SOME PROPERTIES OF PIECEWISE SMOOTH FUNCTIONS[1]

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Abstract: Piecewise smooth equations are increasingly important in the numerical treatment of complementarity problems and models of equilibrium. This note brings out a property of the piecewise smooth functions that enter such equations, for instance through penalty expressions.

Keywords: nonsmooth optimization, piecewise smoothness, variational analysis

Dedication: It is a great pleasure to contribute to a volume in honor of Lucien Polak, a long-time friend. While focused on numerical methodology, he has been a strong proponent of the broad view of optimization in which nonlinear programming is only one component in a framework that includes control problems, semi-infinite programming, min-max optimization, and other challenges. His insistence on building a middle ground between conceptual algorithms and implementations, by providing a theory of how to coordinate the various epsilons through innovative approaches to optimality criteria has deeply influenced my thinking.

1 Introduction

Piecewise smooth functions have received considerable attention in the last few years because of applications to solution methodology in optimization, particularly in connection with complementarity problems and variational inequalities more generally. Some of that attention has been directed toward special penalty expressions and their usage, but fundamental research on piecewise smoothness has been conducted as well. Basic background and developments in the subject can be found in [1], [2], [3], [4], [5], [6], [7], [8], and more recently for example in [9].

Intuitively, the notion of a piecewise smooth function is meant to capture the idea of a function whose domain can be partitioned locally into finitely many "pieces" relative on which smoothness holds, and continuity holds across the joins of the pieces. Here smoothness refers to continuous differentiability. The definition of piecewise smoothness, going back to [1], sidesteps direct mention of such pieces, however, since that would raise technically troublesome issues such as their boundary properties and interrelationships. Instead, it works with collections of smooth functions that are defined universally in a local sense, beyond any boundaries of possible pieces.

This definition of piecewise smoothness is helpful and convenient but conceivably could be in conflict with the intuitive notion in some respects. For instance, might it be possible for a function to be piecewise smooth on an open set O in $\mathbb{R}^n$ with $n \geq 2$ and actually smooth at all but one point $\bar{x} \in O$, without in fact being smooth on all of O? If so, we would seem to have something like piecewise smoothness with just one piece, and yet not have smoothness.

The purpose of this brief note is to demonstrate that the situation just described is impossible, and further to shed more light on the structure of piecewise smoothness in general.

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2 Definitions and Results

Definition 1. *A function f on an open set $O \subset \mathbb{R}^n$ is said to be piecewise smooth if it is continuous on O and for each $\bar{x} \in O$ there is a finite collection $\{ f_i\}_{i\in I}$ of $\mathcal{C}^1$ functions defined on a neighborhood of $\bar{x}$ such that, for some $\varepsilon > 0$, one has*

$$f(x) \in \{ f_i(x) \,|\, i \in I\} \text{ when } |x - \bar{x}| < \varepsilon. \tag{1}$$

in which case the notation is used that

$$I(x) = \{ i \in I \,|\, f(x) = f_i(x)\}.$$

Such a collection is said to form a local representation for f at $\bar{x}$, and it is called minimal if no subcollection forms such a representation.

Note that a local representation at $\bar{x}$ serves also as a local representation at each x in some neighborhood of $\bar{x}$.

The featured result, stated below as Theorem 1, will be attained by way of a pair of lemmas having some independent interest. We will go on then, in Theorem 2, to show how the same techniques are able to reconcile even more broadly the intuitive notion of piecewise smoothness with its formulation in Definition 1.

