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MATHEMATICS OF DEBT INSTRUMENT TAXATION

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Abstract. The mathematical principles behind the taxation of bonds and other securities in the form of debt instruments are elucidated. The main tax rules of current importance in portfolio management in the United States are sorted out and expressed in formulas as an aid to financial modeling and computerization. As a foundation, the theory of constant yield to maturity is developed with generality ample enough to cover situations where the dates on which interest is compounded may not be equally spaced. Various inconsistencies are identified in the approaches to tax computation embodied in current regulations, such as the method of fractional exponents and the way of characterizing original issue discount or premium. Consistent alternatives to these approaches are proposed.

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TABLE OF CONTENTS

1. INTRODUCTION

Debt instruments are financial contracts that an issuer sells to a holder for some price, thereby obligating the issuer to pay a nonnegative, future cash stream to the holder. The holder can subsequently transfer ownership to a different holder through sale or exchange.

Issuers and holders can be many kinds of entities, and even if they do not pay taxes themselves, they may be influenced by the effects of taxes in financial markets. It is vital therefore to understand the taxation of debt instruments from a general mathematical perspective, as well as for the sake of calculating what taxes must be paid in a given case. This is a formidable challenge, however, because of the many rules and exceptions that have evolved, and because the legal language in which the rules are stated is far from an expression by mathematical formulas. In addition, the fundamentals of how to attribute a yield rate to a debt instrument relative to the price paid for its cash stream are not well understood by many practitioners or approached consistently in law. This has clouded not only the procedures for determining taxes but the ability of issuers, holders, and governmental authorities to anticipate the potential consequences of code provisions and regulations, especially in the case of new types of securities that might be written.

A central aim of this study is to provide—for U.S. federal taxation—a relatively compact mathematical description[1] which covers the specially patterned instruments that are typically seen. Such a description can serve many uses in finance and accounting. It should assist in building mathematical models of bond trading and portfolio adjustment that attempt to bring in the effects of taxes and open the door to much higher modes of computerization than now prevail. Our intention is to furnish, at the least, the means of dealing mathematically with the tax implications of holding long and short positions in debt instruments of the kinds that now dominate financial markets, and of assessing the tax advantages and disadvantages of new instruments that might be introduced in such markets or set up as contracts between two parties.

At the same time this study is dedicated to elucidating the principles behind tax law as it has emerged in this area, and to forging the sharp tools needed to expose conflicts and uncertainties in statement or intent. It is obviously in the interest of issuers, holders, and governmental authorities alike to have a firmer understanding of what their decisions might lead to. Everyone can benefit from dispelling vagueness such as may enter regulations either through inherent difficulties in interpreting the will of congress or in hopes of combating unforeseen abuses.

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[1] No more than basic algebra is required in order to understand our formulas and their derivation. Calculus enters the proofs of two of the theorems concerned with facts about the formulas, but again, the mathematical expressions of these facts can be appreciated from the algebraic standpoint alone.

Our theoretical focus thus brings us beyond formulas to matters of policy. We arrive in particular at recommending a *policy of mathematical consistency* in tax code and regulations, such as can only be secured through precise mathematical formulation and analysis of all provisions and unsparing attention to their logical coordination.

Because the study is designed in part as a bridge between different communities of professionals, care is taken to develop a consistent terminology. Explanations are sometimes furnished for notions that to one group may be totally familiar, but to another group potentially confusing. The theoretical foundations for the calculation of interest income over time are laid out in detail, since this seems to be a topic suffering from certain mathematical misconceptions which result in ambiguity and unnecessary trouble. Some of these misconceptions have even been enshrined in official rules for determining taxes.

In the United States, the holding of debt instruments can produce taxable income in two ways, through interest or capital gains.[2] For the issuer of a debt instrument, interest income is replaced by interest expenditure, likewise with tax consequences, and of course instead of a capital gain there can always be a capital loss. The chief difficulty lies in determining just how much interest should be deemed to have been received or paid at any time, or how much capital gained or lost. This is far from simple for a number of reasons, both mathematical and legal.

For many years the two types of income were taxed as they had been characterized by the parties. The U.S. Internal Revenue Service (IRS) eventually attempted through litigation to develop two departures from this treatment, both of them motivated by the fact that capital gains were at the time being taxed at only half the rate for interest income, which was having an effect on the way debt instruments were being written. First, the IRS successfully argued that any excess of the face value of a bond, over its issue price, was not entitled to capital gains treatment because there had been no sale or exchange of the bond.[3] Second, the IRS argued that any such excess amount should be taxed as interest on receipt. Congress dealt with the matter in the Internal Revenue Code of 1954. Under that code the holder of a bond issued with discount was required to treat the discount as additional interest income.[4] In 1969,

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[2] The treatment of gains and losses on capital assets is specified in Internal Revenue Code SS1221, 1222, and 1223. Interest income is covered in S61(a), and the deduction of interest expense in S163(a). The timing of tax liabilities and deductions is set forth in S451(a) and S461(a).

[3] Congress reversed this by enacting the predecessor of Internal Revenue Code S1271(a), which generally dictates that all amounts received on maturity be treated as received in exchange for the bond. A reason perhaps was the arbitrariness of distinguishing a return paid by the issuer from a return received on a sale to a third party immediately before maturity.

[4] Only from that point did the IRS begin to enjoy success in litigation on its second point, a

Congress revised the rules to require all holders of corporate bonds issued with discount to treat the discount amount as additional interest received *ratably* (in mathematics: at a linear rate) over the time the bond was outstanding. This was a critical departure, because it initiated the taxation of income before it had literally been received. In each case where holders had to declare constructed receipt of more interest, an equivalent amount could be deducted by issuers as an interest expense against income. Thus, an issuer was allowed to deduct expenses before they had actually been paid, even decades in advance in some common situations. For bonds issued by taxed entities but held by untaxed entities, this amounted to affording substantial tax relief to issuers without a revenue balance from holders.

In 1982, Congress further revised the rules to require that the interest in question, not only corporate bonds but on governmental bonds and most others, be handled as if earned at a constant yield rate on an ever increasing amount of principal over the life of the bond. The taxation of payments that were only implicit—had not actually changed hands—was retained but refined. Since then, Congress and the U.S. Treasury, through its regulations, have tinkered further in an effort to improve the rules and reduce the circumstances in which taxpayers might gain undue advantage from the way interest income and expenditure are identified. The ad hoc nature of these efforts, without a solid mathematical platform for systematically assessing the effects of adopted rules, continues to cast a shadow over the outcome, however, especially with respect to potential implications for taxed versus untaxed issuers and holders.

In each stage of the legal developments, instruments issued in an earlier era retained the tax treatment they already had. Such instruments are still around and actively traded, with the result that various forms of old rules in addition to recent ones are still required in determining *current* taxes. For this reason we must, in our endeavor to set up mathematical formulas for taxation, cope with multiple layers of regulations instead of just the latest. Anyway, the mathematical attention we give to past provisions helps in appreciating present ones.

Classification of debt instruments according to date of issue is very important for this reason, but other divisions make a substantial difference too in the way taxes are computed, as has already come up. Debt instruments are *long-term* when they have more than one year from issuance to maturity, but *short-term* when they have one year or less. An instrument is *governmental* if issued by the federal government, the government of any of the states (or U.S. possessions, or the District of Columbia) or any of their political subdivisions. It is *corporate* if issued by a corporation. All other nongovernmental obligations are *noncorporate*, for instance if issued by a partnership, trust, or individual. An instrument is *tax-exempt* if the interest income associated with it is exempt from federal taxation for all holders. A capital gain or loss on such

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landmark case being United States v. Midland Ross Corp. 381 U.S. 54 (1965).

an instrument nevertheless does have tax consequences.

Beyond the complications arising from this staggering list of distinctions, there are ambiguities, at least from the mathematical perspective, in the fact that tax rules often evolve through precedent rather than general principle. Rules may be couched rather narrowly in terms of the particular instruments occupying attention at the time of writing, such as standard bonds in the case the actions of Congress described above, and their application to some other kind of instrument may not be immediately clear. It is up to the Treasury to clarify any details of how tax law should be implemented, and this is supposed to be done through regulations that are proposed, then adjusted after hearings and public comment, and finally approved and adopted. Unfortunately, this process suffers delays which are a further source of problems.

Congress passes tax laws, which make up the Code, and the IRS is authorized by Congress to issue regulations to implement the Code. The Code is collected in CCH Tax Law Staff (1992a)(1992b), while the Proposed Regulations and (final) Regulations are collected in CCH Tax Law Staff (1993). See Robinson (1986), Staff of the Joint Committee (1984), and Tax Law Staff (1993) for explanations of the Code. Garlock (1993) and Bittker and Lokken (1993) explain the Code and Regulations, while Pratt, Burns, and Kulsrud (1991) gives brief overviews. Each change in the Regulations appears in the IRS Cumulative Bulletin, which we refer to as Commissioner of the IRS (1993).

The voluminous Proposed Regulations of 1986, although long relied upon in practice, did not reach the final stage of approval (by the U.S. Treasury) before being rescinded late in 1992 in favor of a different set of Proposed Regulations. When, or whether, the latest version will eventually be approved is uncertain. However, the IRS has said it will continue to take the 1986 version as authoritative for instruments issued before 22 December 22 1992, when the revised regulations were distributed for comment. Meanwhile, the revised regulations will be considered as "guidelines" for instruments issued since. This means yet another divergence in tax treatment, but one in a state of limbo as of this writing.[5]

In general, all debt instruments issued prior to 1955, and some issued since, are taxed in a straightforward manner in which the holder's interest income and the issuer's interest expense, and their capital gain or loss, are determined from the instrument's *nominal* specification of what portion of the payments that it explicitly provides are interest, and what portion are repayment of principal, as long as this is consistent. Many instruments issued in 1955 or later, however, are taxed relative to a *revised* specification which is associated with the "original issue discount,"

[5] Since the situation is in flux, we shall concentrate in this study on rules given in the IRS Proposed Regulations of 1986 but indicate in footnotes various changes embodied in the pending IRS Proposed Regulations of 1992.

or OID amount, that may be present. The crucial feature in this case, already alluded to, is that additional interest is regarded as paid *implicitly* to the holder as taxable income during the life of the instrument, but automatically reinvested and therefore not *explicitly* made available to the holder until later. The issuer can generally deduct such implicit payments as a current expense.

Implicit interest income or expenditure is similarly associated not only with “original issue discount” but with features known as “market discount,” “acquisition premium,” and “amortizable premium,” all of which will be explained in due course. Purchase or sale at some time between issue and maturity usually requires a recharacterization of the debt, relative to the price involved in the transaction. This leads to a *particularized* specification of the instrument’s content. “Stripping” and “shorting” are ways that essentially new instruments are frequently created out of existing ones, and they too raise challenges in the determination of interest income or expenditure.

These aspects of taxation have their various histories and disparate treatments. A key concept in every case now, though, is that of imputed interest based on a *constant* yield rate over the relevant life of an instrument. It is well understood that real rates of interest go up and down, so that the imputation of a theoretical constant yield rate to a given instrument, relative only to the market circumstances in which it was acquired, is bound to have an artificial quality. The concept is nevertheless relied upon, because it is thought to be the only reasonably simple approach to a difficult problem. Its mathematical underpinnings are therefore crucial to our project from all sides.

Taxation in terms of a constant yield rate is only one of many conceivable approaches to obtaining governmental revenue from debt instruments, and we do not take the position here that it is necessarily better or worse than some other approach, or claim that it can be made entirely problem-free. We do argue strongly, however, that if this approach is going to be followed in tax code and regulations, it must be taken consistently. Indeed, we offer a number of examples, involving novel forms of securities, which reveal serious and unanticipated results of inconsistency. These examples may help also in efforts to break away from habits of thinking about taxes chiefly in terms of standard debt instruments as they now exist. In these ways we hope to contribute to a better grounding of tax policy.

The prescriptions for calculating implicit interest income or expenditure through constant yield to maturity affect the calculation of the holder’s *basis* in the instrument—the amount of investment considered to be outstanding on any given date. In this way they also affect the capital gain (or loss) at the time of sale or redemption.

In this study we begin in Section 2 by exploring the mathematics of simple and compound interest, not only for the sake of later being able to express taxes by formulas, but also in the hopes of building a better framework for understanding the approach adopted by Congress and some of the imperfections that have entered in implementing that approach. Keeping to the core of the subject, we ignore instruments involving variable interest *rates* (in contrast to possibly variable interest payment *amounts*). But we account for the possibility of accrual periods of different lengths over which a constant yield rate generates simple interest between compounding dates. Instruments with irregular accrual period lengths arise for a number of reasons, in particular with short periods at the beginning or end, and their tax treatment has persistently been troublesome.

Section 3 describes the three specifications of a debt instrument's content—nominal, revised, and particularized—that may be needed in general to pin down its tax consequences. In setting up these three specifications and carefully distinguishing them from each other, although not all come into play in every instance, we offer a new approach to ascertaining precisely which interpretation of interest or principal may be involved in a given tax rule. This scheme helps accommodate later to the fact that tax computations usually proceed through a series of adjustments to a basic tax amount, where the adjustments stem from possible differences between the three specifications. In cases where two of the specifications turn out to coincide, the corresponding adjustment simply comes out as zero and falls away, and it is not necessary to pass to some alternative mathematical formula in order to arrive at the correct taxes.

Section 4 examines the concepts of discount and premium along with terms like "stated redemption price at maturity" and "qualified periodic interest payments" which figure prominently in regulations. These terms are misnomers referring to artificial quantities inconsistent with the principle of constant yield to maturity or the true amount of discount or premium at original issue, except in special cases like standard bonds.

Section 5 addresses a number of technical questions that have arisen in connection with irregular accrual periods or accounting shifts to different lengths for such periods. It tries to clear up misconceptions about the "accuracy" of different approaches to figuring yield relative to constant yield to maturity. The method of fractional exponents, often described as exact and presented in this light in regulations, is shown to be only approximate, whereas the method held to be only approximate is actually exact.

Section 6 continues with the mathematics of basis and capital gain. The tax rules for long-term instruments are presented in Section 7 and those for short-term instruments in Section 8. Finally, the taxation of stripped instruments and short positions is taken up in Section 9.

The debt instruments we discuss are assumed not to have been *issued in exchange for property*. Such instruments have still other complications. We omit the rules applicable to special instruments that were part of a *corporate reorganization*. Also, we exclude instruments where there was an original (although perhaps unstated) *intention to call before the date of maturity*, or where *the principal is subject to acceleration* or the payments depend on *contingencies*, so that the future cash stream is not completely fixed in advance. Accordingly we always suppose that the date of maturity is known.

The discussion is phrased mainly in terms of the taxes paid by a holder, who may or not be an "original" holder of the debt instrument.[6] With only minor exceptions the tax implications for an issuer, such as deduction of interest expense, are a mirror image of the implications for an original holder who holds to maturity.[7] Of course the tax treatment of the income effects could nonetheless end up being quite different between issuer and holder, since for instance only one might be a taxed entity. An investor who "strips" an existing instrument or assumes a short position is in effect the issuer of a new instrument.

2. MATHEMATICS OF CONSTANT YIELD

What must be specified in a debt instrument, directly or indirectly, in order for the payments it provides to be unambiguous mathematically and clear in their interpretation as interest or repayment of principal, and at the same time consistent with the principle of constant yield to maturity? We pose this question now on the theoretical level, reserving until later the consideration of whether a given specification passes additional tests under current tax law of acceptability as an adequate description of an instrument's economic content relative to the market in which it was acquired.

The bonds traded on Wall Street typically provide "coupon" payments of equal size every six months, but other financial instruments, such as self-amortizing obligations[8] and certain certificates of deposit,[9]exhibit other patterns of payments. In most cases the dates crucial to

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[6] The term *original holder* has a technical meaning in tax literature (and in this study) which is narrower than might be thought, at least in the case of debt instruments publicly issued in multiple copies. It refers in that case to a holder (other than a dealer or broker) who purchased the debt instrument on the date of issue at the first price at which effective sales took place. For an instrument not issued in multiple copies (and not issued in exchange for property) it refers to the first holder.

[7] For each copy of the instrument the interest expense regarded as incurred by the issuer equals the interest regarded as earned by an original holder, and so forth.

[8] This is the technical term for instruments having the form widely seen in mortgages.

[9] The name usually refers to arrangements where one of the parties is a bank, but similar contracts can be made between other parties as well. The characteristic we have in mind is the payment of an initial sum on which interest is earned over a period of years with regular compounding at a fixed rate, but the interest is automatically added to the principal, so that

the calculation of interest income are equally spaced, but short periods in the tax life of an instrument can appear at the beginning or the end, for instance as a result of purchase, sale or "stripping" on a date after original issue, or a delay in the original issue after the payment dates were set. Also, tax rules sometimes require that an instrument be recharacterized in a manner that may give rise to irregular spacing. It is important therefore that we start with a broad view of how interest income may be specified in a financial contract.

Our goal in this section is to develop the mathematics of compound interest from first principles, so as to have a framework for treating in a unified manner a wide variety of instruments, whether existing or contemplated. We aim at demonstrating along the way that there is no difficulty in accommodating irregular spacing of dates, and that efforts to get around such a circumstance by changing the spacing inevitably produce a different "yield" than the one inherently present along with a different view of the size or pacing of interest income or expenditure.

Interest and Principal. The notion of the *annual yield* of an instrument is central. It refers to the rate of interest paid on a yearly basis (prorated for other periods), expressed as a percentage of the remaining *principal*, i.e., of the amount of debt outstanding and therefore earning interest. The realization of this notion is made complicated by the different approaches to adjusting the amount of principal as time goes on, not to speak yet of different interpretations as to how much principal was invested in the first place.

In general, for any instrument, there are periods in which the principal is regarded as constant and *simple interest* is earned. These are separated by dates on which an explicit or implicit payment occurs. Then the principal may jump to a different level (higher or lower, depending on the circumstances), in which case *compound interest* enters the picture, because simple interest over the next period grows from this different level.

The traditional distinction in finance between simple and compound interest is basic to our mathematical approach. We do not imagine that simple interest is "inaccurate" and should therefore somehow always be reconstructed as accruing nonratably (nonlinearly) in the mode of compound interest computed relative to shorter intervals, for instance daily. That would imply wholesale rejection of the way financial contracts have long been formulated—virtually nothing could be accepted any more as valid in its customary statement. Moreover it would lead to unacceptable ambiguity in knowing how much income an instrument was supposed to provide, because of nonuniqueness in the possibilities for reconstruction. We keep to the position that if a change in the degree of compounding is desired, this should be handled by passing to a *different* specification of the instrument in which the insertion of additional compounding dates

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the holder receives no explicit payment until the end.

is acknowledged outright.

