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PARAMETRICALLY ROBUST OPTIMALITY IN NONLINEAR PROGRAMMING*

A. L. DONTCHEV †, R. T. ROCKAFELLAR‡,§

Abstract. In nonlinear programming, the strong second-order optimality condition and the linearly independent gradient condition have many uses. In particular, the first guarantees that a point is an isolated locally optimal solution, while the second insures the uniqueness of the associated multiplier vector, but other, less stringent assumptions would already be enough for that. In fact, the combination of these two conditions is equivalent to having a powerful stability property beyond just local primal-dual uniqueness. That property is parametric robustness: almost no matter how parameters are introduced into the objective and constraint functions, the dependence of the locally optimal solution and its multiplier vector on the parameters will be single-valued and Lipschitz continuous. Then, moreover, the mapping from parameter vector to solution-multiplier pair will have directional derivatives which meet the high standards of semidifferentiability. At any point and in any direction, the derivative can be calculated by solving an auxiliary problem of quadratic programming.
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In the design and analysis of methods for solving nolinear programming problems in the standard format

$$(\mathcal{P}) \qquad \text{minimize} \quad f_0(x) \quad \text{subject to} \quad f_i(x) \begin{cases} \leq 0 & \text{for} \quad i = 1, \dots, s, \\ = 0 & \text{for} \quad i = s+1, \dots, m, \end{cases}$$

with $\mathcal{C}^2$ functions f_0, $f_1, \dots, f_m$ on $\mathbb{R}^n$, two special properties, demanding much more than just the Kuhn-Tucker conditions, are often imposed at a feasible point $\bar{x}$. These are the *strong second-order optimality condition* and the *linearly independent gradient condition*:

(SSOC) there exists a Kuhn-Tucker vector $\bar{y} = (\bar{y}_1, \dots, \bar{y}_m)$ for $\bar{x}$ such that

$$\begin{cases} \nabla^2 f_0(\bar{x}) + \sum_{i=1}^m \bar{y}_i \nabla^2 f_i(\bar{x}) \text{ is positive definite on the subspace orthogonal} \\ \text{to the gradients } \nabla f_i(\bar{x}) \text{ for } i \in [1, s] \text{ having } \bar{y}_i > 0 \text{ and for } i \in [s+1, m]. \end{cases}$$

(LIGC) the gradients $\nabla f_i(\bar{x})$ of the active constraints at $\bar{x}$ are linearly independent.

Both (SSOC) and (LIGC) can be viewed in terms of second derivatives of the Lagrangian

$$L(x, y) = f_0(x) + y_1 f_1(x) + \cdots + y_m f_m(x) \text{ on } \mathbb{R}^n \times [\mathbb{R}^s_+ \times \mathbb{R}^{m-s}], \tag{1}$$

since the $n \times n$ matrix in (SSOC) is $\nabla^2_{xx} L(\bar{x}, \bar{y})$, whereas the $m \times n$ matrix with the gradients $\nabla f_i(\bar{x})$ as its rows, which enters into (LIGC), is $\nabla^2_{yx} L(\bar{x}, \bar{y})$.

Various results are known which relate the combination of (SSOC) with (LIGC) to stability properties of $(\mathcal{P})$ with respect to parameters affecting the constraint and objective functions,

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†Mathematical Reviews, Ann Arbor, MI 48107-8604
‡Dept. of Mathematics, University of Washington, Seattle, WA 98195-4350
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including at the most basic level primal canonical parameters (which shift the constraints up or down) and dual canonical parameters (which tilt the objective). Such results are shown in [2], [8], [14] and in the books [1], [6] and [9] where more references can be found. Furthermore, (SSOC) and (LIGC) enter the scene when Newton-type methods are employed to numerically solve $(\mathcal{P})$, see e.g. [7, Section 12.4].