Lemma 1. *Suppose f is piecewise smooth on O and that $\{ f_i\}_{i\in I}$ is a minimal local representation for f at the point $\bar{x} \in O$. Then for every $i \in I$ there is an open set O_i such that $\bar{x} \in \operatorname{cl} O_i$ and $f \equiv f_i$ on O_i.*

Proof. Let ε be such that (1) holds for the open ball at $\bar{x}$ of radius ε; denote that ball by B, assuming without loss of generality that $\operatorname{cl} B \subset O$. For each $i \in I$, let $C_i = \{ x \in B \,|\, f(x) = f_i(x)\}$ and $O_i = B \setminus \bigcup_{j\neq i} C_j$. Because f and f_i are continuous, C_i is closed relative to B and therefore O_i is open. Furthermore $\bar{x} \in \operatorname{cl} O_i$, for if not the set $\bigcup_{j\neq i} C_j$ would cover a neighborhood of $\bar{x}$ and f_i would be superfluous in the local representation, contradicting minimality. □

Lemma 2. *Let f be piecewise smooth on O, and let $\{ f_i\}_{i\in I}$ be any local representation for f at $\bar{x} \in O$. If f is differentiable at $\bar{x}$, then there exists $i \in I(\bar{x})$ such that $\nabla f(\bar{x}) = \nabla f_i(\bar{x})$.*

Proof. Without loss of generality we can take the local representation to be minimal. Then in particular, $f(\bar{x}) = f_i(\bar{x})$ for every $i \in I$. Denote the linear functions $\langle \nabla f(\bar{x}), \cdot\rangle$ and $\langle \nabla f_i(\bar{x}), \cdot\rangle$ by g and g_i. Our aim is to show that $g \equiv g_i$ for some i, and we can accomplish that producing an open set on which g and g_i agree.

Let ε be such that (1) holds. Consider any $w \in \mathbb{R}^n$ and any sequence of values $\tau^\nu \searrow 0$. For each ν large enough that $|\tau^\nu w| < \varepsilon$, there exists $i \in I$ with $f(\bar{x} + \tau^\nu w) = f_i(\bar{x} + \tau^\nu w)$. Because I is finite, we can suppose, by passing to a subsequence if necessary, that the same i works for every ν. Then we have

$$\frac{f(\bar{x} + \tau^\nu w) - f(\bar{x})}{\tau^\nu} = \frac{f_i(\bar{x} + \tau^\nu w) - f_i(\bar{x})}{\tau^\nu} \text{ for all } \nu.$$

Hence, on taking limits on both sides, we have $g(w) = g_i(w)$.

This shows that $\{ g_i\}_{i\in I}$ is a local representation of g. Reducing I to a subset J if necessary, we can arrange that $\{ g_i\}_{i\in J}$ is a minimal representation. Then Lemma 1 applies, and we are finished. □

The first part of the proof of Lemma 2 essentially reproduces a result in [4, Proposition 2.1], namely that any piecewise smooth function f represented at $\bar{x}$ by a collection $\{f_i\}_{i\in I}$ is directionally differentiable, with its directional derivative function being piecewise linear and represented by the collection of (linear) directional derivative functions associated with the functions f_i that are active at $\bar{x}$. The new contribution is to feed this into an invocation of Lemma 1. Variants of Lemmas 1 and 2 can be seen also in [3, Prop. 4.1.1 and Prop. 4.1.3].

Theorem 1. *Suppose f is piecewise smooth on $O \subset I\!R^n$ with $n \geq 2$, and in fact that f is smooth on $O \setminus \{\bar{x}\}$ for some $\bar{x} \in O$. Then f must be smooth on O. Moreover for any local representation $\{ f_i\}_{i\in I}$ that is minimal at $\bar{x}$, one has $\nabla f(\bar{x}) = \nabla f_i(\bar{x})$ for all $i \in I$.*

Proof. Consider a minimal local representation $\{f_i\}_{i\in I}$ at $\bar{x}$ and a ball $B = \{ x \,|\, |x - \bar{x}| < \bar{\varepsilon}\}$ in which it operates. For each $i \in I$, let $D_i = \{ x \in B \,|\, f(x) = f_i(x),\ \nabla f(x) = \nabla f_i(x)\}$. Observe that every point $x \in B \setminus \{\bar{x}\}$ belongs to at least one of the sets D_i by Lemma 2 (as applied to x instead of $\bar{x}$).