The annual yield associated with an instrument will be denoted by y. The simple interest earned at rate y by a principal amount $V > 0$ over a time period whose length in years is θ is

$$I = \theta y V. \tag{2.1}$$

Here θ could be a whole number, or it could be a fraction such as $1/2$, $1/4$, $1/12$, or $1/365$. In a period over which this interest is earned, and at the end of which the holder of the instrument is paid C dollars (with $C \geq 0$), the quantity $R = C - I$ is viewed as (partial) repayment of the principal. The new value of the principal, to be used in computing simple interest over the next period, if any, is therefore $V - R$.

If $C = I$, then $R = 0$, and the new value of the principal is the same as before; no compounding takes place. If $C > I$, then $R > 0$, and the new value of the principal is lower, so that simple interest will accumulate slower in the next period than in the current period. It is also possible, however, to have $C < I$, so that $R < 0$. In this event some of the simple interest that was earned has been added to the principal—a *negative* repayment of principal has occurred. Then the new value of the principal is higher, so that simple interest will accumulate faster in the next period. The difference $I - C$, when positive, is an *implicit* payment of interest supplementary to the *explicit* payment C.

Definition 2.1 (full specification of a debt instrument). *By a full specification of a debt instrument (with constant annual yield), we shall mean a mathematically unambiguous indication of the following data:*

(a) *a value $y \geq 0$, the annual yield rate;*

(b) *a finite sequence of calendar dates $i_0 < \cdots < i_k < \ldots < i_m$, which will in general be termed compounding dates, with i_0 the date of issue, i_m the date of maturity, and the intervals $[i_{k-1}, i_k]$ the accrual periods;*

(c) *for each date i_k beyond i_0 the amount $C_k \geq 0$, the explicit payment to the holder for the period ending on that date, with $C_m > 0$.*

(d) *an initial value $V_0 > 0$, the principal on date i_0, with $V_0 \leq C_1 + \cdots + C_m$.*

Such a specification will be called consistent if these elements have the property that when simple interest at the rate y is applied over each accrual period to the principal outstanding at the beginning of the period and then added to that principal, after which the explicit payment C_k on the date i_k ending the period is subtracted off (with the resulting net amount then taken to be the principal at the beginning of the next period, if any), the principal outstanding at the close of the maturity date i_m (ending the final period) will be 0.

The consistency requirement is all important in what follows. To express it precisely in mathematics, we introduce the parameters

$$\theta_k \;=\; \text{length of the } k\text{th accrual period } [i_{k-1}, i_k] \text{ (measured in years)}, \tag{2.2}$$

as derived from the given dates,[10] along with the symbols

$$V_k \;=\; \text{principal outstanding on date } i_k \text{ after adjustment for any payment.} \tag{2.3}$$

The simple interest earned in the kth accrual period is

$$I_k = \theta_k y V_{k-1} \;\text{ for }\; k = 1, \ldots, m, \tag{2.4}$$

and the amount of principal repaid on date i_k is therefore

$$R_k = C_k - I_k = C_k - \theta_k y V_{k-1} \;\text{ for }\; k = 1, \ldots, m. \tag{2.5}$$

Principal then evolves according to the law

$$V_k = V_{k-1} - R_k = V_{k-1} + I_k - C_k, \tag{2.6}$$

which can be written as

$$V_k = \big(1 + \theta_k y\big) V_{k-1} - C_k \;\text{ for }\; k = 1, \ldots, m. \tag{2.7}$$

Consistency means that when this mathematical law is applied, starting with the given amount V_0, it will be true that $V_m = 0$.

If $I_k = C_k$, then $V_k = V_{k-1}$, while if $I_k < C_k$, there is a positive repayment of principal, so that $V_k < V_{k-1}$. But if $I_k > C_k$, there is a negative repayment of principal, so that $V_k > V_{k-1}$. The amount of increase in principal in this case, which is the reinvested interest amount $I_k - C_k > 0$, is an *implicit payment* to the holder for the accrual period in question beyond the explicit payment C_k, which instead represents money that truly changes hands. Such an implicit payment of reinvested interest can occur in particular for periods in which $C_k = 0$.

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[10] Calendar realities often necessitate compromise in how the accrual period lengths θ_k are to be interpreted, especially when the dates i_k are meant to be "equally spaced." For example, if $i_0, \ldots, i_m$ all fall on the same day of the month in successive months, one may take $\theta_k = 1/12$ for all k even though some months are shorter than others. Similarly, the common case of semiannual compounding is described by $\theta_k = 1/2$ for all k despite the fact that the number of days in a year is not divisible by 2 except in a leap year. February 29, if it appears among the dates, is identified with February 28 in years without such a day when the issue comes up in an attempt to set up equal spacing.

It is useful to think of the dates i_k in Definition 2.1 as placed within a full sequence of dates in time, numbered by $i = 0, 1, \ldots$ (no days skipped). Here $i = 0$ refers to a fixed but arbitrary starting date in the past, at least as early as any other date that might come under consideration. This scheme has systematic advantages in keeping track of a portfolio of instruments, but its chief attraction here arises from the need to be exact about events which are crucial to the taxation of a debt instrument and, in the case of acquisition and disposal, can occur at intermediate dates within an accrual period.

The dates i_k need not be equally spaced (although this is commonly the case), nor do the amounts C_k necessarily have to be regarded as earned by the holder as single-sum payments on those dates in order to satisfy Definition 2.1.[11] But C_k must be the total explicitly paid over the accrual period $[i_{k-1}, i_k]$ and yet not be regarded as potentially affecting the principal in the debt until date i_k. It is possible that nothing is explicitly paid out in certain periods, and for this reason C_k is mathematically allowed to be 0 sometimes. In theory the amounts $C_1, \ldots, C_m$ do not have to follow any special pattern in their magnitudes, as long as the consistency requirement in Definition 2.1 is met.

Of course as a special case there could just be two dates i_0 and i_m with no other compounding dates i_k intervening. Tax laws usually require, however, that the dates not be too far apart.[12]

Although Definition 2.1 speaks of i_0 as the date of issue, there is no change in the nature of the mathematics when i_0 is taken instead to be the date of acquisition of the instrument and V_0 the price of acquisition. This will be explained in Section 3, but it is mentioned now because acquisition can occur on any date, and that is one of the main reasons why instruments with unequal spacing between compounding dates inevitably have to come into consideration in any discussion of the meaning of constant annual yield.

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[11] For accrual-basis taxpayers, interest is regarded as earned evenly over the accrual period in question, as will be explained in full in Section 6. Regulations demand that certain instruments be treated relative to a date sequence different from the given one, in which case the payments corresponding to the given periods must be reassigned to the new periods and thus might not truly arrive on the dates that are used in that different sequence. Such "forced respecification" will be explained in Section 5. In the pending IRS Proposed Regulations of 1992, all of C_k must be received on date i_k, and the circumstances that could trigger forced respecification are relaxed. But dates still should not be more than one year apart.

[12] It will be seen in Section 7 that in most cases (in particular for all bonds issued after 1982) accrual periods are not allowed to exceed one year, and for certain kinds of instruments they must be six months. To meet such conditions it is possible to introduce extra compounding dates in the sequence, taking the explicit payments with respect to those dates to be 0. However, a recomputation of the yield y will then be necessary (unless $y = 0$) in order to maintain the properties in Definition 2.1, as explained in Section 4.

The formulas after Definition 2.1 for the evolution of principal underscore the fact that the concept of constant annual yield works straightforwardly regardless of whether accrual periods are *regular*—all of the same length—or *irregular*. The general case is not well treated by textbooks in finance, so we develop its fundamental properties here with care. The central result will be the yield-to-maturity equation in Theorem 2.2.

The fact that Definition 2.1 requires an up-front listing of the dates i_k as part of the very meaning of consistency cannot be overemphasized. In our scheme it makes no sense to speak of first determining the yield of an instrument and only then deciding what accrual periods to use in accounting for its interest stream.[13] The two go hand in hand. While some of the data elements in Definition 2.1 can be inferred unambiguously from the others (under the assumption that a consistent, full specification—with constant yield—is intended), the dates i_k and their spacing generally cannot be so inferred, being instead a matter of opinion.

Often instruments are written not directly in terms of a yield y and overall payments C_k but rather in terms of certain amounts designated as "interest" and "repayment of principal" (the latter including a redemption amount at maturity), which are to be paid on stated dates. Whether or not this information uniquely determines a consistent, full specification in the sense of Definition 2.1 is then a subject for analysis. The stated dates can be taken as the dates i_k, and the payment amounts on those dates interpreted as I_k and R_k (these sometimes possibly zero), with C_k identified as the sum $I_k + R_k$. Then, as long as a value for V_0 is indicated, it will be possible through the yield-to-maturity equation in Theorem 2.2 to infer a value for y and test whether the stated I_k and R_k amounts agree with the ones generated from y, V_0, and the payments C_k by (2.4)–(2.5). If so, a consistent, full specification is at hand. If not, it may nonetheless be possible to achieve such a specification by supplementing the given data in one way or another, for example by enriching the sequence of dates. But in this case the resulting consistent, full specification is not uniquely determined, since it depends on this input, which could be provided in more than one way.[14] If even this does not work, the instrument falls outside the category addressed in this study.[15]

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[13] This pattern appears for example in the IRS Proposed Regulations of 1986, S1.1272–1(b), where it seems though that the principle of constant yield to maturity is not strictly adhered to; see the discussion of the "method of fractional exponents" below and in Section 5.

[14] A strength of the scheme offered here is this clarification of the extent to which an instrument is well specified to begin with, and the degree of arbitrariness in filling in missing details in order to arrive at consistent interpretation, if possible.

[15] This is not to say that such an instrument is necessarily improper in any way, but merely that its treatment may involve other features, like variable interest rates. Such features could be the subject of further mathematical developments, but we have chosen not to undertake them

We move ahead now with the main facts about instruments with constant yield. In developing these facts, the mathematical concept of "net return" must be pinned down beyond the possibility of conflicting interpretations.

Definition 2.2 (net return on a debt instrument). *By the net return on a debt instrument, as associated with a consistent, full specification in the sense of Definition 2.1, we shall mean the amount*

$$C_1 + \cdots + C_m - V_0. \tag{2.8}$$

The instrument will be said to have positive net return when this amount is positive.

Stipulation (d) in Definition 2.1 says that the net return, as defined in this way, cannot be negative. The case of zero net return does arise in practice with so-called zero-coupon bonds, as will be seen in Example 3.3. Note that net return refers to a quantity dependent on an instrument's specification not only through the explicit payment amounts C_k but also through the stated amount of initial principal. When one value of V_0 is replaced by another, as will usually be the case for tax purposes through the supplementary specifications to be described in Section 3, the net return changes as well. Note further that no discounting of future income is involved in this concept, nor does it include any possible effects of capital gain or loss from a subsequent sale. The analysis for now is based on holding an instrument to maturity. The framework for analyzing capital gains will be set up in Section 6.

The theorem stated next confirms that any consistent, full specification of a debt instrument will bring with it the kinds of mathematical consequences that must be demanded for the concept to be acceptable. By attending to the criteria provided by Definition 2.1, we can be sure of not having to face, perhaps in special circumstances not yet thought of, any case where interest payments might not be positive, or where the amount of principal said to be outstanding might dip below zero.

Theorem 2.1 (positivity of principal and interest). *Relative to a consistent, full specification of a debt instrument in the sense of Definition 2.1, the principal values V_k satisfy*

$$V_k > 0 \ \text{ for } \ k = 0, 1, \ldots, m-1 \ \text{ (but } V_m = 0\text{).} \tag{2.9}$$

Thus, $I_k > 0$ for $k = 1, \ldots, m$, unless $y = 0$ (in which case $I_k = 0$ for all k), the latter being true if and only if the instrument fails to have positive net return. In general,

$$\begin{aligned} V_k &= R_{k+1} + \cdots + R_m \\ &= \big(C_{k+1} + \cdots + C_m\big) - \big(I_{k+1} + \cdots + I_m\big) \ \text{ for } \ k = 0, 1, \ldots, m-1 \end{aligned} \tag{2.10}$$

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here.

with $R_m > 0$, and in particular

$$C_1 + \cdots + C_m - V_0 \;=\; I_1 + \cdots + I_m. \tag{2.11}$$

Proof. As already noted, $V_m = 0$ by the consistency requirement in Definition 2.1. From (2.7) for $k = m$ we therefore have

$$V_{m-1} = \big(1 + \theta_m y\big)^{-1} C_m > 0, \tag{2.12}$$

because $C_m > 0$ in part (c) of Definition 2.1. Since $R_m = V_{m-1} - V_m$ by definition, we get $R_m > 0$ also. Working backwards from (2.12) using (2.7) in the form of the relation

$$V_{k-1} = \big(1 + \theta_k y\big)^{-1} \big[C_k + V_k\big], \tag{2.13}$$

we see that whenever $V_k > 0$ we also have $V_{k-1} > 0$, since $C_k \geq 0$. Thus, all the principal values V_k before date i_m must be positive. Equation (2.10) follows from the fact that first $0 = V_m = V_{m-1} - R_m$, then $V_{m-1} = V_{m-2} - R_{m-1}$, and so forth, using (2.5). Equation (2.11) is obtained from (2.10) by taking $k = 0$. $\square$

(Note: the square symbol at the end of the preceding line is used to mark the end of a mathematical proof. This way a signal is given that the text is about to resume in ordinary mode. Some readers may wish to skip over the details of the mathematical arguments in the formal proofs, at least at first pass.)

As explained, Theorem 2.1 assures us that no mathematical surprises lurk in Definition 2.1. The sign conditions on the payments C_k and the stipulation that $V_m = 0$ (i.e., consistency) are enough in themselves to exclude weird situations such as negative principal due to overpayment to the holder. The conditions guarantee that the amounts of interest earned and the amounts of principal repaid add up as they should. Equation (2.11) asserts that the net return in Definition 2.2 is the same as the total of all the interest payments associated with the specification. This helps to legitimize the terminology, since otherwise the total of all interest payments might come up as a competing notion of "net return." Uncertainty might then enter over whether that would have to be treated separately, and thus over whether words were being used appropriately in the theory.

Theorem 2.1 says in this way that, by the end, the holder gets back the entire amount V_0 of initial principal, and that everything else explicitly received is properly accounted for as interest, whether fully made available when earned or only later. This mathematical conclusion is worth paying careful attention to, because a number of provisions in tax regulations will later emerge as based perhaps on misapprehensions about whether an instrument properly represents the income it provides a holder in its face value description. According to Theorem 2.1, there is no cause for worry if the description constitutes (directly or indirectly, but unambiguously) a consistent, full specification in the sense of Definition 2.1. This result provides important justification for the concept of such a specification and points to a better way of addressing concerns than the problem-laden regulatory approach that must now be coped with in determining interest income, which we will get to in Section 4.

The centerpiece of the theory of constant yield is the following statement.

Theorem 2.2 (general yield-to-maturity equation)**.** *For any sequence of dates i_k, payment amounts C_k, and initial value V_0 satisfying conditions (b), (c), and (d) of Definition 2.1 there exists a unique annual yield rate y for (a) such that a consistent full specification of the debt instrument is achieved. This value y, which is positive if and only if the instrument has positive net return in the sense of Definition 2.2 (otherwise both y and the net return are zero), is the unique solution to the equation*

$$V_0 = \frac{C_1}{(1+\theta_1 y)} + \frac{C_2}{(1+\theta_1 y)(1+\theta_2 y)} + \cdots + \frac{C_m}{(1+\theta_1 y)(1+\theta_2 y)\cdots(1+\theta_m y)}. \tag{2.14}$$

Proof. The fact that equation (2.14) follows from the conditions in Definition 2.1 is obtained by applying the relation in (2.13) repeatedly, starting from (2.12). (First, substitute the right side of (2.12) for V_{m-1} in the form of (2.13) with $k-1 = m-2$. Second, substitute the right side of that into the form of (2.13) with $k-1 = m-3$. Continuing this way for $m-3$ more steps, arrive at (2.14).)

Now, assuming only the data and conditions in (b)(c)(d) of Definition 2.1, consider the right side of (2.14) as defining a function $f(y)$ of a variable y in the infinite interval $[0,\infty)$ of conceivable yield rates. This function f is continuous, because it is a sum of rational functions in which the denominators are positive over the interval in question. The function f is decreasing in the sense that if $0 \le y' < y''$, then $f(y') > f(y'')$. (The latter holds because $(1+\theta_k y') < (1+\theta_k y'')$ for all k, and the values C_k are nonnegative with C_m actually positive.) Furthermore $f(0) = C_1 + \cdots + C_m$, while $\lim_{y\to\infty} f(y) = 0$. These facts imply through the standard inverse function theorem in elementary calculus that for any value V_0 in the interval $\big(0,\, C_1 + \cdots + C_m\big]$ there exists a unique y in the interval $[0,\infty)$ such that $f(y) = V_0$. Equation (2.14) thus has a unique solution y, as claimed, and this solution is $y = 0$ if and only if $V_0 = C_1 + \cdots + C_m$, i.e., the instrument has zero net return. The yield value y provides a sequence of principal values $V_1, \ldots, V_m$ that are generated from V_0. From (2.7) we have

$$V_1 = (1+\theta_1 y)V_0 - C_1, \quad V_2 = (1+\theta_2 y)V_1 - C_2, \quad \ldots \quad V_m = (1+\theta_{m-1} y)V_{m-1} - C_m.$$

Substituting the right side of (2.14) for V_0 in the first of these relations, we get

$$V_1 = \frac{C_2}{(1+\theta_2 y)} + \frac{C_3}{(1+\theta_2 y)(1+\theta_3 y)} + \cdots + \frac{C_m}{(1+\theta_2 y)(1+\theta_3 y)\cdots(1+\theta_m y)}.$$

Next, substituting this expression for V_1 in the relation $V_2 = (1+\theta_2 y)V_1$, we get

$$V_2 = \frac{C_3}{(1+\theta_3 y)} + \frac{C_4}{(1+\theta_3 y)(1+\theta_4 y)} + \cdots + \frac{C_m}{(1+\theta_3 y)(1+\theta_4 y)\cdots(1+\theta_m y)}.$$

Continuing inductively in this manner we deduce that

$$V_k = \frac{C_{k+1}}{(1+\theta_{k+1} y)} + \frac{C_{k+2}}{(1+\theta_{k+1} y)(1+\theta_{k+2} y)} + \cdots + \frac{C_m}{(1+\theta_{k+1} y)(1+\theta_{k+2} y)\cdots(1+\theta_m y)} \quad \text{for each } k \le m-1, \tag{2.15}$$

and in particular, as the case where $k = m-1$, that $V_{m-1} = (1+\theta_m y)^{-1} C_m$. When this expression for V_{m-1} is substituted into the evolution equation $V_m = (1+\theta_m y)V_{m-1} - C_m$, which is the case of (2.7) with $k = m$, we obtain $V_m = 0$. We conclude that when the unique value y obtained from the data in (b)(c)(d) of Definition 2.1 is taken as the value in (a) so as to have a full specification of a debt instrument, this specification meets the test of consistency. □

Theorem 2.2 provides mathematical support that is absolutely essential for a sound treatment of interest and principal in terms of constant annual yield. Among the features most to be noted are the theorem's generality. In order to be sure of the existence of an annual yield rate with all the properties that notion should entail, only the tests in Definition 2.1 have to be passed. The focus in discussing any particular instrument with a possibly loose description merely has to be narrowed, by inferring or supplying additional details if necessary, to the stage where a single specification in the sense of that definition has been identified. Aside from this, there are no requirements. All imaginable debt instruments are covered, no matter how strange their pattern of payments in comparison with today's familiar instruments, as long as they fit with Definition 2.1.