Here we present a fresh statement of the connection with solution stability, highlighting it in the strongest possible way. Further, we bring out significant consequences for the sensitivity analysis of locally optimal solutions to $(\mathcal{P})$. These developments emerge from the tactic of addressing a wide class of parameterizations of $(\mathcal{P})$ simultaneously, instead of looking at only a single, given parameterization assumed to meet various requirments, as has typically been the approach in sensitivity analysis.

Definition 1 (admissible parameterizations). *By an admissible parameterization of* $(\mathcal{P})$ *will be meant a choice of an open neighborhood* W *of a vector* $\bar{w}$ *in some space* $\mathbb{R}^d$ *together with a collection of* C^2 *functions* $\bar{f}_i : W \times \mathbb{R}^n \to \mathbb{R}$ *such that*

$$\bar{f}_i(\bar{w}, \cdot) = f_i \quad \text{for } i = 0, 1, \ldots, m. \tag{2}$$

Each admissible parameterization corresponds to an embedding of $(\mathcal{P})$ in a family of nonlinear problems which depend on $w \in W$:

$$(\bar{\mathcal{P}}(w)) \qquad \text{minimize } \bar{f}_0(w,x) \text{ in } x \text{ subject to } \bar{f}_i(w,x) \begin{cases} \leq 0 & \text{for } \; i = 1, \ldots, s, \\ = 0 & \text{for } \; i = s+1, \ldots, m, \end{cases}$$

When $w = \bar{w}$, we get $(\mathcal{P})$. The Lagrangian for $(\bar{\mathcal{P}}(w))$, namely

$$\bar{L}(w,x,y) = \bar{f}_0(w,x) + y_1 \bar{f}_1(w,x) + \cdots + y_m \bar{f}(w,x) \tag{3}$$

reduces likewise to $L(x,y)$ when $w = \bar{w}$.

For any $w \in W$, we can consider with the aid of the parameterized Lagrangian $\bar{L}$ the set

$$S(w) = \{\text{Kuhn-Tucker pairs } (x,y) \text{ for } (\bar{\mathcal{P}}(w))\} \tag{4}$$

and thereby get a set-valued mapping $S : w \mapsto S(w)$ from W to $\mathbb{R}^n \times \mathbb{R}^n$. This is the *stationarity* mapping associated with the given parameterization. Note that $S(\bar{w})$ consists of all the Kuhn-Tucker pairs $(\bar{x}, \bar{y})$ for $(\mathcal{P})$ itself; the triple $(\bar{w}, \bar{x}, \bar{y})$ belongs to the graph of S.

Our focus is on circumstances in which $\bar{x}$ is locally optimal and such local optimality is preserved when passing from $(\bar{w}, \bar{x}, \bar{y})$ to nearby elements (w,x,y) of the graph of S. The following concept will help us to express the exact property we wish to have. We will say that S has a *Lipschitz localization* at $\bar{w}$ for $(\bar{x}, \bar{y})$ if there exist neighborhoods U of $\bar{w}$ and V of $(\bar{x}, \bar{y})$ such that the mapping $s : w \mapsto S(w) \cap V$ is single-valued and in fact is a Lipschitz continuous function of $w \in U$. Observe that, geometrically, the graph of s is the intersection of $U \times V$ with the graph of S in $W \times \mathbb{R}^n \times \mathbb{R}^m$.

Definition 2 (parametric robustness of solutions). *A vector* $\bar{x}$ *will be called a robust locally optimal solution to* $(\mathcal{P})$ *if* $\bar{x}$ *is a feasible solution having a multiplier vector* $\bar{y}$ *associated with it the Kuhn-Tucker conditions and is such that for every admissible parameterization of* $(\mathcal{P})$ *and its associated stationarity mapping* S*,*

(a) S *has a Lipschitz localization* s *at* $\bar{w}$ *for* $(\bar{x}, \bar{y})$*, for which*

(b) there exist neighborhoods W_0 of $\bar{w}$ and X_0 of $\bar{x}$ such that, if $w \in W_0$ and $s(w) = (x, y)$, then x is the unique locally optimal solution to $(\bar{\mathcal{P}}(w))$ in X_0.

