages were envisioned, not just one, as here. Artificial (exogenous) bounds on contract sales were needed in order to establish the existence of an equilibrium.

In an influencial development in 1985, Cass [4] and Werner [33] independently obtained the existence of an equilibrium in a model in which prices and contracts were all in terms of "money" (so-called *nominal* contracts), but they did not include bounds on money supply. As noted by Magill and Shafer [26], that introduced indeterminancy in which there was no way to relate the money in any one state with the money in another. A proposal of Magill and Quinzii [25] to remedy this required putting the money into a multi-layered scheme of transactions quite different from it behaving as a good like here.

In the models of Radner, Cass and Werner, even though they incorporate an uncertain and potentially multi-stage future, each state s is still isolated, just as in the Walrasian formulation that was the focus of Arrow and Debreu. All the goods that become available in a state s, through endowments or, in some versions, production, must be consumed in that same state. There is no carry-over of goods whatsoever, not to speak of there being money that might be saved.

An important question concerns information. How are the agents supposed to know, at time 0, what will happen in the markets at a later time? The models implicitly require some such knowledge, because *it is in the initial state* $s = 0$ *that the agents already have to plan their buying and selling of goods in the future states* $s > 0$, and their budgets for such planning depend on the prices of goods in those states as well on the prices in the financial market.

An alternative model of *temporary equilibrium*, as explained by Grandmont [**?**], sets aside decisions about trading goods in future markets. Agents, in their financial planning, act only on "anticipations" of future prices. This is an interesting proposal, but in getting down to serious mathematical details, such as the specifics of "antipations," it appears to run into thorny difficulties. Fundamentally there is a lack of feedback about future supplies and demands of goods and therefore no way for agents, through some interchange, to tune their "anticipations" to an approximation of reality.

The point we believe should be kept in mind is that the future markets in the models of Radner, Cass and Werner, as well as here, cannot truly be claimed to take place in the future. This is an inevitable consequence of the notion of a Walrasian broker for those markets. That broker is entrusted with matching future supply and demand for goods on the basis of plans for buying and selling goods that the agents fix *in the present*. This broker operation must therefore be in the present as well, not in the future! We see it as a model of information exchange in which an agent can gain insights into what other agents may be hoping to do. That exchange generates feedback about prices, allowing the agents to get a better handle on how to plan for the future.

As a matter of fact, in our view there is no need even to think that, when the real future arrives, the agents will definitely carry out these plans. Circumstances may have changed beyond those taken into account when the plans were made. Our interpretation of equilibrium modeling with an uncertain future is thus a compromise between the traditional "sequential market" approach and that of "temporary equilibrium." However it brings skepticism about the virtues of incorporating more than just a few stages of the future. That would require a further proliferation of Walrasian brokers trying to glean planning information in the present, which would stretch the information-gathering concept too far.

5 Pattern of development

The model laid out here fits mathematically as a special case of a much more extensive model in our paper [?], and the existence theorem here follows essentially as a corollary of the existence theorem there. Although for that reason the details of its proof are superfluous for the present paper, an indication of the main ideas and their articulation in a nontraditional context may be illuminating.

The strategy is to characterize the existence of equilibium as a so-called *variational inequality* problem. Such formulations have gained in importance areas of application related to optimization, as in the book of Facchinei and Pang [11]. They rely on many advances in variational analysis, available for example in [31], as an enlargement of convex analysis [30]. Variational inequalities have previously been utilized in general economic equilibrium only in the one-stage models in our papers [?, ?].

Along with providing support for existence arguments, variational inequalities have other properties worthy of attention from economists. A variational inequality problem constitutes a paradigm of analysis that can be compared with the classical paradigm of "n equations in n unkonwns" but is able to cover vastly larger territory. An advanced theory is now available for studying perturbations and stability of their solutions [8]. Furthemore, they offer prospects of computational support which could lead to numerical experimentation with econometric models of equilibrium.

In order to proceed from a money-supported equilibrium, as defined above, to a variational inequality formulation, it is crucial to introduce auxiliary variables which complete the picture of "an n-dimensional condition in n unknowns." These variables are the Lagrange multipliers associated with the agents' budget constraints in optimality. Their values become part of the solution along with the prices and plans for market transactions; we speak then of a *enhanced* equilibrium.

The first step in arguing toward the existence of such an equilibrium is obviously then to make sure that Lagrange multipliers will be on hand. This follows from the concavity of utility and the assumption of ample survivability, which guarantees applicability of the Slater condition as a constraint qualification. The next step is to use the Lagrange multipliers in expressing necessary and sufficient conditions for optimality in an agent's maximization problem. The expression is achieved in the form of a *saddle point* condition on the associated Lagrangian function (which, by the way, would not be possible if utility were merely quasi-concave).

The saddle point condition can be translated in turn to a variational inequality of functional type. The market clearing condition in equilibrium can, on the other hand, be expressed through complementary slackness as a variational inequality of geometric type concerning the cones of normal vectors to a nonnegative orthant. These variational inequalities, as subconditions, can then be combined into a single composite variational inequality.

The composite variational inequality obtained in this way fully stands for an enhanced equilibrium, but it suffers from unboundedness which prevents immediate application of a criterion for the existence of a solution. An appeal to truncations must then be made, as is familiar in virtually all theory about the existence of economic equilibrium, even it if has to be carried out in a different way than in the past.

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