On the other hand, once Theorem 2.2 has been applied and the existence of a constant annual yield rate has thereby been ascertained, we know it is the *only* rate value fitting the common sense prescriptions. Any other value *must* be out of harmony with financial truth, maybe even to the extent of carrying hidden absurdities (as some examples in Sections 4 and 5 later will illustrate). This is important because it furnishes a standard that any method proposed for determining annual yield must measure up to. If the number the method comes up with as the yield is always the same as the one in Theorem 2.2, fine. Otherwise, the method cannot be correct.

The existence of a unique solution y to the general yield-to-maturity equation (2.14) should, of course, be seen separately from the issue of how the solution may actually be calculated from given values of V_0 and C_k and the period lengths θ_k. Apart from some elementary situations, the solution will not be expressible by any direct algebraic formula and will have to be computed by numerical methods. This is as true when the accrual periods are regular as well as when they are irregular; no extra difficulty has been created by the moving to the level of generality in Theorem 2.2. The numerical procedures needed for the computation are mathematically available, but the financial calculators now in use have only been programmed for special cases. Of course, this could easily be remedied.

The yield-to-maturity equation (2.14) is the mathematical core of the methodology in this study. In its general form it appears relatively unknown in the financial community, but the case of equally spaced dates is widely familiar and acknowledged as the basis for current approaches to the taxation of interest income and expense.

Equally Spaced Dates. For regular instruments, where the compounding dates i_k are equally spaced,[16] the accrual period lengths all have the same size: $\theta_k = \theta$ for $k = 1, \dots, m$. The yield-to-maturity equation (2.14) then takes the special form

$$V_0 = \frac{C_1}{(1+\theta y)} + \frac{C_2}{(1+\theta y)^2} + \cdots + \frac{C_m}{(1+\theta y)^m}. \tag{2.16}$$

Annual compounding corresponds to $\theta = 1$, semiannual to $\theta = 1/2$, quarterly to $\theta = 1/4$, and daily to $\theta = 1/365$. The residual equation (2.15) specializes similarly to

$$V_k = \frac{C_{k+1}}{(1+\theta y)} + \frac{C_{k+2}}{(1+\theta y)^2} + \cdots + \frac{C_m}{(1+\theta y)^{m-k}}. \tag{2.17}$$

Other Approaches to Characterizing Yield. In place of the yield-to-maturity equation in Theorem 2.2 a different equation, involving fractional exponents, is often seen in tax literature and regulations. This equation, which will be discussed in Section 5, is not a correct yield-to-maturity equation relative to the data given, but it can be regarded as one relative to some altered specification with additional compounding dates.[17] As mentioned already, our position is that if the insertion of more compounding dates is deemed desirable, this should be carried out explicitly. Otherwise, doubts enter about the magnitude of various quantities, such as the

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[16] See Footnote 10.

[17] Although such an interpretation is possible, the different yield rate so obtained is not actually employed as it correctly would be in that context, because the inserted dates are suppressed; see Section 5.

yield rate, and the soundness of the mathematical theory is compromised. Indeed, the use of the method of fractional exponents typically designates a higher value than y as the yield rate (Theorem 4.2). In some situations this value is not unique, because it is influenced by how many extra dates are involved, which is left up to party doing the calculation.

Deferred Interest. An important question is whether an amount $I_k - C_k > 0$ of implicit interest should be regarded as "really" paid by the issuer to the holder on date i_k. Tax regulations are inconsistent in their view of this, as will be seen in Section 4 (in particular Example 4.1), but ultimately they do generally categorize such implicit interest amounts as current taxable income to the holder and current deductible expense to the issuer. An alternative in theory would be to handle these amounts as *deferred* payments of interest—having tax consequences only at a later date when money actually changes hands.

Mathematically rigorous formulas, building an extra layer on the formulas already laid out in this section, can easily be set down for determining exactly when the deferred payments are finally made and in what sizes. In this extension of the theory the principal amount V_k that is outstanding at the end of date i_k is furnished with an additional representation as

$$V_k = W_k + D_k, \tag{2.18}$$

where W_k is the amount of the *original* principal that is outstanding, while D_k is the amount of deferred interest yet to be paid; initially $W_0 = V_0$ and $D_0 = 0$, and thereafter always $W_k \leq V_0$ and $D_k \geq 0$. In accordance with this representation each explicit payment C_k has an expression

$$C_k = J_k + S_k, \tag{2.19}$$

where J_k is the amount of interest explicitly paid on date i_k, and S_k is the amount of *original* principal repaid on date i_k and is *never negative*—in contrast to R_k, which is negative whenever interest is being reinvested. Some of J_k may be interest earned in the current period, but other portions of J_k may be interest finally being handed over that was earned in one or more earlier periods. The formulas governing J_k, S_k, W_k, and D_k, are

$$
\begin{aligned}
J_k &= \begin{cases} C_k & \text{if } D_{k-1} + I_k \geq C_k, \\ D_{k-1} + I_k & \text{if } D_{k-1} + I_k < C_k, \end{cases} \\[2ex]
S_k &= \begin{cases} 0 & \text{if } D_{k-1} + I_k \geq C_k, \\ C_k - J_k \leq 0 & \text{if } D_{k-1} + I_k < C_k, \end{cases} \\[2ex]
W_k &= W_{k-1} - S_k \leq W_{k-1}, \\[2ex]
D_k &= \begin{cases} D_{k-1} + I_k - C_k & \text{if } D_{k-1} + I_k \geq C_k, \\ 0 & \text{if } D_{k-1} + I_k < C_k. \end{cases}
\end{aligned}
\tag{2.20}
$$

These recursive formulas are uniquely dictated by the principle of assigning as much as possible of each explicit payment C_k to payment of current interest and then, if some of C_k is still left over, to payment of deferred interest. Only if a fraction of C_k remains after all current and all deferred interest has been explicitly been paid out is that remainder characterized as repayment of original principal. One always has

$$W_k \;=\; V_0 - [\text{all } \textit{original} \text{ principal repaid through period } k]. \tag{2.21}$$

3. SPECIFICATIONS UNDERLYING TAXATION

In general, the tax-determining formulas to be presented in Section 7 require knowledge of up to three potentially different specifications in the sense of Definition 2.1 for the same debt instrument. These specifications, which we refer to as the nominal, revised, and particularized specifications, will be explained here with examples. Roughly, the "nominal" specification refers to the original, stated description of the instrument, the "revised" to its actual economic content at time of issue, and the "particularized" to its economic content relative to the market circumstances of its acquisition by a particular investor at a subsequent date.

Nominal Specification. From now on, each debt instrument under discussion will be assumed to have a face value description which is a consistent, full specification in the sense of Definition 2.1.[18] This will be called its *nominal specification*, and the term "nominal" will also

[18] Only the data in (b)(c)(d) of Definition 2.1 need be given; the unique corresponding yield y is then automatically supplied through the equation in Theorem 2.2 and can be computed by numerical methods. On the other hand, if only the data in (a)(b)(c) are furnished the corresponding value of V_0 can be obtained from the same equation.

be attached to all the quantities like yield, interest payments and principal repayments that are associated with it. Such quantities will continue to be denoted by the symbols used in Section 2. (Asterisks and double asterisks will mark the symbols used for the revised and particularized specifications introduced below.)

Example 3.1 (self-amortizing obligation, nominal specification). *Suppose the accrual periods all have the same length θ and the payments $C_1, \dots, C_m$ are all equal to the same amount $C > 0$. Any two of the three numbers C, V_0 and y determines the third through the specialized yield-to-maturity equation (2.16) (under the stipulation that $0 < V_0 \leq mC$ and $y \geq 0$). From equation (2.17) with*

$$x = (1 + \theta y)^{-1} \tag{3.1}$$

we have $V_k = C(x + \cdots + x^{m-k})$ but on the other hand $V_{k-1} = C(x + \cdots + x^{m-k} + x^{m-k+1})$, so that

$$V_{k-1} - V_k = Cx^{m-k+1}.$$

Here $0 < x^{m-k+1} \leq 1$ because $0 < x \leq 1$. Since $I_k = C_k - R_k$ in (2.5) (now with $C_k = C$), whereas $R_k = V_{k-1} - V_k$ from (2.6), the expression just obtained for $V_{k-1} - V_k$ gives us

$$I_k = (1 - x^{m-k+1})C \geq 0 \;\text{ and }\; R_k = x^{m-k+1}C > 0 \;\text{ for }\; k = 1, \dots, m. \tag{3.2}$$

The amounts V_k and I_k decrease with k, while the amounts R_k increase with k.

Example 3.2 (standard bond, nominal specification). *Suppose the accrual periods all have the same length θ and the payments $C_1, \dots, C_{m-1}$ are all equal to the same number $C \geq 0$, but $C_m = V_0 + C$. In the notation (3.1) the yield-to-maturity equation (2.16) then takes the form*

$$V_0 = C(x + \cdots + x^m) + V_0 x^m,$$

so that $C/V_0 = (1 - x^m)/(x + \cdots + x^m)$. Because the numerator in this ratio can be factored as $1 - x^m = (1 - x)(1 + x + \cdots + x^{m-1})$ while the denominator can be factored as $x + \cdots + x^m = x(1 + x + \cdots + x^{m-1})$, the ratio can be reduced to $C/V_0 = (1 - x)/x = \theta y$. Thus,

$$y = \frac{C}{\theta V_0}. \tag{3.3}$$

In (2.7) we therefore have $V_k = \big[1 + (C/V_0)\big]V_{k-1} - C$ for $k = 1, \dots, m-1$. This equation implies $V_1 = V_0$, and then recursively that $V_k = V_0$ for $k = 1, \dots, m-1$, although $V_m = 0$. It follows next through (2.4) and (2.6) that

$$I_k = C \;\text{ for all }\; k, \;\text{ whereas }\; R_k = 0 \;\text{ for }\; k = 1, \dots, m-1 \;\text{ but }\; R_m = V_0. \tag{3.4}$$

The value C is called the coupon amount in the bond.

Example 3.3 (zero-coupon bond, nominal specification). *A zero-coupon bond is a standard bond in the sense of Example 3.2 for which the coupon amount is $C = 0$. Regardless of the sequence of dates i_k that might be indicated (perhaps the only dates are i_0 and i_m), the annual yield is $y = 0$. The only positive explicit payment is at the end, when the amount V_0 is paid back. One has*

$$\begin{aligned} I_k &= 0 \ \text{ for } \ k = 1, \ldots, m, \\ R_k &= 0 \ \text{ for } \ k = 1, \ldots, m-1, \ \text{ with } \ R_m = V_0. \end{aligned} \tag{3.5}$$

On the face of things, there is no interest income generated at all; the net return in the sense of Definition 2.2 is zero.

To the uninitiated, a zero-coupon bond may look like a worthless investment because, relative to its nominal specification, it is said to provide no income. But there is more to it than that. The explanation will proceed when we come to the "revised specification."

Example 3.4 (CD-like instrument, nominal specification). *Suppose that the accrual periods all have the same length θ, and that the payments $C_1, \ldots, C_{m-1}$ are all 0, but that the final payment is $C_m = V_0 + N$ for some amount $N > 0$.*[19] *The amount N is the net return in the sense of Definition 2.2. The yield-to-maturity equation in form (2.16) is*

$$V_0 \ = \ \frac{V_0 + N}{(1 + \theta y)^m}.$$

It can be solved algebraically for the yield rate y:

$$y \ = \ \frac{1}{\theta}\left[\left(1 + \frac{N}{V_0}\right)^{1/m} - 1\right].$$

From (2.5) we have negative repayments of principal in every period except the last,

$$R_k = -I_k = -\theta y V_{k-1} \ \text{ for } \ k = 1, \ldots, m-1.$$

In other words, the interest received by the holder is always reinvested automatically. A positive repayment of principal does occur for the final period, however. All of the final payment C_m except for the amount

$$I_m = C_m - V_{m-1} = V_0 + N - (1 + \theta y)^{m-1} V_0 = N - \big[(1 + \theta y)^{m-1} - 1\big] V_0$$

is repayment of principal (although some of it may be beyond the original principal, because of representing new principal added from interest that was paid out along the way.)

[19] This is the pattern for a common form of CD (certificate of deposit) issued by banks, but it could also be followed by bonds issued by other entities.

The contrast between Examples 3.3 and 3.4 is that in Example 3.4 the yield is positive, not zero. The initial principal V_0 in Example 3.4 is less than the final payment C_m, whereas in Example 3.3 it is the same. We shall see that for tax purposes, however, that a zero-coupon bond typically ends up being recharacterized as a CD-like instrument through replacing the nominal value V_0 in Example 3.3 by a revised value which is lower. CD-like instruments thus exhibit the kind of pattern of implicit interest payments that tax rules now hold up as representing receipt of taxable income by the holder. For further comparison, see Example 4.1.

Revised Specification. The nominal initial value V_0 of a debt instrument may not reflect its true economic content at time of issue. A better indicator of this content is the amount paid by an original holder of the instrument.[20] We denote this amount by V_0^* and call it the *revised initial value*, or the *issue value*.[21] It represents what the instrument was worth in the market prevailing at time of issue.

It might expected that V_0^* ought to appear in the face description of the instrument in place of V_0, but custom dictates that V_0 is a standard amount (like \$100,000), whereas V_0^* is an odd amount determined only at the time of initial sale, often by auction. Anyway, a simple substitution of V_0^* for V_0 is not possible without other shifts in the specification of the instrument. This is because of the interrelationships between initial principal, explicit payments, compounding dates, and annual yield that must be satisfied in order for the specification to meet the test of consistency laid down in Section 2.

The way the difficulty is handled is to view the specification of the instrument as revised in the following manner which respects the conditions in Definition 2.1 and therefore provides another *consistent*, full specification with constant yield to maturity. The explicit payment amounts C_k are retained along with the given compounding dates i_k[22] But V_0 is replaced by V_0^* in tandem with replacing y by the new yield value y^* obtained as the unique solution to the shifted yield-to-maturity equation

$$V_0^* = \frac{C_1}{(1+\theta_1 y^*)} + \frac{C_2}{(1+\theta_1 y^*)(1+\theta_2 y^*)} + \cdots + \frac{C_m}{(1+\theta_1 y^*)(1+\theta_2 y^*)\cdots(1+\theta_m y^*)}, \tag{3.6}$$

or in the case of equally spaced dates with $\theta_k = \theta$,

$$V_0^* = \frac{C_1}{(1+\theta y^*)} + \frac{C_2}{(1+\theta y^*)^2} + \cdots + \frac{C_m}{(1+\theta y^*)^m}. \tag{3.7}$$

[20] For the definition of an original holder, see Footnote 6.

[21] In tax regulations this is generally called the *issue price*.

[22] As long as these are frequent enough and regular enough not to trigger forced respecification as described in Section 5. The circumstances under which this would happen will be dealt with in Section 7.

This value y^* is the *revised yield* for the instrument. It generates the sequence of *revised principal values*

$$V_k^* = (1+\theta_k y^*)V_{k-1}^* - C_k \ \text{ for } \ k = 1,\dots,m, \tag{3.8}$$

and a correspondingly different decomposition of the explicit payments C_k into *revised interest* amounts

$$I_k^* = \theta_k y^* V_k^* \ \text{ for } \ k = 1,\dots,m, \tag{3.9}$$

and *revised principal repayment* amounts

$$R_k^* = C_k - I_k^* = V_{k-1}^* - V_k^*. \ \text{for} \ k = 1,\dots,m. \tag{3.10}$$

These quantities satisfy all the relationships developed earlier, except for the change in notation from plain symbols to symbols adorned with asterisks. As an application of Theorem 2.1, the values V_k^* for $k < m$ are all positive (but $V_m^* = 0$), and

$$\begin{aligned} V_k^* &= R_{k+1}^* + \cdots + R_m^* \\ &= \big(C_{k+1} + \cdots C_m\big) - \big(I_{k+1}^* + \cdots + I_m^*\big) \ \text{ for } \ k = 0,1,\dots,m-1, \end{aligned} \tag{3.11}$$

with $R_m^* > 0$. In particular

$$C_1 + \cdots + C_m - V_0^* \ = \ I_1^* + \cdots + I_m^*, \tag{3.12}$$

this quantity being the *revised net return* on the instrument. The most important thing to bear in mind is that when $V_0^* \neq V_0$ there may be implicit payments of interest under the revised specification where none were called for under the nominal specification. This stems from the observation that in cases where $I_k \leq C_k$ one could nevertheless have $I_k^* > C_k$ if $I_k^* > I_k$.

The theorem stated next describes the exact relationship between the interest payments under the nominal and revised specifications of a given instrument. It shows how the associated differences in annual yield create corresponding differences in *cumulative* payment of interest, although not necessarily between individual payments.