We are ready now to state and prove the first of the two theorems in this paper.

Theorem 1 (parametric robustness in nonlinear programming). *In problem $(\mathcal{P})$, a vector $\bar{x}$ is a robust locally optimal solution if and only if (SSOC) holds with (LIGC).*

Proof. Any admissible parameterization as in Definition 1 can be augmented by canonical primal and dual parameters so as to be an *extended canonical* parameterization with

$$\begin{aligned} w' = (w, v, u) \text{ in } W' = W \times \mathbb{R}^n \times \mathbb{R}^n,\ \bar{w}' = (\bar{w}, 0, 0), \\ \bar{f}_0'(w', x) = \bar{f}_0(w, x) + \langle v, x \rangle \\ \bar{f}_i'(w', x) = \bar{f}_i(w, x) + u_i \text{ for } i = 1, \dots, m. \end{aligned} \tag{5}$$

For the corresponding family of problems $(\bar{\mathcal{P}}(w')) = (\bar{\mathcal{P}}(w, v, u))$, in which we

$$\text{minimize } \bar{f}_0(w, x) + \langle v, x \rangle \text{ subject to } \bar{f}_0(w, x) + u_i \begin{cases} \leq 0 & \text{for } i = 1, \dots, s, \\ = 0 & \text{for } i = s+1, \dots, m, \end{cases} \tag{6}$$

we get a stationarity mapping $S : w' = (w, v, u) \mapsto S(w') = S(w, v, u)$ going from $W \times \mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n \times \mathbb{R}^m$.

In [2, Theorem 6], we obtained a result like the theorem we are now involved with proving, but in terms of extended canonical parameterizations only. We have to demonstrate now that the same thing holds in terms of all admissible parameterizations.

Because every extended canonical parameterization is in particular an admissible parameterization with parameter vector w' as in (5), and conditions (SSOC) and (LIGC) are already necessary for those parameterizations in w' to have the properties (a) and (b) of Definition 2 through [2, Theorem 6], it is clear that (SSOC) and (LIGC) are also necessary for $\bar{x}$ to be a robust locally optimal solution to $(\mathcal{P})$ in the present setting.

For the converse, suppose that the stationarity mapping $S : (w, v, u) \mapsto S(w, v, u)$ for an extended canonical parameterization as in (6) has the properties (a) and (b) of Definition 2. Then the submapping $S_0 : w \mapsto S(w, 0, 0)$ has those properties as well. Since the stationarity mapping associated with any admissible parameterization can be identified with such a submapping, the sufficiency of (SSOC) and (LIGC) for extended canonical parameterizations to enjoy the properties in Definition 2, by [2, Theorem 6], implies the sufficiency of conditions for all admissible parameterizations.

Next we look more closely at the Lipschitz localizations s that exist by Theorem 2 and explore their differentiability properties. The *directional derivative* of s at a point $w \in \mathbb{R}^d$ with respect to a vector $w' \in \mathbb{R}^d$ is defined of course by

$$s'(w; w') = \lim_{\varepsilon \searrow 0} \left[s(w + \varepsilon w') - s(w)/\varepsilon \right], \tag{7}$$

when the limit exists. If it exists for every w', and if moreover the mapping $w' \to s'(w; w')$ furnishes a first-order approximation to s at w in the sense that

$$s(w + w') = s(w) + s'(w; w') + o(|w'|), \tag{8}$$

then s has the property of *semidifferentiability* at w which was defined and characterized in various ways in [15, Chap.7].