For each $\varepsilon \in (0, \bar{\varepsilon})$ let S_ε be the sphere $\{ x \,|\, |x - \bar{x}| = \varepsilon\}$. On S_ε, the mapping ∇f is continuous; it agrees with ∇f_i on $D_i \cap S_\varepsilon$, with these sets covering S_ε by the observation already made. Because ∇f_i is continuous on D_i, which contains $\bar{x}$, we have

$$\limsup_{\varepsilon \searrow 0} \nabla f(S_\varepsilon) \subset \{ \nabla f_i(\bar{x}) \,|\, i \in I\}.$$

Here the image sets $\nabla f(S_\varepsilon)$ are connected, because connectedness is preserved under continuous mappings and the spheres S_ε are themselves connected; here we use the assumption that $n \geq 2$.

It follows that the sets $\nabla f(S_\varepsilon)$, which are uniformly bounded, must converge to a particular element of $\{ \nabla f_i(\bar{x}) \,|\, i \in I\}$ (inasmuch as small neighborhoods of $\{ \nabla f_i(\bar{x}) \,|\, i \in I\}$ isolate its finitely many points). Thus, ∇f has a continuous extension from $O \setminus \{\bar{x}\}$ to O.

By way of the mean value theorem, as applied over segments $[\bar{x}, \bar{x} + \tau w]$, this ensures that f is differentiable at $\bar{x}$, hence in fact $\mathcal{C}^1$ on O. This justifies the first claim in the theorem. Applying Lemma 1, we then get the second claim as well. □

Beyond the situation addressed in Theorem 1, there are other insights provided by our patterns of argument.

Definition 2. *A function f on an open set O in $I\!R^n$ will be called smooth relative to P, a closed subset of O, if $P = \mathrm{cl}(\mathrm{int}\, P)$, f is smooth on $\mathrm{int}\, P$, and the gradient mapping ∇f on $\mathrm{int}\, P$ can be extended continuously to P.*

Theorem 2. *If f is piecewise smooth on O, then any point $\bar{x} \in O$ has a neighborhood covered by a finite collection of closed sets $\{P_i\}_{i\in I}$ in O such that $\bar{x} \in P_i$ and f is smooth relative to P_i.*

Proof. Consider a minimal local representation $\{f_i\}_{i\in I}$ as in Lemma 1 and corresponding sets C_i and O_i in the proof of Lemma 1, noting that $\mathrm{int}\, C_i \supset O_i \neq \emptyset$. Take $P_i = \mathrm{cl}\,\mathrm{int}\, C_i$. Then obviously $P_i = \mathrm{cl}\,\mathrm{int}\, P_i$ as well, and $\bar{x} \in P_i$ because $\bar{x} \in \mathrm{cl}\, O_i$. Moreover $P_i \subset \mathrm{cl}\, C_i$, so f agrees with f_i on P_i. We have f_i smooth relative to P_i, and therefore f is smooth relative to P_i.

Finally it must be established that the sets P_i cover a neighborhood of $\bar{x}$. This will be done for the ball B in the proof of Lemma 1. Suppose there were a point $x_0 \in B$ not belonging to any P_i. Then some closed ball B_0 around x_0 within B would be disjoint from $\operatorname{int} C_i$ for every $i \in I$. Since B_0, like B, is covered by the sets C_i, it must actually be covered by the sets $C_i \setminus \operatorname{int} C_i$, which are closed relative to B_0 and have empty interior. By the Baire category theorem, however, it is impossible for a compact set with nonempty interior to be expressed as the union of finitely many (or even countably many) closed subsets having empty interior. The contradiction confirms that every point of B belongs to at least one P_i. □

Theorem 2 implies of course that f is continuously differentiable on an open set $O' \subset O$ that has the same closure as O. That property of piecewise smoothness was previously obtained in [3, Prop. 4.1.5], likewise with an argument that appeals to the Baire category theorem.

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