Theorem 3.1 (implicit interest under the revised specification)**.** *Under the revised specification, in comparison with the nominal specification, one has*

$$\begin{aligned} V_0 - V_0^* &= (I_1^* - I_1) + (I_2^* - I_2) + \cdots + (I_m^* - I_m) \\ &= (R_1 - R_1^*) + (R_2 - R_2^*) + \cdots + (R_m - R_m^*). \end{aligned} \tag{3.13}$$

If $V_0^* < V_0$, one has $y^* > y$, and the interest paid over the remaining future is always higher:

$$I_k^* + \cdots + I_m^* \ > \ I_k + \cdots + I_m \ \textit{ for } \ k = 1,\dots,m, \tag{3.14}$$

If $V_0^* > V_0$, one instead has $y^* < y$, and

$$I_k^* + \cdots + I_m^* \ < \ I_k + \cdots + I_m \ \textit{ for } \ k = 1,\dots,m, \tag{3.15}$$

Proof. Equation (3.13) is immediate from equations (2.11) and (3.12), along with (2.5) and its counterpart (3.10). The strict monotonicity of the function f underlying the yield-to-maturity equation in the proof of Theorem 2.2 implies that if everything is kept fixed in an instrument's specification except that the initial value is replaced by a lower value, then the yield goes up. This establishes the relationships claimed between y and y^*. More generally, we may compare the formula

$$\begin{aligned} V_{k-1} = \frac{C_k}{(1+\theta_k y)} + & \frac{C_{k+1}}{(1+\theta_k y)(1+\theta_{k+1} y)} \\ & + \cdots + \frac{C_m}{(1+\theta_k y)(1+\theta_{k+1} y)\cdots(1+\theta_m y)}, \end{aligned} \tag{3.16}$$

obtained from (2.15) with the corresponding formula with respect to the revised specification, which is

$$\begin{aligned} V^*_{k-1} = \frac{C_k}{(1+\theta_k y^*)} + & \frac{C_{k+1}}{(1+\theta_k y^*)(1+\theta_{k+1} y^*)} \\ & + \cdots + \frac{C_m}{(1+\theta_k y^*)(1+\theta_{k+1} y^*)\cdots(1+\theta_m y^*)}, \end{aligned} \tag{3.17}$$

to see that when $y^* > y$ we have $V^*_{k-1} < V_{k-1}$ for $k = 1, \ldots, m$, but when $y^* < y$ the opposite holds. Since $I_k + \cdots + I_m = C_k + \cdots + C_m - V_{k-1}$ by Theorem 2.1, and by the same token $I^*_k + \cdots + I^*_m = C_k + \cdots + C_m - V^*_{k-1}$, the relations in (3.14) and (3.15) (according to whether y^* is higher or lower than y) are correct. □

Example 3.5 (self-amortizing obligation, revised specification). *The case of $V_0 - V_0^* > 0$ for a self-amortizing obligation*[23] *in Example 3.1 has the effect of shifting the interpretation of the constant payment amounts C for each period more towards interest and away from repayment of principal. The case of $V_0^* - V_0 > 0$ has the opposite effect. To verify this mathematically, first combine the formula $I_k = \theta y V_{k-1}$ with the formula for V_{k-1} in (3.16), specialized to the case where $\theta_k = \theta$ and $C_k = C$ for all k, to obtain*

$$\frac{I_k}{\theta y} = \frac{C}{(1+\theta y)} + \frac{C}{(1+\theta y)^2} + \cdots + \frac{C}{(1+\theta y)^{m-k+1}}.$$

In terms of $x = 1/(1+\theta y)$, for which $\theta y = (1-x)/x$, we get from this the equation

$$\begin{aligned} \frac{I_k}{C} &= \left[\frac{1-x}{x}\right]\big(x + x^2 + \cdots + x^{m-k+1}\big) \\ &= \big(1-x\big)\big(1 + x + \cdots + x^{m-k}\big) = 1 - x^{m-k+1}, \end{aligned}$$

[23] This situation typically arises when the holder of the obligation requires additional "points" from the party that assumes the obligation.

so that

$$I_k \;=\; C\left[1 - \frac{1}{(1+\theta y)^{m-k+1}}\right].$$

By a parallel route we arrive at

$$I_k^* \;=\; C\left[1 - \frac{1}{(1+\theta y^*)^{m-k+1}}\right].$$

From a comparison of these results it is clear that $I_k^* > I_k$ for all k when $y^* > y$, but the opposite holds when $y^* < y$.

Example 3.6 (standard bond, revised specification). *For a typical bond, as in Example 3.2, the yield-to-maturity equation to be solved for the revised yield y^* has the special form*

$$V_0^* = C\left[\frac{1}{(1+\theta y^*)} + \frac{1}{(1+\theta y^*)^2} + \cdots + \frac{1}{(1+\theta y^*)^m}\right] + V_0\left[\frac{1}{(1+\theta y^*)^m}\right]. \tag{3.18}$$

When V_0^* is different from V_0, the solution y^* to this equation cannot be expressed by an algebraic formula in terms of V_0^*, V_0 and C; there is no analog for y^* of the simple relation $y = C/\theta V_0$. Nonetheless y^* can be determined by numerical methods. Furthermore

$$I_k^* - I_k = I_k^* - C > 0 \;\text{ for all } k \;\text{ when } V_0^* < V_0, \tag{3.19}$$

and these implicit interest amounts add up then to the difference between the nominal initial value and the revised initial value,

$$[I_1^* - C] + \cdots + [I_m^* - C] = V_0 - V_0^*. \tag{3.20}$$

Equation (3.20) specializes (3.13) to the fact that $I_k = C$ in this case (see Example 3.2), while the proof of (3.19) goes as follows. We begin with the version of (3.17) that corresponds to the data under consideration, namely

$$V_{k-1}^* = C\left[\frac{1}{(1+\theta y^*)} + \frac{1}{(1+\theta y^*)^2} + \cdots + \frac{1}{(1+\theta y^*)^{m-k+1}}\right] + V_0\left[\frac{1}{(1+\theta y^*)^{m-k+1}}\right]. \tag{3.21}$$

Letting u stand for $1/(1+\theta y^*)$ so that $\theta y^* = (1-u)/u$, and noting the algebraic relation $\big[(1-u)/u\big]\big(u + u^2 + \cdots + u^{m-k+1}\big) = 1 - u^{m-k+1}$, we can write (3.21) as

$$\theta y^* V_{k-1}^* = C\big(1 - u^{m-k+1}\big) + \theta y^* V_0 u^{m-k+1},$$

where $\theta y^* V_{k-1}^* = I_k^*$ and $V_0 = C/\theta y$. Thus,

$$\frac{I_k^*}{C} = 1 - u^{m-k+1} + \frac{y^*}{y} u^{m-k+1} = 1 + \left[\frac{y^*}{y} - 1\right] u^{m-k+1}.$$

Since $u > 0$ the right side of this equation is larger than 1 when $y^* > y$ (as is true by Theorem 3.1 when $V_0^* < V_0$), so $I_k^*/C > 1$ in that case. This establishes (3.19). The counterpart to this result if instead $V_0^* > V_0$ is the inequality

$$I_k - I_k^* = C - I_k^* > 0 \text{ for all } k \text{ when } V_0^* > V_0, \tag{3.22}$$

which is proved by the same argument. In this case the coupon amounts C would no longer be entirely interest but would include some repayment of principal. The sum of such implicit repayment amounts would equal the premium amount $V_0^* - V_0 > 0$.

Example 3.7 (zero-coupon bond, revised specification). *Under the revised specification of a zero-coupon bond, the case of Example 3.6 where $C = 0$, implicit interest appears when $V_0^* < V_0$ despite the fact that in the nominal specification there was no interest income indicated at all. On each date i_k beyond i_0 one will have $I_k^* > 0$, whereas $I_k = C = 0$. The implicit interest payments will add up to the difference between the nominal initial value and the revised initial value:*

$$I_1^* + \cdots + I_m^* \;=\; V_0 - V_0^*, \tag{3.23}$$

as seen from (3.12).

In effect, a zero-coupon bond is converted under the revised specification into a debt instrument like a certificate of deposit under its nominal specification as in Example 3.4.

Example 3.8 (CD-like instrument, revised specification). *If a certificate of deposit, or like instrument as described in Example 3.4, where all interest is automatically reinvested, were somehow acquired by an original holder for an amount $V_0^* \neq V_0$, the only effect would be to replace V_0 by V_0^* in the same yield formula in order to get the revised yield y^*. The character of the instrument would otherwise be unchanged; the interest received in each period would automatically be reinvested in the next, until the final period, when the final increment of interest is returned with all the accumulated principal, some old and some new.*

Particularized Specification. The revised specification addresses the fact that the nominal specification of an instrument may have little to do with its real workings as a producer of interest income, since the issue value V_0^* may differ significantly from the nominal initial value V_0. But this degree of analysis only covers an original holder, or the issuer. The economic evaluation of an instrument acquired subsequently by some other holder at a price higher or lower than V_0^*, needs to be pursued from a different angle. This is what we do next.

Consider a holder who obtains the instrument on a date

$$a = \text{ date of acquisition,} \qquad i_0 \le a < i_m, \tag{3.23}$$

which may or may not be one of the compounding dates i_k, for a certain price

$$P_a = \text{ cost of acquisition to the holder,} \qquad P_a > 0. \tag{3.24}$$

Let $i_{\overline{k}}$ denote the next compounding date after a, so that

$$i_{\overline{k}-1} \;\le\; a \;<\; i_{\overline{k}}. \tag{3.25}$$

As far as this holder is concerned, the economic content of the instrument resides in the fact that the investment P_a on date a produces payments $C_{\overline{k}}, \ldots, C_m$ on the remaining dates $i_{\overline{k}}, \ldots, i_m$ in the life of the instrument. The *particularized yield* for this particular holder is therefore defined to be the unique value y^{**} obtained from Theorem 2.2 for this pattern of payments. In other words, y^{**} is the unique solution to the equation

$$\begin{aligned} P_a = \frac{C_{\overline{k}}}{(1+\theta_a y^{**})} + \frac{C_{\overline{k}+1}}{(1+\theta_a y^{**})(1+\theta_{\overline{k}+1} y^{**})} + \cdots \\ + \frac{C_m}{(1+\theta_a y^{**})(1+\theta_{\overline{k}+1} y^{**})\cdots(1+\theta_m y^{**})}, \end{aligned} \tag{3.26}$$

where

$$\theta_a = \text{ length of the period from date } [a, i_{\overline{k}}]. \tag{3.27}$$

By such means we have a uniquely determined *particularized specification* of the debt instrument relative to acquisition by the holder in question. Observe that the power of Theorem 2.2 in being able to handle unequally spaced compounding dates is put to use even if the original dates i_k for the instrument are equally spaced, because the first accrual period from a to $i_{\overline{k}}$ will be shorter than the other periods unless the acquisition took place precisely on date $i_{\overline{k}-1}$. This particularized specification is achieved without resorting to the "method of fractional exponents," which will be explained in Section 5. (That method would tend to distort the economic content, as will be demonstrated in Theorem 5.1.)

To facilitate comparison between the consequences of the particularized specification and the nominal and revised specifications—all three of which may ultimately be needed in determining taxes on a given instrument—it will be convenient not to renumber the compounding dates in the particularized specification but to keep them as they are. We therefore symbolize the *particularized principal value* on the remaining dates i_k by V_k^{**} and the corresponding *particularized interest payment* by I_k^{**} and the *particularized principal repayment* by I_k^{**}. Thus,

$$\begin{aligned} V_{\overline{k}}^{**} &= \big(1+\theta_a y^{**}\big) P_a, \\ V_k^{**} &= \big(1+\theta_k y^{**}\big) V_{k-1}^{**} - C_k \ \text{ for } \ k = \overline{k}+1, \ldots, m, \end{aligned} \tag{3.28}$$

where $V_m^{**} = 0$. Accordingly

$$\begin{aligned} I_{\overline{k}}^{**} &= \theta_a y^{**} P_a, \\ I_k^{**} &= \theta_k y^{**} V_{k-1}^{**} \text{ for } k = \overline{k}+1, \dots, m, \end{aligned} \tag{3.29}$$

while

$$R_k^{**} = C_k - I_k^{**} \text{ for } k = \overline{k}, \dots, m. \tag{3.30}$$

Once more, these quantities satisfy all the relationships developed earlier; only the double asterisks notation is different. Theorem 2.1 tells us that the values V_k^{**} for $\overline{k} \le k < m$ are all positive (but $V_m^{**} = 0$), and

$$\begin{aligned} V_k^{**} &= R_{k+1}^{**} + \cdots + R_m^{**} \\ &= \big(C_{k+1} + \cdots C_m\big) - \big(I_{k+1}^{**} + \cdots + I_m^{**}\big) \text{ for } k = \overline{k} \dots, m-1, \end{aligned} \tag{3.31}$$

with $R_m^{**} > 0$. We have

$$C_1 + \cdots + C_m - V_0^{**} \;=\; I_{\overline{k}+1}^{**} + \cdots + I_m^{**}, \tag{3.32}$$

and this quantity is the *particularized net return* on the instrument.

Example 3.9 (standard bond, particularized specification). *In the case of a standard bond as in Example 3.2 with accrual periods of length θ the particularized yield-to-maturity equation (3.26) becomes*

$$\begin{aligned} P_a = \frac{C}{(1+\theta_a y^{**})} &+ \frac{C}{(1+\theta_a y^{**})(1+\theta y^{**})} \\ &+ \frac{C}{(1+\theta_a y^{**})(1+\theta y^{**})^2} + \cdots + \frac{C+V_0}{(1+\theta_a y^{**})(1+\theta y^{**})^{m-\overline{k}}}. \end{aligned} \tag{3.33}$$

This equation uniquely determines the particularized yield y^{**}. The particularized principal value on the next compounding date $i_{\overline{k}}$ is then

$$V_{\overline{k}}^{**} = \big(1+\theta_a y^{**}\big) P_a - C_{\overline{k}}, \tag{3.34}$$

while the corresponding interest and principal payments are

$$I_{\overline{k}}^{**} = \theta_a y^{**} P_a, \qquad R_{\overline{k}}^{**} = C - I_{\overline{k}}^{**}. \tag{3.35}$$

Thereafter one has

$$\begin{aligned} V_k^{**} &= \big(1+\theta y^{**}\big) V_{k-1}^{**} - C \text{ for } k = \overline{k}+1, \dots, m-1, \\ I_k^{**} &= \theta y^{**} V_{k-1}^{**} \text{ for } k = \overline{k}+1, \dots, m, \\ R_k^{**} &= C - I_k^{**} \text{ for } k = \overline{k}+1, \dots, m-1, \end{aligned} \tag{3.36}$$

and finally $V_m^{**} = 0$ and $R_m^{**} = V_0 + C - I_m^{**}$.

Deferred Interest Under Different Specifications. The notion of deferred interest, which was developed mathematically at the end of Section 2, can be applied not only to the nominal specification of an instrument but also to the revised specification or any particularized specification. The formulas are valid with the addition of one or two asterisks on the symbols, whichever is appropriate.

4. DISCOUNT AND PREMIUM

In ordinary parlance, something is purchased at *discount* if the amount paid for it is less than its nominally listed price. It is purchased at *premium* if the amount paid for it is greater. The *discount* or *premium* that is involved is the price difference in either case. Discount and premium are concepts crucial in taxation, and they underlie both the revised and the particularized specifications of a debt instrument.

An instrument has been *issued with discount* if $V_0^* < V_0$, or in other words if an original holder paid less for it than its nominal initial value. It has been *issued with premium* if instead $V_0^* > V_0$. These circumstances are covered by the facts in Theorem 3.1, from which we extract the following for emphasis.

Theorem 4.1 (discount and premium in relation to yield)**.**

(a) *An instrument has a higher yield under its revised specification than under its nominal specification if and only if it has been issued with discount. The extra income to an original holder over its life comes out then as the discount amount* $V_0 - V_0^* > 0$.

(b) *An instrument has a lower yield under its revised specification than under its nominal specification if and only if it has been issued with premium. The lost income to an original holder over its life comes out then as the premium amount* $V_0^* - V_0 > 0$.

This is totally in harmony with the idea that V_0^*, instead of V_0, constitutes the amount of principal initially invested by an original holder of the instrument, i.e., the starting "basis" for such a holder. Similar effects involving the particularized specification appear when an instrument is acquired by another holder at a later date, but for now we focus only on the revised specification.

A natural policy of taxation would be to require the additional interest quantity $V_0 - V_0^* > 0$ in the discount case of Theorem 4.1(a) to be declared as taxable income, but to allow the lost interest $V_0^* - V_0 > 0$ quantity in the premium case of Theorem 4.1(b) to be deducted—in some appropriate pattern of amounts and timing. This is more or less what turns out to be the case under the tax rules now in effect, but the regulations are weighed down by complications which obscure the philosophy and create inconsistencies.

These inconsistencies are so serious that it seems unlikely the features creating them could, or would, be upheld, if challenged. The rest of this section is aimed primarily at demonstrating this unfortunate state of regulatory affairs. In the end, we will take the position that the hard facts in Theorem 4.1 must govern all interpretations of discount and premium. The reader interested mainly in the results could skip ahead now to the next section.

Qualified Interest Payments and OID. An essential quantity in current taxation is that which is called *original issue discount*, or OID. This is defined by[24]

$$\text{OID} \;=\; \text{SRPM} \;- V_0^* \;\text{ when this difference is positive,} \tag{4.1}$$

where SRPM is an amount called the "stated redemption price at maturity" of the instrument, whose exact meaning will need to be examined at length. Of course if SRPM $= V_0$, then OID $= V_0 - V_0^*$, which is the true amount of discount at original issue. The trouble is that the regulations on this matter, if taken for what they now say,[25] assign to SRPM an artificial value which could well be higher or lower than V_0 in many cases, as will come to light shortly.[26]

A notion which needs to be examined in parallel, although the attention paid to it in the literature on taxation has been far less, is that of *original issue premium*, or OIP, with the definition

$$\text{OIP} \;=\; V_0^* - \;\text{SRPM} \;\text{ when this difference is positive.} \tag{4.2}$$

Again, if SRPM $= V_0$ we have OIP $= V_0^* - V_0$, and all is natural. The problem to be faced, though, is that such a natural outcome is not ensured by the Tax Code on "amortizable bond premium."[27]

The term "stated redemption price at maturity" does not really appear in the Tax Code in connection with premium; there the statutory wording instead is the "amount payable at maturity." Whether this is supposed to mean the same thing has not been clarified in regulations, but if not, an instrument could have both "original issue discount" and "original issue premium"

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[24] There is a tendency to use the initials "OID" to refer besides to various portions of the quantity in (4.1) as it is accounted for, or to the general idea behind it, or even to interest not necessarily having anything to do with any discount. Mathematics cannot operate under such loose language, so we restrict our usage to the single sense of this definition.