Theorem 2 (solution sensitivity under robustness). *Let $\bar{x}$ be a robust locally optimal solution to $(\mathcal{P})$ with multiplier vector $\bar{y}$, and let s be a Lipschitz localization of the stationarity mapping S associated with an admissible parameterization of $(\mathcal{P})$ as in Definition 1. Then s is semidifferentiable at all w in some neighborhood of $\bar{w}$. Furthermore, a formula for the directional derivatives $s'(w; w')$ is available: the pair $(x', y') = s'(w; w')$ is characterized by x' being the unique optimal solution, and y' the unique associated multiplier vector, for the following problem of quadratic programming with respect to w and the pair $(x, y) = s(w)$,*

$$\mathcal{P}'(w;w') \qquad \begin{array}{c} \text{minimize } w'\nabla^2_{wx}\bar{L}(w,x,y)x' + \dfrac{1}{2}x' \cdot \nabla^2_{xx}\bar{L}(w,x,y)x' \text{ in } x' \text{ subject to} \\[2ex] \nabla_w f_i(w,x)w' + \nabla_x f_i(w,x)x' \begin{cases} \leq 0 \ \text{for} \ i \in [1,s] \ \text{ with } f_i(w,x) = 0, \ y_i = 0 \\ = 0 \ \text{for} \ i \in [1,s] \ \text{ with } f_i(w,x) = 0, \ y_i > 0, \\ = 0 \ \text{for} \ i = s+1, ..., m \end{cases} \end{array}$$

Proof. Recall that for a convex set $Z \subset \mathbb{R}^N$ and a point $\bar{z} \in Z$ the normal cone $N_Z(\bar{z})$ consists of all vectors $v \in \mathbb{R}^N$ such that $v \cdot (z - \bar{z}) \leq 0$ for all $z \in Z$; when $\bar{z} \notin Z$, $N_Z(\bar{x})$ is taken to be the empty set. The *variational inequality* for Z and a single-valued mapping $F : \mathbb{R}^N \to \mathbb{R}^N$ is the condition on $\bar{z}$ that

$$-F(\bar{z}) \in N_Z(\bar{z}), \text{ or equivalently, } \bar{z} \in Z \text{ and } F(\bar{z}) \cdot (z - \bar{z}) \geq 0 \text{ for all } z \in Z. \tag{9}$$

In the footsteps of Robinson [14], we can pose the Kuhn-Tucker conditions for the parameterized nonlinear programming problem $(\mathcal{P}(w))$ as the parameterized variational inequality

$$-F(w,z) \in N_Z(z) \ \text{ with} \begin{cases} z = (x,y), \quad Z = \mathbb{R}^n \times [\mathbb{R}^s_+ \times \mathbb{R}^{m-s}], \\ F(w,z) = \big(\nabla_x \bar{L}(w,x,y), -\nabla_y \bar{L}(w,x,y)\big). \end{cases} \tag{10}$$

The stationarity mapping S associated with the family of problems $(\mathcal{P}(w))$ is then the solution mapping $w \mapsto \{z| - F(w,z) \in N_Z(z)\}$. Our attention is concentrated on the localized graphical behavior of this mapping around $\bar{w} \in W$ and $\bar{z} = (\bar{x}, \bar{y})$ in $S(\bar{w})$.

Much is known about the behavior of solution mappings for variational inequalities in general. In our paper [3], we showed that if the parameterization by $w \in \mathbb{R}^d$ is *ample* (a standard of "richness"), then the graph of S is a type of Lipshitz manifold of dimension d around $(\bar{w}, \bar{z}) \in \mathbb{R}^d \times \mathbb{R}^N$ which, in a geometric sense, has a first-order approximation given by its graphical derivative $DS(\bar{w}|\bar{z})$, the graph of which is similarly a Lipschitz manifold of dimension d around $(0,0)$. The case where these local Lipschitz manifolds turn into the graphs of single-valued Lipschitz continuous mappings corresponds to S having a Lipschitz localization at $\bar{w}$ for $\bar{z}$, which in fact is equivalent under ample parameterization to $DS(\bar{w}|\bar{z})$ having a Lipschitz localization s at 0 for 0; then we have semidifferentiality with $s'(\bar{w}; w') = DS(\bar{w}|\bar{z})(w')$. (For all this, see [3, Secs. 6,7].)