[25] S1.1273-1(b), implementing Code S1273(a)(2).

[26] For purposes of this discussion, we try therefore to draw a fine line between "original issue discount" and "OID" on the one hand, meaning the quantity in (4.1), and the true "discount associated with original issue," meaning $V_0 - V_0^*$ when that is positive.

[27] S171(b)(1).

at the same time, and provisions for treating it would be at loggerheads. To avoid such an absurdity, we interpret that it does mean the same thing.[28] This interpretation lies behind the appearance of SRPM in (4.2) as well as in (4.1).

Of course, such a patch-up between OID and OIP only goes so far. To the extent that SRPM may differ from V_0, conflicts are still bound to rear up from Theorem 4.1. For instance, in cases where $V_0 < V_0^* <$ SRPM an instrument would have "original issue discount" in the official sense, even though it was really issued with premium in the economic sense, and its yield, as determined through the principle of constant yield to maturity relative to its issue price, is actually *lower* than its nominal yield; cf. Theorem 4.1(b). Likewise, in cases where SRPM $< V_0^* < V_0$ an instrument would have "original issue premium" despite having been issued at discount and having a yield that is *higher*.

The formula that defines SRPM, toward which we are proceeding, depends heavily on another notion, that of

$$Q_k \;=\; \text{the } \textit{qualified} \text{ interest payment on date } i_k$$

to a holder under an instrument's nominal specification. "Qualified" seems to suggest some higher standard of legitimacy in payment, in contrast presumably to forms or amounts that might be thought of as more likely to facilitate tax avoidance. The words of Congress on which the concept rests speak of *"any interest based on a fixed rate, and payable unconditionally at fixed periodic intervals of one year or less during the entire term of the debt instrument."*[29] In legal language, saying that something is payable "unconditionally" merely excludes situations where the times or amounts of the payments might be uncertain in advance and subject to contingencies; code and regulations make this clear in other contexts as well as the one under discussion. Therefore, it is difficult to see why the interest amounts I_k in the nominal specification (this being a full, consistent specification in the sense of Definition 2.1, as far as we are concerned here) might not themselves be interpreted as paid in such a sense,[30] at least if the date spacing is regular and the accrual periods do not exceed one year.

As a matter of fact, before the cited statutory wording took effect in 1985 the role of Q_k in the formulas below *was* taken by I_k in treatments of discount and premium. There was no trouble

[28] Such a view with respect to the IRS Proposed Regulations of 1986 has been taken also by Garlock [1, p. 399]. It appears outright in S1.1273-1(c) of the pending IRS Proposed Regulations of 22 December 1992.

[29] Internal Revenue Code S1273(a)(2), effective for debt instruments issued after 1984.

[30] Or if not those amounts I_k, parts of which may be implicit payments, then the explicitly paid amounts J_k defined at the end of Section 2, although that would mathematically be a lot more complicated.

over whether the interest payments associated with an instrument's nominal specification might somehow be unqualified, if they made sense in the traditional framework of finance. Implicit payments were as good as explicit ones.

That could still be the situation today, and many of the problems soon to be described would have been avoided, if it were not for certain elaborations which have since entered the regulations. In particular, the IRS Proposed Regulations of 1986 substitute for "payable unconditionally" the words "actually and unconditionally payable."[31] With this, the IRS seems possibly to have taken the position that implicit payments, such as occur when earned interest is automatically reinvested, are not "actual" payments.[32] But why insist on some subtle difference among shades of actuality of payment, when the real question is simply how much *taxable income* should be regarded as received in total by a holder of an instrument in each accrual period? If implicit payments are going to be regarded in the end as taxable payments, which will be the result in motivating cases like the zero-coupon bond in Example 4.1 below, what is the point? The entire trend of tax law connected with discount bonds has been in the direction of making implicit income includable as current income, so efforts to distinguish between "qualified" and "unqualified" reveal a philosophical contradiction as well. Ironically, the departure of "qualified" interest from the economic accrual (as embodied in the revised specification in the amounts I_k^*) creates and legitimizes opportunities for the very tax avoidance it was intended to suppress, as several examples will soon illustrate.

Anyway, whatever the intention behind "qualified" interest, the definition provided in regulations, as far as it can be understood mathematically,[33] is flawed to the point of not meeting an identifiable goal. As we shall see, interest explicitly received with impeccable credentials can fail to be qualified in this way, while repayments of principal can wind up being characterized as payments of qualified interest. Only a return to identifying Q_k with I_k will be able to clear the problem up.

The official definition of the amounts Q_k starts from the quantities furnished by the nominal

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[31] In S1.1273(b)(ii). These regulations came out only on 2 April 1986 but effective for debt instruments issued after 1984.

[32] Such a supposition is reinforced by the pending IRS Proposed Regulations of 1992, which in turn offer the substitute wording "unconditionally payable in cash or in property (other than debt instruments of the issuer) at least annually at a single fixed rate." This is in S1.272-3(c). Interest that reappears as new principal after having being declared as income might be thought of as giving rise to new debt instruments of the issuer. Insistence of that viewpoint, however, would require treating many single instruments as bundles of smaller instruments, which would be a terrible nuisance without visible benefits.

[33] Nowhere is the mathematical prescription fully spelled out. Our presentation of it has been gleaned from examples in the regulations and conversations with tax authorities.

specification of the instrument, in particular the yield y and initial principal V_0, and with them the complete sequence of principal values V_k derived from the corresponding representation $C_k = I_k + R_k$ of each explicit payment. (These quantities are thus evidently taken as "correct" as far as they go.) From the sequence of principal values $V_0, V_1, \ldots, V_{m-1}$, which we know from Theorem 2.1 to be positive, a sequence of what might be called "potential yield rates"

$$y_k = \frac{C_k}{\theta_k V_{k-1}} \text{ for } k = 1, \ldots, m, \tag{4.3}$$

is calculated,[34] and then an "operative yield rate" is defined by

$$\widehat{y} = \text{ smallest of } y_1, \ldots, y_m. \tag{4.4}$$

Finally, the qualified interest payments Q_k are taken to be given by[35]

$$Q_k = \theta_k \widehat{y} V_{k-1} \text{ for } k = 1, \ldots, m \tag{4.5}$$

By virtue of (4.3) and (4.4) it is clear that[36]

$$Q_k \leq C_k \text{ for } k = 1, \ldots, m. \tag{4.6}$$

With the qualified interest payments Q_k defined, we can pass at once to the official definition provided for the "stated redemption price at maturity," which is

$$\text{SRPM} \;=\; [C_1 - Q_1] + \cdots + [C_m - Q_m]. \tag{4.7}$$

This can straightaway be contrasted with the expression

$$V_0 \;=\; [C_1 - I_1] + \cdots + [C_m - I_m], \tag{4.8}$$

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[34] For some instruments it may be necessary, before proceeding, to respecify in such a way as to achieve equal spacing of dates; for more on this see Section 5 and the compendium of rules in Section 7. The IRS Proposed Regulations of 1992, in S1.1273-1(c)(1), relax the requirement of equal spacing of compounding dates in determining SRPM. That does not mean that instruments issued before they came out escape the problem, however.

[35] The 1986 Proposed Regulations use the full term *qualified periodic interest payments*, and these have become known as QPIP's. In the pending IRS Proposed Regulations of 1992, S1.1273–1(c), the term *qualified stated interest payments* is used instead, but the meaning appears to be slightly different: the use of the word "stated" seems to dictate that the nominal yield rate y should be included among the y_k's in calculating $\widehat{y}$ in (4.4). This would guarantee that $\widehat{y} \leq y$. With such a modification in the way of determining $\widehat{y}$, the values of the Q_k's shift in some situations as well.

[36] The seemingly altered definition for Q_k in the IRS Proposed Regulations of 1992 would ensure also at least that $Q_k \leq I_k$; see the preceding footnote.

which derives from Theorem 2.1, see (2.10). The conclusion is that

$$\begin{aligned} \text{SRPM} \;&= V_0 \;\text{ when }\; Q_k = I_k \;\text{ for }\; k = 1, \ldots, m, \\ &\qquad\qquad\qquad \text{(or equivalently, when } \widehat{y} = y). \end{aligned} \tag{4.9}$$

It is worth repeating that if the interest payments I_k associated with a consistent, full specification were to be regarded as "qualified," regardless of the extent to which they might be explicit or implicit (but are treated as made by the issuer to the holder for the periods ending on the dates i_k), as they were until the 1986 Proposed Regulations (applying to debt instruments issued after 1984), most of the inconsistencies that will be displayed would fade away. To the extent that a discrepancy between SRPM and V_0 is able to develop out of the currently adopted definition, it must of course be visible through noncancellation of certain differences $I_k - Q_k$ in the equation

$$\text{SRPM} \;=\; V_0 + [I_1 - Q_1] + \cdots + [I_m - Q_m], \tag{4.10}$$

which like (4.9) comes from comparison of (4.7) with (4.8).

For a reference point let it be observed right away that

$$\text{SRPM} \;=\; V_0 \;\text{ for a standard bond,} \tag{4.11}$$

since standard bonds come under the sway of the criterion in (4.9) (see Example 3.1). No doubt this important case gives the model from which the words "stated redemption price at maturity" originated. For other kinds of instruments, though, the amount SRPM as defined in (4.7) need not have been "stated" or have any interpretation as a "price," nor is it likely to have any connection with "redemption at maturity." It may fail to have any good meaning at all.

There are many peculiarities to be noticed here. The "operative yield rate" $\widehat{y}$ involved in defining the "qualified" payments has no standing as a proper yield rate for the instrument in question along the lines of constant yield to maturity. It does not satisfy a yield-to-maturity equation relative to the given data (unless it happens to agree with y), nor is it regarded as affecting the evolution of principal. Why then should it be thought to provide a better standard for determining interest received "at a fixed rate," by its use in the formula $Q_k = \theta_k \widehat{y} V_{k-1}$, than the solidly justified yield y in the formula $I_k = \theta_k y V_{k-1}$? The rationale for assigning any significance to the ratios y_k in (4.3) in the imputation of interest is unclear, especially when the procedure apparently acknowledges from the outset that C_k may correctly consist in part of a principal repayment R_k (in particular when $k = m$).

The crux of the matter is that the "operative" yield $\widehat{y}$ is not based on recognized financial principles, nor can it even be computed on its own, since it depends, at least tacitly, on first accepting a background level of correctness in the yield rate y. Questions do have to be faced by the IRS about how to handle instruments that may be only partially or inconsistently described, but the answers ought to grow out of a carefully worked out approach first of all to instruments for which the specification is full and consistent, as we limit attention to here. The constant yield rate y associated with an instrument and its nominal specification under the firm mathematics of Definition 2.1 should take precedence over the rate $\widehat{y}$ with its shaky derivation.

If that were followed, it would be true that SRPM $= V_0$ and OID $= V_0^* - V_0$ as needed for agreement with economic realities; this is apparent from (4.9). If not, as now is the case, there are endless discomforting repercussions.

Example 4.1 (zero-coupon bonds versus CD-like instruments)**.** *On the one hand, consider a zero-coupon bond that pays M dollars at the end of ten years and is purchased by an original holder for a price of P dollars, where $P < M$. On the other hand, consider a ten-year CD-like instrument as in Example 3.4 that is acquired without discount by an original holder for P dollars and has a yield rate r such that the final pay-back is M. The sequence of dates i_k is to be the same in both cases.*

For the zero-coupon bond we have $V_0 = M$ and $y = 0$, but $V_0^* = P$ and $y^* = r$. On each intermediate date i_k the explicit payment is $C_k = 0$ with $I_k = 0$, but $I_k^* > 0$. Under the nominal specification nothing happens, but under the revised specification there are implicit interest payments I_k^* which are automatically reinvested; these happen to be of the magnitude

$$\theta r(1+\theta r)^{k-1}P \ \text{ for } k = 1,\dots,m. \tag{4.12}$$

We have SRPM $= V_0 = M$ in line with (4.10), so that OID $= M - P = V_0 - V_0^*$ through definition (4.1). The quantity designated as OID therefore conforms with the true discount at original issue. In taxation, as will be seen in Section 7, the holder is obliged to declare the amount in (4.12) as taxable income for the period ending on date i_k.

In the CD-like instrument these same features are displayed openly. This time $V_0 = P$ and $y = r$, so that the nominal interest amounts I_k themselves have the magnitudes in (4.12). Again these are paid implicitly and automatically reinvested, but already the nominal specification stipulates that they are income to the holder. Since the instrument is acquired at its nominal value, there is no discount at original issue: $V_0^* = V_0$. The revised specification coincides with the nominal specification, and $I_k^* = I_k$, so that the adjustments $I_k^* - I_k$ to account for discount at original issue are all 0*.*

It would seem that the taxation of the two instruments should come out the same. In the end it will turn out basically that it does, but in the meantime an oddity arises. The CD-like instrument has $\widehat{y} = 0$, because $C_k = 0$ when $1 \le k < m$, and it therefore has $Q_k = 0$ on all dates

i_k; on the other hand, $C_m = M$. Through (4.7) then, SRPM $= M$. It follows in (4.1) that OID $= M - P > 0$. This bond thus is deemed to have "original issue discount" although there was no actual discount in price at original issue.

Since the taxation of the two instruments in Example 4.1 will eventually come out the same despite the difference in treatment, the inconsistency in this particular illustration mainly has the quality of a glitch in terminology. Nonetheless it does fuel confusion about the meaning of things. The holder, instead of being encouraged for straightforwardness, comes under a kind of suspicion and is forced to recast the facts in a way that does not fit with economic truth.[37]

The situation gets definitely odder when the possibility is contemplated that the CD-like instrument in Example 4.1 might for some reason be acquired at a price P' higher than P, although still less than M,

$$P \; < \; P' \; < \; M.$$

Then $V_0^* = P'$, so that $V_0^* > V_0$ and there is premium at original issue in the amount of $P' - P > 0$. The yield rate y^* based on constant yield to maturity, relative to the issue price P' and the explicit payments provided, is necessarily *lower* than y in this case; see Theorem 4.1(b). Nonetheless, the instrument is officially said to exhibit "original issue discount," because OID $= M - P' > 0$ in this case.

Example 4.2 (principal repayments misconstrued as qualified interest). *Consider an instrument of novel form that, with an initial investment V_0 and yield y, compounds interest semiannually for ten years, but in each compounding date i_k before maturity explicitly pays out not only the full amount of interest earned over the period ending on that date, but also an equal amount as repayment of principal. One has*

$$C_k = 2I_k = 2(\theta y)V_{k-1} \;\; \textit{for} \;\; k = 1, \ldots, m-1 \;\; \textit{with} \;\; \theta = \tfrac{1}{2}, \tag{4.13}$$

and consequently

$$V_k = (1 + \tfrac{1}{2}y)V_{k-1} - C_k = (1 - \tfrac{1}{2}y)V_{k-1} \;\; \textit{for} \;\; k = 1, \ldots, m-1, \tag{4.14}$$

but $C_m = (1 + \frac{1}{2}y)V_{m-1}$, so that $V_m = 0$. (It is assumed that $y < .5$, so that principal stays positive until the end.) We find from (4.3) and formula (4.13) that

$$y_k = \frac{2(\frac{1}{2}y)V_{k-1}}{\frac{1}{2}V_{k-1}} = 2y \;\; \textit{for} \;\; k = 1, \ldots, m-1,$$

[37] Also, in filing taxes the holder of the CD-like instrument must include the interest in the category of "original issue discount" with its special forms, instead of just ordinary interest income.

whereas

$$y_m = \frac{(1+\frac{1}{2}y)V_{m-1}}{\frac{1}{2}V_{m-1}} = 2 + y > 2y,$$

and this provides through definitions (4.4) and (4.5) that

$$\widehat{y} = 2y, \text{ so } Q_k = 2I_k = C_k = I_k + R_k \text{ for } k = 1, \dots, m. \tag{4.13}$$

Half of each so-called qualified interest payment Q_k is therefore really return of principal.[38]

Furthermore, from (4.10) and (4.13) we have for this instrument that

$$SRPM = V_0 - I_1 - \cdots - I_m < V_0, \quad \text{in fact} \quad SRPM = (1 - \tfrac{1}{2}y)^m V_0. \tag{4.14}$$

Thus, there is "original issue premium" in the amount of

$$OIP = \left[1 - (1 - \tfrac{1}{2}y)^m\right] V_0$$

even for an original holder who acquires at price equaling the nominal value V_0.

A disturbing consequence in this situation is that if the instrument in Example 4.2 were acquired by an original holder for a price $V_0^* < V_0$, representing a true discount at original issue, but with $V_0^* >$ SRPM, it would officially be deemed to have the "original issue premium" amount OIP $= V_0^* -$ SRPM > 0 and would escape taxation of the supplementary interest payments $I_k^* - I_k$ that add up to the discount amount $V_0 - V_0^*$.[39] It would turn out that the holder would instead have official sanction for deducting from income various amounts characterized as extra interest expense incurred.

To drive the point home about how inadequate regulations have been in identifying the possibilities they are supposed to prevent, because of uneven mathematical foundations, we give a small, numerical example. This concerns a hypothetical instrument with features similar to those in Example 4.2, but more extreme. A massive potential for tax advantage is clearly revealed.

Example 4.3 (negative taxable interest income in every period).[40] *Consider a two-year instrument with nominal initial investment $V_0 = \$100,000$ and yield $y = .10$, compounded annually,*

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[38] This is relative to S1.1273-1(a) and (b) of the IRS Proposed Regulations of 1986, still regarded as authoritative for instruments issued up to 22 December 1992. The pending IRS Proposed Regulations of 1992 seem to exclude this particular outcome; see Footnote 33.

[39] Under the IRS Proposed Regulations of 1986; see the preceding footnote.

[40] This example would presumably no longer operate under the pending IRS Proposed Regulations of 1992; see Footnotes 33 and 34. Whether instruments of such character were ever written under the regulations in effect until recently is unknown; the change in the regulatory circumstances appears to be a "lucky accident" in this respect. If such instruments were not written, that is of course testimony to the lack of appreciation for mathematical perspectives even on the part of those who would like to turn tax rules to their advantage.

which pays out $C_1 = \$100,000$ *at the end of the first year and* $C_2 = \$11,000$ *at the end of the second, when it matures. There are three dates involved,* i_0, i_1, *and* i_2, *spaced one year apart; the constant period length is* $\theta = 1$.

This is a full specification in the sense of Definition 2.1, but is it consistent? The test is made according to what happens to the principal. At the end of the first year one has

$$I_1 = \$10,000, \qquad R_1 = \$90,000, \qquad V_1 = \$10,000,$$

while at the end of the second year the figures are

$$I_2 = \$1,000, \qquad R_2 = \$10,000, \qquad V_2 = \$0.$$

The fact that $V_2 = \$0$ *confirms consistency.*

To compute next the quantity that has been designated in regulations as the SRPM, the beginning step is to determine the two potential yield rates y_1 *and* y_2 *associated with the payment stream. These are*

$$y_1 = \frac{C_1}{\theta V_0} = \frac{100,000}{100,000} = 1.00, \qquad y_2 = \frac{C_2}{\theta V_1} = \frac{11,000}{10,000} = 1.10.$$

The operative yield rate is deemed therefore to be $\widehat{y} = 1.00$, *and the corresponding "qualified periodic interest payments" are*

$$Q_1 = \theta \widehat{y} V_0 = \$100,000, \qquad Q_2 = \theta \widehat{y} V_1 = \$10,000.$$

This means that the "stated redemption price at maturity" is officially considered to be

$$\text{SRPM} = [C_1 - Q_1] + [C_2 - Q_2] = \$1,000.$$

Suppose now that the instrument has been acquired by an original holder for a price $V_0^* =$ $\$99,018$, *which corresponds to a revised specification in which the yield is* $y^* = .11$. *Under this revised specification the payment* C_1 *is interpreted as giving*

$$I_1^* = (.11)(\$99,018) = \$10,892, \qquad R_1^* = \$89,108, \qquad V_1^* = \$9,910,$$

while the payment C_2 *results in*

$$I_2^* = (.11)(\$9,910) = \$1,090, \qquad R_2^* = \$9,910, \qquad V_2^* = \$0.$$

The interest adjustments $I_1^* - I_1 = \$892$ *and* $I_2^* - I_2 = \$90$ *add up to the true discount at original issue, which is* $V_0 - V_0^* = \$982$.

It might seem that the holder should declare as taxable income the amounts I_1^* *for the first year and* I_2^* *for the second. But because* SRPM $< V_0^*$, *regulations construe the instrument as not having "original issue discount." Instead, there is "original issue premium" in the amount of* OIP $= V_0^* -$ SRPM $= \$98,018$.

As will emerge from the rules laid out in Section 7, taxation goes forward in this case relative to the nominal specification, but in addition the holder is permitted to utilize the so-called premium amount in one way or another as an offset. In one approach,[41] *the premium could be amortized by the straight-line method, with half of it deducted in each year. This means that the holder could declare the net interest income from the instrument to be* $\$10,000 - 49,009 = -\$39,009$ *in the first year and* $\$1,000 - 49,009 = -\$48,009$ *in the second. Even if the pseudo-premium on the discount bond in Example 4.3 were not handled in such a remarkable way, there would be a gratuitous tax advantage available to the holder*[42] *through the fact that amortization of bond premium has always been optional to the holder. Because the instrument would fall into the category of having "original issue premium," it would be shunted to taxation on its nominal specification rather than its revised specification. The total interest income deemed to be received from it would then be* $I_1 + I_2 = \$11,000$, *instead of* $I_1^* + I_2^* = \$11,982$. *The difference in these interest amounts, equaling the true discount of* \$982 *at original issue, would escape being taxed as ordinary income. Perhaps it could be conceived, though, as a capital gain at maturity: it is redeemed then for* \$0 *when its basis is* $-\$982$, *so that a gain of* \$982 *might be inferred.*

This outcome is less dramatic (for this particular example), but from a theoretical standpoint it still is impressive in showing that the rules do not accomplish what they set out to do. With ingenuity, an entire array of instruments might, until very recently, have been formulated and issued in such a manner as to exhibit OIP instead of OID, and thereby keep a portion of real interest income out of the category of ordinary income.

Installment Obligations. Another inconsistency in the tax picture, again entering from the way "qualified" interest has been conceived, is found in what constitutes an *installment obligation*. One might expect that term would be applied to any instrument which in its nominal specification has $R_k > 0$, or equivalently $C_k > I_k$, on some date i_k before maturity. But the

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[41] Applicable to instruments issued before 28 September 1985.

[42] *This would be true if the instrument were issued any time under the tenure of the IRS Proposed Regulations of 1986, i.e., up to 22 December 1992. After that, the value of SRPM calculated for in this example would presumably be different.*

definition given in regulations[43] is that it refers instead to an instrument for which $C_k > Q_k$ on some date i_k before maturity. Once more, there is nothing wrong when Q_k is identified with I_k, but if not there is trouble.

The next example indicates how an instrument can end up being called an installment obligation officially, when the prescription for Q_k in current regulations is followed, even if principal is never returned before maturity under the instrument's nominal specification.

Example 4.4 (an installment obligation with no early return of principal). *Consider a three-year instrument with constant yield rate* $y = .20$ *compounded annually, starting from* $V_0 =$ \$1000, *which for the first year returns all the interest earned, but for the second year only returns half of it, reinvesting the other half. This means that* $C_1 = \$200$ *(so* $I_1 = \$200$, $R_1 = \$0$, $V_1 = \$1000$*) but then* $C_2 = \$100$ *(while* $I_2 = \$200$, $R_2 = -\$100$, $V_2 = \$1100$*). For the third year, the instrument provides* $C_3 = (1 + y)V_2 = \$1320$ *(with* $I_3 = \$220$ *and* $R_3 = \$1100$*; then* $V_3 = \$0$*). This furnishes a consistent, full specification in the sense of Definition 2.1 with period length* $\theta = 1$.

To determine the "qualified" interest payments Q_k, *the lowest of the ratios*

$$y_1 = \frac{C_1}{\theta V_0} = \frac{200}{1000}, \qquad y_2 = \frac{C_2}{\theta V_1} = \frac{100}{1000}, \qquad y_3 = \frac{C_3}{\theta V_2} = \frac{1320}{1100},$$

is selected as $\widehat{y}$ *through (4.4); thus* $\widehat{y} = .10$. *From the definition* $Q_k = \theta \widehat{y} V_{k-1}$ *we then obtain*

$$Q_1 = \$100, \qquad Q_2 = \$100, \qquad Q_3 = \$110.$$

Because $Q_1 < C_1$, *the instrument is considered to be an installment obligation, even though* $R_1 = 0$ *and* $R_2 < 0$.

Incidentally, SRPM $= [C_1 - Q_1] + [C_2 - Q_2] + [C_3 - Q_3] = \$1310$ in Example 4.4. This makes little sense relative to the facts, according to which the final payment of \$1320 consists of \$1000 in original principal, \$100 in new principal (from interest income in the second year that was reinvested), and \$220 in interest. The inequality $C_1 > Q_1$, which is the only feature forcing the instrument to be labeled as an installment obligation, obviously has nothing to do with any kind of repayment of principal at the end of the first year.

For another surprise, look again at Example 4.3. The instrument described in this example *fails to be an installment obligation* under the definition, because there is only one intermediate date before maturity, namely i_1, and we have $Q_1 = C_1$; *yet almost all the principal is paid back in the first year*, with R_1 being 90% of V_0.

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[43] S1.1273–1(b)(2)(i) of the IRS Proposed Regulations of 1986.

Adjusted Issue Price. We have seen in Section 3 that the revised principal value V_k^* on date i_k is a quantity that evolves from the issue price V_0^* by the rule

$$V_k^* = V_{k-1}^* + I_k^* - C_k \text{ for } k = 1, \dots, m-1, \text{ where } I_k^* = \theta_k y^* V_{k-1}^*. \tag{4.15}$$

Very close to this in tax literature is the concept of the *adjusted issue price* on date i_k, which we denote by $\widehat{V}_k^*$. This is described in regulations as evolving by the rule

$$\widehat{V}_k^* = \widehat{V}_{k-1}^* + O_k - [C_k - Q_k] \text{ for } k = 1, \dots, m-1, \text{ with } \widehat{V}_0^* = V_0^*, \tag{4.16}$$

where O_k represents "unqualified" interest—thought to be earned above the "qualified" Q_k amount—as computed for the kth period from

$$O_k = \begin{cases} \widehat{I}_k^* - Q_k & \text{if } \widehat{I}_k^* \geq Q_k, \\ 0 & \text{if } \widehat{I}_k^* < Q_k, \end{cases} \qquad \text{where } \widehat{I}_k^* = \theta_k y^* \widehat{V}_{k-1}^*. \tag{4.17}$$

This interest amount O_k is considered to be the portion of income that a holder should declare for the kth period as coming from "original issue discount" (hence our symbol O_k for denoting it), whereas $C_k - Q_k$ is perhaps a putative return of principal. The total interest income believed to be received by the holder in the kth period is $Q_k + O_k$.

Formulas (4.16) and (4.17) are tailored for use on certain classes of instruments with "original issue discount" as in (4.1). Even for such instruments, however, trouble can arise. Just as earlier, the main culprit lies in taking Q_k to mean anything other than I_k. But there is another mathematical villain now in the cutoff rule in (4.17), which keeps the tax adjustment for the OID portion from going negative. While that provision might seem offhand to be a simple precaution against potential abuse, its actual effect is some cases to force the adjusted issue price to deviate from the revised principal value and lose its economic grounding. When that happens the door is opened to all sorts of difficulties.

The exact nature of the problem is set down in the following theorem. The main conclusion is that an imbalance in accounting for the OID quantity arises directly in association with any circumstance in which an interest payment under the revised specification does not cover all the interest regarded as "qualified."

Theorem 4.2 (potential imbalance in adjusted issue price and OID).

(a) *If an instrument in its revised specification has the property that $I_k^* \geq Q_k$ on all intermediates dates i_k (i.e., for $k = 1, \dots, m-1$), then its adjusted issue price and revised principal value always coincide:*

$$\widehat{V}_k^* = V_k^* \text{ for } i = 0, 1, \dots, m,$$

and in particular $\widehat{V}_m^* = V_m^* = 0$. *Then moreover*

$$O_1 + O_2 + \cdots + O_m \;=\; \text{OID}.$$

(b) *But if there is a date* i_{k^*} *on which* $I_{k^*}^* < Q_{k^*}$, *then*

$$\widehat{V}_k^* > V_k^* \text{ for } k = k^*, \ldots, m.$$

In the event of such an imbalance the adjusted issue price cannot reach 0 *at maturity: it will be true in particular that*

$$\widehat{V}_m^* > 0, \text{ although } V_m^* = 0.$$

In fact the gap between $\widehat{V}_k^*$ *and* V_k^* *will successively widen on each date* i_k *after* i_{k^*}. *If there are such dates (i.e., if* $k^* < m$*), one will have*

$$O_1 + O_2 + \cdots + O_m \;>\; \text{OID}.$$

Thus, the payments in which the OID *quantity is supposed to be parceled out will overshoot.*

Proof. Observe first that because $\widehat{V}_0^* = V_0^*$ by definition, we start out with $\widehat{I}_1^* = I_1^*$. If this quantity is not exceeded by Q_1, we get $O_1 = I_1^* - Q_1$ from (4.17) and therefore $\widehat{V}_1^* = \widehat{V}_0^* + [I_1^* - Q_1] - [C_1 - Q_1] = V_0^* + I_1^* - C_1 = V_1^*$ from (4.16) and then (4.15). So it continues: we have from $\widehat{V}_1^* = V_1^*$ that $\widehat{I}_2^* = I_2^*$, and if this quantity is not exceeded by Q_2 we obtain $O_2 = I_1^* - Q_2$ from (4.17), so that $\widehat{V}_2^* = V_2^*$ from (4.16) and (4.15). In following this pattern we deduce that as long as $I_k^* \geq Q_k$ we get $\widehat{V}_k^* = V_k^*$. This could go all the way to the end, and we would reach the first conclusion in the theorem, moreover with

$$\begin{aligned} O_1 + \cdots + O_m &= [I_1^* - Q_1] + \cdots + [I_m^* - Q_m] = [I_1^* + \cdots + I_m^*] - [Q_1 + \cdots + Q_m] \\ &= [C_1 + \cdots + C_m - V_0^*] - [Q_1 + \cdots + Q_m] \\ &= [C_1 - Q_1] + \cdots + [C_m - Q_m] - V_0^* \;=\; \text{SRPM } - V_0^* \;=\; \text{OID}, \end{aligned} \tag{4.18}$$

where the equality on the second line utilizes (3.12), and the last two equalities invoke the definition (4.7) of SRPM and the definition (4.1) of OID. This proves (a).

Otherwise the pattern of having $\widehat{V}_k^* = V_k^*$ only continues to a date i_{k^*} (take it to be the first such) on which $I_{k^*}^* < Q_{k^*}$. At that stage we have $\widehat{V}_{k^*-1}^* = V_{k^*-1}^*$, so that $\widehat{I}_{k^*}^* = I_{k^*}^*$ still, but $O_k = 0$. Then $\widehat{V}_{k^*}^* = \widehat{V}_{k^*-1} + 0 - [C_{k^*} - Q_{k^*}]$ in contrast to $V_{k^*}^* = V_{k^*-1}^* + I_{k^*}^* - C_{k^*}$. By subtracting the second relation from the first, we obtain

$$\widehat{V}_{k^*}^* - V_{k^*}^* = Q_{k^*} - I_{k^*}^* > 0,$$

which gives $\widehat{V}^*_{k^*} > V^*_{k^*}$. Our aim next is to show that once this strict inequality has been encountered it will be maintained henceforth. For this we write (4.15) and (4.16) in general in the form

$$\begin{aligned} V^*_k - V^*_{k-1} &= [I^*_k - Q_k] - [C_k - Q_k], \\ \widehat{V}^*_k - \widehat{V}^*_{k-1} &= O_k + [C_k - Q_k], \end{aligned}$$

and subtract the first equation from the second, obtaining

$$\begin{aligned} \big[\widehat{V}^*_k - V^*_k\big] - \big[\widehat{V}^*_{k-1} - V^*_{k-1}\big] = F_k \\ \text{with } F_k = O_k - [I^*_k - Q_k] &= \begin{cases} \big[\widehat{I}^*_k - Q_k\big] - \big[I^*_k - Q_k\big] & \text{if } \widehat{I}^*_k \geq Q_k, \\ 0 - \big[I^*_k - Q_k\big] & \text{if } \widehat{I}^*_k < Q_k, \end{cases} \\ &= \begin{cases} \theta_k y^* \big[\widehat{V}^*_{k-1} - V^*_{k-1}\big] & \text{if } \widehat{I}^*_k \geq Q_k, \\ Q_k - I^*_k & \text{if } \widehat{I}^*_k < Q_k. \end{cases} \end{aligned} \tag{4.19}$$

Arguing now from $k = k^* + 1$ on, we note that if $\widehat{V}^*_{k-1} > V^*_{k-1}$, then $\widehat{I}^*_k > I^*_k$ out of comparison with the interest formulas in (4.15) and (4.16). It will be shown from this that the inequality $\widehat{V}^*_{k-1} > V^*_{k-1}$ implies $F_k > 0$ in (4.19). In the case of (4.19) where $\widehat{I}^*_k \geq Q_k$, we have $F_k = \theta_k y^* \big[\widehat{V}^*_{k-1} - V^*_{k-1}\big] > 0$. On the other hand, in the case of (4.19) where $\widehat{I}^*_k < Q_k$ we have $F_k = Q_k - I^*_k > 0$ because $Q_k - I^*_k > Q_k - \widehat{I}^*_k$ through the inequality $\widehat{I}^*_k > I^*_k$ (as a consequence of $\widehat{V}^*_{k-1} > V^*_{k-1}$). Since either way we get $F_k > 0$, we are able to conclude from the first equation in (4.19) that

$$\widehat{V}^*_k - V^*_k > \widehat{V}^*_{k-1} - V^*_{k-1} > 0.$$

In other words, the inequality $\widehat{V}^*_{k-1} > V^*_{k-1}$ is inherited in the next period as $\widehat{V}^*_k > V^*_k$. Thus, as claimed, once a gap enters between the adjusted issue price and the revised principal value, it holds to the end.

The argument shows in fact that the gap between the adjusted issue price and the revised principal value keeps widening after it forms. It widens on date i_k by the positive amount F_k in (4.19).

Furthermore in this case, from having $\widehat{I}^*_k > I^*_k$ for all $k > k^*$, although the two interest quantities are equal for $k \leq k^*$, we will have through definition (4.17) that $O_k > I^*_k - Q_k$ for all $k > k^*$, although $O_k = I^*_k - Q_k$ for all $k \leq k^*$. Then in tracing the calculation in (4.18) we find that, as long as $k^* < m$, the first equality is replaced by ">" while all the other equalities continue to hold. This establishes the final inequality claimed in (b). □

It may be imagined that the discrepancy in the second case in Theorem 4.2 can never arise, but that conjecture is false, as we now demonstrate. First we look at an example that comes from further study of the instrument in Example 4.3.

Example 4.5 (surplus basis after maturity).[44] *The adjusted issue price behaves as follows for the discount bond in Example 4.3. It starts out with* $\widehat{V}_0^* = V_0^* = \$99,018$*, this being the issue price, in comparison to* $V_0 = \$100,000$*. Since the instrument's yield to maturity is* $y^* = .11$ *relative to this price of purchase by an original holder, we get* $\widehat{I}_1^* = I_1^* = \$10,892$*, and because* $Q_1 = \$100,000 > \widehat{I}_1^*$ *we have* $O_1 = 0$*. Then*

$$\widehat{V}_1^* = \widehat{V}_0^* + O_1 - [C_1 - Q_1] = \$99,018,$$

since $C_1 = Q_1$*; this is in contrast to*

$$V_1^* = V_0^* + I_1^* - C_1 = \$9,910.$$

In the next stage, $\widehat{I}_2^* = \$10,892$*; because* $Q_2 = \$10,000 < \widehat{I}_2^*$*, we obtain* $O_2 = \widehat{I}_2^* - Q_2 = \$892$*. From the fact that* $C_2 = \$11,000$*, we end up with*

$$\widehat{V}_2^* = \widehat{V}_1^* + O_2 - [C_2 - Q_2] = \$98,910$$

as the adjusted issue price at maturity—after all payments have been received. This contrasts with $V_2 = \$0$ *and* $V_2^* = \$0$*.*

If the holder were taxed according to "yield to maturity" as imagined to be realized through this pattern generated from the adjusted issue price (instead of in an economically correct manner from the revised specification), it might seem that the amounts $\widehat{I}_1^* = \$10,892$ *and* $\widehat{I}_2^* = \$10,892$ *would be viewed as the taxable income received for the two years. This would differ markedly in total from the amounts* $I_1^* = \$10,892$ *and* $I_2^* = \$1,090$ *dictated under the revised specification. A potentially compensating feature, however, would appear in the residual of* $\$98,910$ *after maturity. Presumably this could be interpreted as a capital loss.*[45]*If so, the net result would hugely be to the holder's advantage.*

Rather than $\widehat{I}_1^*$ *and* $\widehat{I}_2^*$*, though, the interest income deemed to be received would be* $Q_1 + O_1 = \$100,000$ *for the first year and* $Q_2 + O_2 = \$10,892$ *for the second, which is far more out of line with* I_1^* *and* I_2^* *and harder than ever to justify in any sense as resulting from "constant yield to maturity." While the holder would be unhappy with such an interpretation of income,*

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[44] Footnote 38 again applies.