In our format (10), any extended canonical parameterization by $w' = (w, v, u)$ obtained from an admissible parameterization of $(\mathcal{P})$ by w as in (6) is an admissible parameterization in the sense of [3], in particular. Along the same lines of reasoning as in our proof of Theorem 1 above, we can reduce the proof of Theorem 2 to the case of such parameterizations. Since we are already in the context of a Lipschitz localization s of S at $\bar{w}$ for $\bar{z} = (\bar{x}, \bar{y})$, we are left only with verifying that the formula for $s'(\bar{w}; w')$ provided by [3, Theorem 7.1] for polyhedral Z comes down to the formula claimed here. (The corresponding formula for $s'(w; w')$ at nearby $w \neq \bar{w}$ will follow then automatically from our context.)

In that formula for an amply parameterized variational inequality over a polyhedral convex set in general, one looks at the auxiliary variational inequality

$$-G(w',z')\in N_K(z') \text{ for} \begin{cases} G(w',z') = \nabla_w F(\bar{w},\bar{z})w' + \nabla_z F(\bar{w},\bar{z})z', \\ K = \{z'\in T_C(\bar{z}) | z' \perp F(\bar{x},\bar{z})\}, \end{cases} \tag{11}$$

where $T_Z(\bar{z})$ is the tangent cone to Z at $\bar{z}$; then K is the *critical cone* at $\bar{z}$. The solution mapping $S' : w' \mapsto \{z'| - G(w',z') \in N_K(z')\}$, which has $(0,0)$ in its graph, turns out to be the graphical derivative mapping $DS(\bar{w}|\bar{z})$, according to [3, Theorem 7.1]. The task facing us is to confirm that when we apply this to the variational inequality (10) we get the stationarity mapping for the parameterized quadratic programming problems $(\mathcal{P}'(w;w'))$.

It is easy to see that when F is the mapping in (10), then G takes the form

$$\begin{aligned} G(w',z') = \big(&\nabla^2_{xw}\bar{L}(\bar{w},\bar{x},\bar{y})w' + \nabla^2_{xx}\bar{L}(\bar{w},\bar{x},\bar{y})x' + \nabla^2_{xy}\bar{L}(\bar{w},\bar{x},\bar{y})y', \\ &-\nabla^2_{yw}\bar{L}(\bar{w},\bar{x},\bar{y})w' - \nabla^2_{yx}\bar{L}(\bar{w},\bar{x},\bar{y})x' - \nabla^2_{yy}\bar{L}(\bar{w},\bar{x},\bar{y})y'\big), \end{aligned} \tag{12}$$

but $\nabla^2_{yy}\bar{L}(\bar{w},\bar{x},\bar{y}) = 0$. (Here $\nabla^2_{xx}\bar{L}(\bar{w},\bar{x},\bar{y}) = \nabla^2_{xx}L(\bar{x},\bar{y})$, etc., but we keep to the notation of $\bar{L}$ so as to relate better to the intension that the same formulas can later be applied to other (w,x,y) instead of $(\bar{w},\bar{x},\bar{y})$.) With respect to the index sets

$$\begin{aligned} I_- &= \{i\in[1,s] |\ \bar{f}_i(\bar{w},\bar{x}) < 0,\ \bar{y}_i = 0\}, \\ I_0 &= \{i\in[1,s] |\ \bar{f}_i(\bar{w},\bar{x}) = 0,\ \bar{y}_i = 0\}, \\ I_+ &= \{i\in[1,s] |\ \bar{f}_i(\bar{w},\bar{x}) = 0,\ \bar{y}_i > 0\}, \end{aligned}$$

the cone $K \subset \mathbb{R}^n \times \mathbb{R}^m$ is described by

$$(x',y')\in K \iff x'\in\mathbb{R}^n \text{ and } y'_i \begin{cases} = 0 & \text{for} \quad i\in I_- \\ \geq 0 & \text{for} \quad i\in I_0, \\ \text{free} & \text{for} \quad i\in I_+ \end{cases}$$