[45] This amount represents the difference between the holder's "basis" going into the maturity date i_2, which is set up for purpose of determining capital gain or loss as will be explained in Section 6, and the payment received then on retirement, which is regarded by law as cash received "in exchange for" the instrument, cf. S1271(a) of the Internal Revenue Code. On the face of things, less value was received when all was over than the value that had still been deemed present, and the difference is then a capital loss.

the issuer might be glad, because these amounts could be taken as deductible interest expense; an instrument with these provisions could perhaps be issued by a taxed entity for purchase by an untaxed entity, who would not need to worry about the other side of the tax picture. Even if the issuer were obliged to declare \$98,910 *as a capital gain, the outcome would still be advantageous, not only because the expense declared in the first year would outweigh the reverse effect in the second year, but through the time value of money and the possibility of capital gains being taxed at a lower rate than interest.*

But inconsistencies pile on inconsistencies and add further twists. A special provision for handling adjusted issue price in the final accrual period requires that the final amount O_m *be taken not from the general formula in (4.17) but to be whatever quantity is needed so as to make* $\widehat{V}_m$ *come out to be zero at maturity—as long as the instrument is not an installment obligation.*[46] *It may have been taken for granted that the instrument under investigation is an installment obligation, and therefore sidesteps this provision, because it returns most of its initial principal in the first period under its nominal specification. Yet actually, as noted earlier, it is not an installment obligation in the sense given in regulations, since* $C_1 = Q_1$. *Thus, the surplus basis of* \$98,910 *is ephemeral. The issuer would not have to declare a capital gain after all. The tax advantage would be hugely in the issuer's favor.*

On the other hand, the erasing of the final surplus in the adjusted issue price does not necessarily prevent the holder from using the discrepancy between adjusted issue price and revised principal value to some advantage. It merely prevents doing so on the date of maturity. There is still some potential in the fact that $\widehat{V}_1^* = \$99,018$, *whereas* $V_1^* = \$9,910$. *From the theory of constant yield to maturity it is known that the second of these figures is the amount which, at the revised yield rate* $y^* = .11$, *would result in receiving the payment* $C_2 = \$11,000$ *at the end of the next (and final) period. Under the assumption that financial markets have been stable over the first period, this comes out to be more or less the price at which the holder could sell it right after date* i_1; *let us suppose it is sold exactly at this price. The difference of* $\$99,018 - 9,910 = \$89,108$ *would then be written down as the holder's capital loss. For the particular instrument at hand, the net would not be in favor of the holder, but what is there to exclude the possibility that for slightly different data written into the bond the result might go the other way? When the "basis" value of an instrument differs, under regulations, so grossly from its economic value, serious repercussions may be expected.*

[46] See the IRS Proposed Regulations of 1986, S1.1273(c)(2)(iii). This feature was simply designed to catch errors in arithmetic that may have occurred earlier in including interest. It was not based on any understanding that the adjusted issue price might turn out to be positive after maturity.

In the end, though, there is still another point of inconsistency to remember. None of the mentioned approaches to tax consequences is correct, because the instrument, although purchased at discount, is officially regarded as a premium instrument with OIP $= \$98,018$. *For such an instrument, regulations prescribe that the adjusted issue price never enters taxation at all. The nominal specification is to be used instead, as explained earlier.*

One could balk at these strange consequences and dismiss them on the grounds that they are only due to having assigned values to Q_1 and Q_2 that are ridiculously far from the interest payments I_1 and I_2 called for under the instrument's nominal specification, and certainly not the intent of the regulations. But that is just our point. The intent of the regulations can only be interpreted from what they appear to say. We think they ought to indicate simply that Q_k should always be taken to be I_k.

Nevertheless, a less extreme example may be helpful in making clear that an imbalance between $\widehat{V}_k^*$ and V_k^* can occur even in more homely circumstances.

Example 4.6 (misbehavior of adjusted issue price in an almost standard bond). *Consider a 20-year bond with $V_0 = \$100,000$ and yield $y = .10$, compounded annually, which is just like a standard bond, except that a small portion of the next-to-last interest payment is reinvested. The explicit payments are $C_k = \$10,000$ for $k = 1,\dots,18$, but $C_{19} = \$9,000$ and $C_{20} = \$111,100$. This is a consistent, full specification in the sense of Definition 2.1 with*

$$I_k = \$10,000, \quad R_k = \$0, \quad V_k = \$100,000 \text{ for } k = 1,\dots,18,$$
$$I_{19} = \$10,000, \quad R_{19} = -\$1,000, \quad V_{19} = \$101,000,$$
$$I_{20} = \$10,100, \quad R_{20} = \$101,000, \quad V_{20} = \$0.$$

To what extent are these interest payments "qualified"? The "operative" yield rate is $\widehat{y} = .09$, so that

$$Q_k = \$9,000 \text{ for } k = 1,\dots,19, \text{ and } Q_{20} = \$9,090.$$

The "stated redemption price at maturity" is therefore believed to be

$$\text{SRPM} \;=\; 18[10,000 - 9,000] + [9,000 - 9,000] + [111,100 - 9,090] = \$120,010,$$

even though the principal is always $\$100,000$ except in the final period, when it is $\$101,000$ and then returned.

Suppose this bond is issued for the price $V_0^* = \$119,641$, which corresponds in the revised specification to a market yield rate of $y^* = .08$. In this case it is issued in fact at a substantial premium, but because $V_0^* <$ SRPM it is believed instead to have original issue discount to the extent of

$$OID = \$370.$$

Its treatment for purposes of taxation should then presumably (under rules described later) follow the adjusted issue price and the interest generated out of that, rather than simply the revised specification. But if so, a strange result occurs, as shown in the accompanying table. The adjusted issue price $\widehat{V}_k^*$ keeps pace with the revised principal value V_k^* until $k = 13$, when it starts to deviate. The gap then gets wider and wider (in confirmation of Theorem 4.2(b)), and in the end—after all payments have been received for year 20, there is a remainder of $\widehat{V}_{20}^* = \$3,494$.

The cause of the discrepancy is seen in the column giving the differences $\widehat{I}_k^* - Q_k$. These reach 0 in year 12 and then go negative; they give O_k until then, but once they are negative, O_k is 0. The total these amounts is therefore the total of the positive numbers in this column: we have

$$O_1 + \cdots + O_{20} \;=\; \$3,863.$$

This is way in excess of the $\$329$ that it is supposed to account for.

In each period the holder is officially regarded as receiving the interest amounts Q_k and O_k. Through year 12 we have $Q_k + O_k = Q_k + (\widehat{I}_k^* - Q_k) = I_k^*$, because $\widehat{I}_k^* = I_k^*$, and so the income agrees with that under the revised specification. But thereafter $Q_k + O_k > I_k^*$, and there is overreporting of interest income in comparison with the revised specification—by the amounts displayed in the last column.

The total of the overreported income amounts under the scheme based on the adjusted issue price $\widehat{V}_k^*$ is \$4,113. This is partly compensated, however, by the final basis value of $\widehat{V}_{20} = \$3,494$, which presumably could be taken as a capital loss. All three values can then be used to their associated streams of interest income. The numerical results are displayed in Table 4.1.

Year	V_k^*	I_k^*	$\widehat{V}_k^*$	$\widehat{I}_k^*$	$\widehat{I}_k^* - Q_k$	$(Q_k + O_k) - I_k^*$
0	119,641		119,641			
1	119,212	9,571	119,212	9,571	571	0
2	118,749	9,537	118,749	9,537	537	0
3	118,249	9,500	118,249	9,500	500	0
4	117,709	9,460	117,709	9,460	460	0
5	117,125	9,417	117,125	9,417	417	0
6	116,495	9,370	116,495	9,370	370	0
7	115,816	9,320	115,816	9,320	320	0
8	115,080	9,265	115,080	9,265	265	0
9	114,287	9,206	114,287	9,206	206	0
10	113,429	9,143	113,429	9,143	143	0
11	112,504	9,074	112,504	9,074	74	0
12	111,504	9,000	111,504	9,000	0	0
13	110,424	8,920	110,504	8,920	−80	80
14	109,258	8,834	109,504	8,840	−160	166
15	107,999	8,741	108,504	8,760	−240	259
16	106,639	8,640	107,504	8,680	−320	360
17	105,170	8,531	106,504	8,600	−400	469
18	103,584	8,414	105,504	8,520	−480	586
19	102,870	8,287	105,504	8,440	−560	713
20	0	8,230	3,494	8,440	−650	860

Table 4.1 Misbehavior of Adjusted Issue Price.

Of course, any tax disadvantage to an original holder of such an almost standard bond would be a tax advantage to the issuer.

Trouble When Negative Offsets Are Disallowed. A problem would remain in the definition of the adjusted issue price even if the qualified payments Q_k were to be identified always with the interest payments I_k. It lies in the cutoff feature of formula (4.17), where O_k is taken to be 0 when the amount of interest being subtracted overpowers the amount $\widehat{I}_k^*$ at hand.[47] The first time in the life of an instrument that a negative value of $\widehat{I}_k^* - Q_k$ is replaced by 0, a gap is created between $\widehat{V}_k^*$ and V_k^* which can never be removed, and which will result in surplus "basis" after maturity, as proven in Theorem 4.2.

Can this come into play when I_k is substituted everywhere for Q_k in determining the evolution of $\widehat{V}_k^*$? In other words, can it happen even in the aftermath of such a substitution that $\widehat{I}_k^* < I_k$ on some date i_k? The next example establishes that it can, moreover on the date i_1 for some instruments, when $\widehat{I}_k^*$ still agrees with I_k^*.

Example 4.7 (a discount instrument paying *less* interest initially)**.** *Consider a ten-year CD-like instrument with nominal value* $V_0 = \$10,000$ *and constant yield* $y = .12$*, to be compounded semiannually, which reinvests all interest until the end (cf. Example 3.4). This has* $C_k = 0$ *for* $k = 1, \ldots, 19$*, but* $C_{20} = \$32,071$*. Suppose it is acquired by an original holder for the price of* $V_0^* = \$8,274$*, which corresponds to a yield of* $y^* = .14$*. This instrument has been issued with discount, the true amount of discount being* $V_0 - V_0^* = \$1,726$*, although* OID $= \$23,797$ *by official definition. Be that as it may, the most interesting feature concerns the interest payments as time goes on.*

Under the nominal specification, the interest deemed to be received by the holder on date i_k *is* $I_k = (1.06)V_{k-1}$*, whereas under the revised specification it is* $I_k^* = (1.07)V_{k-1}^*$*. The observation to make is that although the interest rate in the second case is higher in the second case, it is being applied to a lower amount of principal in the beginning. These two tendencies create a tug-of-war in the product, and whether the amount* I_k^* *will come out larger than the amount* I_k *is not a foregone conclusion. We do know from (3.13) in Theorem 3.1 that*

$$(I_1^* - I_1) + (I_2^* - I_2) + \cdots + (I_{19}^* - I_{19}) + (I_{20}^* - I_{20}) = V_0 - V_0^* = \$1,722, \tag{4.20}$$

but this ultimate balance might be attained with some of the initial difference terms negative, as long as that is made up later by some of the subsequent difference terms being more positive. Here we have exactly such a situation. We calculate that $I_1^* = (1.07)8,274 = \$571$*, whereas* $I_1 = (1.06)(10,000) = \$600$*. Thus* $I_1^* - I_1 = -\$29$*.*

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[47] This is written into the IRS Proposed Regulations of 1986; see S1.1272-1(c)(2)(1)(B). Fortunately it is absent, however, from the corresponding part of the pending IRS Proposed Regulations of 1992, which is S1.1272-1(b)(1)(iii).

Furthermore, with Q_k *replaced by* I_k *in (4.16) and (4.17) we get* $\widehat{V}_1^* = 8,274+0-[0-600] = \$8,874$, *in contrast to* $V_1^* = 8274 + 571 = \$8,845$. *By Theorem 4.2(b) it must be true that* $\widehat{V}_k^* > V_k^*$ *for* $k = 2,\ldots,20$ *as well. The total* $O_1 + \cdots + O_{20}$ *will end up exceeding the discount amount* \$1722 *it is supposed to account for.*

Although the discrepancy in perception of interest in Example 4.7 is not very large, and can only operate in early periods before compensation sets in to achieve the balance in (4.20), the effect on adjusted issue price is permanent. With such an inconsistency admitted, there is no telling what might be squeezed out of the tax situation. A disadvantage to a holder can be erased when the holder is an untaxed entity, but resurface as an advantage to a taxed issuer.

Conclusions—How To Proceed. In response to the problems seen in this section, we hold that the only reasonable interpretation of the qualified interest payments Q_k is that they should be the amounts I_k, even when these are partly explicit, and that the adjusted issue price must be correspondingly be identified with the revised principal value V_k^*.[48] Furthermore, "stated redemption price at maturity" should be identified with V_0, and OID with $V_0 - V_0^*$, when positive. In consequence of identifying the qualified interest payment on date i_k with I_k, an "installment obligation" will be an instrument having a principal repayment $R_k > 0$ on some date before maturity. *This is how we shall interpret things in the rest of the study.* The risky and mathematically perplexing inconsistencies connected with the treatment of discount and premium in the regulations, as they now seem to read, will thereby be avoided.

On a deeper level, though, what is the real root of so many clashes? Could there be other problems of such a serious nature lurking in the regulations, or even in the Tax Code itself? Only with the sharp tools of mathematical analysis can this possibly be found out, and only through attention to sound theory based on consistent mathematical formulations of economic ideas can the potential for nasty surprises be checked. We hope that this study will be seen as a step in the right direction.

5. DATE SPACING TECHNICALITIES

Even with the understanding reached at the end of Section 4 on the need for reinterpreting certain aspects of tax rules that are mathematically unworkable, not to speak of incompatible with what must have been intended, there are difficult problems in code and regulations which must be confronted before we can proceed to actual formulas for computing taxes. In looking at

[48] With the cutoff feature for offsets dropped, cf. the preceding footnote and the discussion to which it applies.

these we shall see how the theory developed in Section 2 can be put to the test in distinguishing whether a proposed method of calculating yield respects the given features of an instrument or distorts its interest stream in one way or another.

Fortunately, the worst complications belong to the past. They are still required for the taxation of instruments issued in the past that have not yet matured, but not for instruments issued under the present regulatory regime. But much of what we are about to say has the side aim also of clearing up persistent confusion in practice about yield computations.

There seems to be a widespread misconception, reflected to a degree in various provisions in tax regulations—which are the cause of our concern here—that equal spacing of the dates $i_0, \ldots, i_m$ in a debt instrument is ultimately in some way *prerequisite* to a correct determination of constant annual yield. In this notion, counter to the facts in Theorem 2.2 (as realized for instance in Example 3.9), irregular accrual periods have to be broken down through the introduction of supplementary compounding dates so as to make them regular, at least temporarily. Some elaborations of the idea envision mistakenly that a yield rate computed in such a manner might be utilized to determine the "correct" stream of interest income to be regarded as received over the *given* (irregular) accrual periods, and that once this has been done, the supplementary dates could thereafter be suppressed without incurring inconsistencies. According to Theorem 2.2, however, the given dates already determine *uniquely* a correct yield rate and stream of interest payments. Unless a proposed method of calculation comes up with this same yield, something has to be out of line.

There is no difficulty with placing extra dates in the sequence $i_0, \ldots, i_m$ so as to achieve equal spacing. (In practice the smallest number of such dates would be used, unless the aim is to pass to daily compounding.) But this does mean a *change from the given specification of a debt instrument to a new specification* for which not only the annual yield rate, but also the interest stream that is deemed to flow from it, will be *different*. The device is not an inconsequential mathematical tactic; it has definite effects on the perception of income and therefore on the amount and timing of taxes that might be owed on that income. Because of its unfortunate role in tax regulations and literature, we need to examine it in some detail.

At stake is a theory that does a consistent job of defining the quantities of importance in finance and does not merely leave them to the opinion of parties who may favor one approach to calculation over another. Only with such a theory firmly in grasp is it possible to know which figures for interest and outstanding principal are exact or approximate, and in the latter case, whether the degree of approximation is satisfactory.

In the following analysis we keep to the basic notation of the nominal specification, but the ideas carry over equally to the revised specification and any particularized specification.

Definition 5.1 (fully compatible accrual length)**.** *A value* $\overline{\theta}$ *will be said to give a fully compatible accrual period length relative to the date sequence* $i_0, \ldots, i_k, \ldots, i_m$ *if each accrual period* $[i_{k-1}, i_k]$*, when not already of length* $\overline{\theta}$*, can be divided into equal subperiods of such length,*[49]

$$\theta_k = n_k \overline{\theta} \ \text{ for } k = 1, \ldots, m, \ \text{ where } n_k \text{ is a whole number.} \tag{5.1}$$

Under the slightly weaker condition where this property holds for all but the initial and final accrual periods, i.e., where

$$\theta_k = n_k \overline{\theta} \ \text{ for } k = 2, \ldots, m-1, \ \text{ where } n_k \text{ is a whole number, and } m > 2, \tag{5.2}$$

$\overline{\theta}$ *will be said to give an internally compatible accrual period length.*

For now we focus on full compatibility, but internal compatibility is a condition prominent in certain tax regulations which will have to be dealt with also.