where a variable is said to be "free" when it can take on any real value. Since $\nabla^2_{yx}\bar{L}(\bar{w},\bar{x},\bar{y})$ is the matrix having the gradients $\nabla_x\bar{f}(\bar{w},\bar{x})$ as its rows, whereas $\nabla^2_{xy}\bar{L}(\bar{w},\bar{x},\bar{y})$ has them as its columns, the variational inequality in (11) takes, through (12), the form that the equation

$$\nabla^2_{xw}\bar{L}(\bar{w},\bar{x},\bar{y})w' + \nabla^2_{xx}\bar{L}(\bar{w},\bar{x},\bar{y})x' + \sum\nolimits_{i=1}^{m} y'_i \nabla_x \bar{f}_i(\bar{w},\bar{x}) = 0 \tag{13}$$

holds along with the conditions, in the notation $l_i(w',x') = \nabla_w \bar{f}_i(\bar{w},\bar{x},\bar{y})w' + \nabla_x\bar{f}_i(\bar{w},\bar{x})x'$, that

$$\begin{array}{llll} y'_i = 0, & l_i(w',x') \text{ free} & & \text{for} ;i\in I_-, \\ y'_i \geq 0, & l_i(w',x') \leq 0, & y'_i l_i(w',x') = 0 & \text{for},;i\in I_0, \\ y'_i \ \text{ free}, & l_i(w',x') = 0 & & \text{for} ;i\in I_+. \end{array} \tag{14}$$

All that is left is the observation that (13) and (14) are the Kuhn-Tucker conditions for the auxiliary problem $(\mathcal{P}'(\bar{w};w'))$.

It follows from the formula in Theorem 2, by the way, that $s'(w;w')$ is a piecewise linear function of w'. Of course, because s is Lipschitz continuous around $\bar{w}$, we actually know, from Rademacher's theorem on almost everywhere differentiability, that $s'(w;w')$ is a linear function of w' for almost every w in a neighborhood of w.

We should note that a formula for the directional derivative of s in the case of extended canonical perturbations was given in [16] and more explicitly in [13] where an earlier result of Jittorntrum (1884) was cited.

Beyond the framework of basic nonlinear programming, there are related results on primal-dual solution stability under perturbations which have appeared in [12] and [4]. A characterization of primal solution stability by itself, without an accompanying multiplier vector, has been obtained in a very general setting in [11] through high-powered techniques in variational analysis. The conditions there are difficult to specialize in fine detail, however, because of the lack, at present, of an adequate calculus for certain second-derivative objects on which the characterization relies. Finally, we mention that a broad introduction to the properties of solution mappings to variational inequalities and other similar relations is now available in [5] together with a number of new results in that area.

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Asen L. Dontchev - received his MSc Degree in 1971 and his PhD in 1974, both at Warsaw University of Technology. He then joined the Institute of Mathematics at the Bulgarian Academy of Sciences. Since 1990 he is with Mathematical Reviews in Ann Arbor, Michigan. He is also Lecturer at the University of Michigan. His current research interests include stability, sensitivity and approximations in variational problems as well as singular perturbations and numerical analysis in optimal control. He is recipient of 1995 Mathematics Prize of the Bulgarian Academy of Science.

Ralph T. Rockafellar - completed his undergraduate work at Harvard in 1957, and his graduate work in 1963 at Harvard as well. After brief stays in Germany and Texas, he settled in University of Washington, Seattle. He is also Adjunct Research Professor at University of Florida. His current research interests are focused on variational analysis and its applications such as development of mathematical foundations for analytical and numerical treatment of variational problems, optimization methodology for modeling large-scale problems in decision or control over time, stochastic programming and its applications to finance, risk theory and economics. R.T. Rockafellar has received a number of honors, such as the Dantzig Prize (1982), Von Neumann Citation (1992), Lanchester Prize (1998), John von Neumann Theory Prize (1999) and honorary doctorates at University of Groningen (Netherlands), University of Montpelliers (France) University of Chile (1998) and University of Alicante (2000).