Method of Extra Dates. The approach we are about to describe has long been promulgated in finance education and is very commonly used in practice when the question arises of calculating yield. It has been adopted in tax regulations as an acceptable method for that purpose, although it amounts to replacing a given instrument with its well determined yield (in the sense of Theorem 2.2) by a differently specified instrument having a higher yield rate, as we shall explain.

For any fully compatible accrual period length $\overline{\theta}$, think of extra compounding dates being inserted in the sequence to mark the boundaries of the subperiods mentioned in Definition 5.1. Take the explicit payments due for the subperiods ending on these extra dates to be 0. The augmented sequence of dates could be renumbered, but without going through that notational exercise, it can be appreciated that a constant annual yield value relative to the altered data is uniquely determined through Theorem 2.2. Moreover, the yield-to-maturity equation needed for that purpose is merely the version with equal spacing. A consistent, full specification *different from the one already available* is thereby achieved. Denote the annual yield in this altered specification by $\overline{y}$.

The altered yield value $\overline{y}$ can be characterized within the original scheme of dates. In the altered scheme, the principal at the end of each subperiod of length $\overline{\theta}$ that does not terminate in one of the original dates i_k is merely multiplied by the factor $(1 + \overline{\theta}\overline{y})$; there is nothing to

[49] Again the compromises in Footnote 10 may intervene. In particular daily accrual periods can always be used; then $\overline{\theta} = 1/365$. Sometimes there are further compromises, like the 30-day month, or the 360-day year.

deduct because the explicit payment for such a subperiod has been taken to be 0. Therefore, over the duration of an original period of length θ_k that has been divided into n_k subperiods of length $\overline{\theta}$, the principal that was present at the beginning just gets multiplied by $(1+\overline{\theta}\overline{y})^{n_k}$ before the amount C_k is subtracted off. From this it is evident that the effect of the alteration on the yield-to-maturity equation in Theorem 2.2 for the original dates is simply to replace each factor $(1+\theta_k y)$ by $(1+\overline{\theta}\overline{y})^{n_k}$. In other words, $\overline{y}$ is the unique solution to

$$V_0 = \frac{C_1}{\left(1+\overline{\theta}\overline{y}\right)^{n_1}} + \frac{C_2}{\left(1+\overline{\theta}\overline{y}\right)^{n_1+n_2}} + \cdots + \frac{C_m}{\left(1+\overline{\theta}\overline{y}\right)^{n_1+\cdots+n_m}}, \tag{5.3}$$

in contrast to y being the unique solution to (2.14).

Theorem 5.1 (effects of the method of extra dates). *Suppose a consistent full specification of a debt instrument as in Definition 2.1 with positive net return as in Definition 2.2 has accrual periods that are not all equal in length. Then for any auxiliary accrual period length $\overline{\theta}$ fully compatible with the specified date sequence $i_0, \ldots, i_k, \ldots, i_m$ in the sense of Definition 5.1, the annual yield $\overline{y}$ in the altered specification having accrual periods that are all of length $\overline{\theta}$ will be smaller than the original yield:*

$$\overline{y} \;<\; y. \tag{5.4}$$

The interest income over the life of the instrument will be the same with respect to the altered specification, but some of it will be received in implicit payments of interest for the subperiods ending on the extra dates, regarded as automatically reinvested.

Furthermore, receipt of interest income, even as aggregated over the original accrual periods, will definitely be different relative to the altered specification. With $\overline{I}_k$ denoting the total interest received over the original (unpartitioned) interval $(i_{k-1}, i_k]$ relative to the altered specification, one will necessarily have $\overline{I}_k > I_k$ for some dates i_k beyond i_0, but $\overline{I}_k < I_k$ for others.

Proof. Because the instrument has positive net return, the yield y under the original specification must be positive (as follows from Theorem 2.1), and likewise the yield $\overline{y}$ under the altered specification must be positive. When the expression $(1+\overline{\theta}\overline{y})^{n_k}$ is multiplied out algebraically (the well known "binomial expansion"), the initial terms are $1 + n_k\overline{\theta}\overline{y}$. Any remaining terms are positive (because $\overline{y} > 0$). Therefore,

$$(1+\overline{\theta}\overline{y})^{n_k} > 1 + n_k\overline{\theta}\overline{y} = 1 + \theta_k\overline{y} \;\text{ unless }\; n_k = 1.$$

Invoking this estimate in equation (5.3) we see that

$$V_0 \; < \; \frac{C_1}{(1+\theta_1\overline{y})} + \frac{C_2}{(1+\theta_1\overline{y})(1+\theta_2\overline{y})} + \cdots + \frac{C_m}{(1+\theta_1\overline{y})(1+\theta_2\overline{y})\cdots(1+\theta_m\overline{y})}.$$

The combination of this inequality with the equation in (2.14) tells us that $\overline{y} < y$. Still, we must have

$$\overline{I}_1 + \cdots + \overline{I}_m = I_1 + \cdots + I_m \tag{5.5}$$

because by Theorem 2.1 (as applied to the altered specification as well as the original specification) both sides equal the quantity $C_1 + \cdots + C_m - V_0$, the net return.

Next, for each of the original dates i_k, denote by $\overline{V}_k$ the amount of principal outstanding on that date relative to the altered specification, after subtraction of the explicit payment then. We have

$$\overline{V}_k = \big(1+\overline{\theta}\overline{y}\big)^{n_k}\overline{V}_{k-1} - C_k \;\text{ for }\; k = 1,\ldots,m, \;\text{ with }\; \overline{V}_0 = V_0 \tag{5.6}$$

and also

$$\overline{I}_k = \big[\,\big(1+\overline{\theta}\overline{y}\big)^{n_k} - 1\,\big]\overline{V}_{k-1} \;\text{ for }\; k = 1,\ldots,m. \tag{5.7}$$

These formulas may be compared with the ones for V_k and I_k in (2.3) and (2.4). For the amounts $\overline{I}_k$ all to agree with the amounts I_k, we would have to have

$$\big(1+\overline{\theta}\overline{y}\big)^{n_k} = \big(1+\theta_k y\big) = \big(1+n_k\overline{\theta}y\big) \;\text{ for all }\; k,$$

which would mean that

$$y = \big[\,\big(1+\overline{\theta}\overline{y}\big)^{n_k} - 1\,\big]/n_k \;\text{ for all }\; k.$$

This is impossible unless all the multiples n_k are all the same. Hence $\overline{I}_k \neq I_k$ for at least one k. But the equation (5.5) also holds, so there must be some dates i_k on which $\overline{I}_k > I_k$ and others on which $\overline{I}_k < I_k$. □

In the very broad context of Theorem 5.1 it cannot be said that the interest income is necessarily shifted in one direction or the other. Of course the distortion might only be minor.[50] The important thing for theory nonetheless is that a shift does occur. In its consequences the method of extra dates definitely deviates from employing the given specification of an instrument directly. The altered specification at least conforms to basic guidelines, however, since it is another consistent, full specification of the instrument.

If a helpful mathematical simplification were achieved through the method of extra dates, it could anyway be regarded as providing a convenient *estimate* of yield, which might be good

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[50] A numerical example will be furnished later in this section; see Example 5.1.

enough for tax purposes. But the equation to be solved for $\overline{y}$ clearly requires more effort to understand and set up than the one for y in Theorem 2.2, and numerical solution is likely to be provided by a computer anyway. In the meantime, theory suffers if "yield" is merely regarded as any one out of some "reasonable range" of numbers associated with an instrument through various schemes of calculation. Whether a value is reasonable can only be answered in the background of knowing first which value is the correct one mathematically.[51]

Forced Respecification. In certain situations which will be discussed in Section 7, the description of a debt instrument may be considered unacceptable under the Tax Code even if it is a consistent, full specification. This may be triggered because the compounding dates are deemed to be too far apart. Or it may result from the accrual periods being deemed to be too irregular.[52]

The consequence then is that the instrument must be respecified. This may simply mean applying the method of extra dates to achieve equal spacing relative to a fully compatible accrual period length $\overline{\theta}$ that does not exceed a certain size (e.g. a year or half a year, depending on the circumstances). If the original accrual periods were already equal in length, such a change would not shift income from one such period to another, but the inserted compounding dates would be new occasions for the receipt of implicit interest or repayment of principal. The pattern in which interest is regarded as arriving over the course of each of the original accrual periods would therefore be shifted.[53]

Although respecification relative to the sequence of dates is really only necessary in practice in situations where the revised specification or a particularized specification have to come into

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[51] This may illustrate a philosophical difference between legal language and mathematical language. Both are motivated by a need to say things without the vagueness and multiplicity of meanings common in ordinary speech and exalted in literature. They are designed for purposes where meanings must not be left in doubt. Legal language is accustomed, of course, to the realities of human affairs, where words are not adequate to all possibilities; when disputes do arise a judge can settle them. Mathematics, while having no pretense of applying to most human affairs, has the capability of deciding the actual truth or falsehood of a surprisingly wide range of assertions, provided that its procedures and standards are dutifully respected.

[52] The pending IRS Proposed Regulations of 1992 go far in erasing this whole problem—for newly issued instruments. But they appear to do this by focusing even more on the "method of fractional exponents," explained below, as the way around irregularities. It will be seen that this method is an outgrowth of the method of extra dates, but lacks its theoretical consistency.

[53] In comparison with straight-line accrual of interest income in the form of simple interest over each of the original periods $[i_{k-1}, i_k]$ (see Section 6), the effect would be to shift interest income within each of the original periods from the beginning toward the end. This could make a difference in tax payments according to where the payment dates might lie in these intervals. A postponement of interest is advantageous for the holder of an instrument but disadvantageous for the issuer, who may be unable to claim interest expense until later.

play, it is essential for the mathematical structure of the formulas to be presented in Section 7 that we treat the matter universally, as it applied to any specification. All three kinds of specifications introduced in Section 3 could be involved in one of these formulas for a given instrument, and unless a common sequence of dates can be used, the mathematics could become unmanageable. Once more we continue with the basic notation of the nominal specification, although it is not the only one to which the conclusions will apply.

For an instrument with irregular accrual periods, but having more than two periods ($m > 2$), the rules involving forced respecification usually allow passage to an accrual period length $\overline{\theta}$ that is not fully compatible but just internally compatible in the sense of Definition 5.1. (Ordinarily $\overline{\theta}$ would have to be the largest such length that does not exceed a certain size.) Then it is only the intermediate accrual periods that necessarily get divided into equal subperiods. *The initial period* $[i_0, i_1]$ *is divided into as many subperiods of length* $\overline{\theta}$ *as possible; if there is a fraction left over, it is placed at the beginning. The final period* $[i_{m-1}, i_m]$ *is similarly divided, but with any fraction placed at the end.* The result is a sequence of dates for which the accrual periods are all equal of length $\overline{\theta}$, except that a shorter period may occur at the beginning, or the end, or both.

The specification of a new sequence of dates is insufficient in itself for determining how interest income should be recomputed. The pattern of explicit payments relative to these dates must also be indicated. In the case of passage to an internally compatible accrual period length as just described, this is done as in the method of extra dates to achieve equal spacing. The explicit payments for subperiods ending in the extra dates are taken to be 0. Original payment amounts C_k are reassigned exclusively to the subperiods ending in the *original* dates i_k.

Once this has been done, the corresponding constant annual yield rate $\overline{y}$ can be calculated by solving the yield-to-maturity equation (2.14) in Theorem 2.2. The respecification then becomes a consistent full specification for the debt instrument in the sense of Definition 5.1, and the interest payments associated with the instrument relative to the new accrual periods (including the possibly shorter periods at the beginning and end) are unambiguously determined. Unfortunately, as if the process of forced respecification were not complicated enough,[54] tax regulations currently fail to provide clear recognition of the uniquely determined rate $\overline{y}$ or explain how to get it. Instead, approaches to recalculating interest income are suggested that at best arrive at approximate payment amounts, although without saving any effort over that required for a correct calculation. Such an approach is the method of fractional exponents described

[54] While stipulations that accrual periods should not be too long may be justified, motivation for the insistence on achieving equal accrual periods in a respecified instrument, at least internally, is lacking. Perhaps the procedure has arisen through misunderstanding of what is needed for a consistent treatment of taxation in terms of constant annual yield.

below.

For one class of debt instruments which will be encountered in Section 7, regulations call for respecification in terms of a sequence of dates starting with the date of issue and spaced exactly one year apart,[55] except that the final accrual period, ending on the date of maturity, can be shorter. Other features of the required respecification are left in doubt, however. While the new date sequence is clear—let us indicate it by $i'_0, i'_1, \ldots, i'_{m'-1}, i'_{m'}$ where $i'_0 = i_0$ and $i_{m'} = i_m$ —we need to know what explicit payments are to be regarded as received by a holder for each of the respecified accrual periods $[i'_{k'-1}, i'_{k'}]$. There is no simple answer,[56] but the solution we adopt here is to take the amount $C'_{k'}$ for each such period to be the sum of the amounts C_k in the original specification for which $i'_{k'-1} < i_k \leq i'_{k'}$ (hence 0 if none of the original dates i_k falls within the new period in question). Once this hurdle is past, we are able again to calculate a constant annual yield rate $\overline{y}$ on the basis of Theorem 2.2 and thereby achieve a respecification which is a consistent full specification of the debt instrument in the sense of Definition 5.1.

Joint Respecification in the Nominal, Revised, and Particularized Sense. In situations where the tax rules to be dealt with in Section 7 require forced respecification of a debt instrument, we shall take this for our purposes as meaning simply that *the original sequence of dates and explicit payments is entirely supplanted by the altered data*—for all three specifications, nominal, revised and particularized. We continue therefore to employ the usual symbols for interest and other quantities associated with these specifications, as if the earlier versions had never existed.

Method of Fractional Exponents. A currently much-used approach to calculating an interest income stream for an instrument with irregular accrual periods is the method of fractional exponents. This is an ad hoc procedure which leads to interest payment amounts which differ from the ones derived from direct application of the principle of constant annual yield through Theorem 2.2, yet it offers no mathematical simplifications.[57]

The method of fractional exponents has mainly been invoked for instruments having equal accrual periods except for a short period at the beginning or end, or both, such as have been

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[55] See Footnote 10.

[56] Garlock [1, p. 92] holds the view that the prescription in question is inconsistent with any reasonable calculation of constant yield to maturity. All methods certainly distort the interest accrual of the nominal specification.

[57] Nevertheless it has widely been recommended in tax literature and is featured in the software now available on financial calculators, while techniques for solving the correct yield-to-maturity equation (2.14) are neglected.

seen to arise in particular through the process of forced respecification explained above. We shall describe it in more general terms, however, because that will make its relationship to other approaches clearer.[58] Let

$$\tilde{\theta} = \text{ the largest of the lengths } \theta_1, \ldots, \theta_m, \tag{5.8}$$

so that

$$\theta_k/\tilde{\theta} \ \leq \ 1 \ \text{ for } \ k = 1, \ldots, m. \tag{5.9}$$

The ratio $\theta_k/\tilde{\theta}$ expresses the length of the kth accrual period as a fraction of the reference length $\tilde{\theta}$. This ratio must equal 1 for at least some period by virtue of (5.8), but on the other hand it will be less than 1 for some other period, or we would be back in the case of regular periods and have no need for the maneuver about to be explained.

The method of fractional exponents takes the equation

$$\begin{aligned} V_0 = \frac{C_1}{(1+\tilde{\theta}\tilde{y})^{p_1}} &+ \frac{C_2}{(1+\tilde{\theta}\tilde{y})^{p_2}} + \cdots + \frac{C_m}{(1+\tilde{\theta}\tilde{y})^{p_m}}, \\ \text{where } p_1 &= \big(\theta_1/\tilde{\theta}\big) \\ p_2 &= \big(\theta_1/\tilde{\theta}\big) + \big(\theta_2/\tilde{\theta}\big) \\ &\cdots \\ p_m &= \big(\theta_1/\tilde{\theta}\big) + \cdots + \big(\theta_m/\tilde{\theta}\big) \end{aligned} \tag{5.10}$$

as defining an annual yield rate $\tilde{y}$ to be associated with the instrument. In other words, each of the factors $(1+\theta_k y)$ in the true yield-to-maturity equation (2.14) is replaced by $(1+\tilde{\theta}\tilde{y})^{\theta_k/\theta}$. The evolution of principal is considered then to be given by

$$\widetilde{V}_k = \big(1+\tilde{\theta}\tilde{y}\big)^{\theta_k/\tilde{\theta}}\widetilde{V}_{k-1} - C_k \ \text{ for } \ k = 1, \ldots, m, \ \text{ with } \ \widetilde{V}_0 = V_0. \tag{5.11}$$

The corresponding interest payments are

$$\widetilde{I}_k = \big[\, \big(1+\tilde{\theta}\tilde{y}\big)^{\theta_k/\tilde{\theta}} - 1 \,\big]\, \widetilde{V}_{k-1}, \tag{5.12}$$

while principal repayments are

$$\widetilde{R}_k = C_k - \widetilde{I}_k = \widetilde{V}_{k-1} - \widetilde{V}_k. \tag{5.13}$$

The evolution rule (5.11) obviously does not correspond to utilizing the rate $\tilde{y}$ to generate simple interest over the designated accrual periods, since for that purpose the factor $(1+\theta_k\tilde{y})$

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[58] The pending IRS Proposed Regulations of 1992 elevate it to this wider applicability anyway.

would be indicated. Therefore, the interest payments cannot be seen as corresponding to a consistent, full specification of the instrument (in the sense of Definition 2.1) in which the date sequence is $i_0, \ldots, i_k, \ldots, i_m$. But the following properties do hold anyway:

$$\widetilde{V}_k > 0 \text{ for } k = 1, \ldots, m-1, \text{ but } \widetilde{V}_m = 0, \tag{5.14}$$

$$\begin{aligned} \widetilde{V}_k &= \widetilde{R}_{k+1} + \cdots + \widetilde{R}_m \\ &= \big(C_{k+1} + \cdots C_m\big) - \big(\widetilde{I}_{k+1} + \cdots + \widetilde{I}_m\big) \text{ for } k = 0, 1, \ldots, m-1, \end{aligned} \tag{5.15}$$

with $R_m > 0$, and in particular

$$C_1 + \cdots + C_m - V_0 \ = \ \widetilde{I}_1 + \cdots + \widetilde{I}_m. \tag{5.16}$$

These properties can be established in direct consequence of equation (5.10), but they also grow out of a conceptual relationship between the method of fractional exponents and respecification in terms of daily compounding. This will be seen in the proof of the next theorem.

Theorem 5.2 (inaccuracy of the method of fractional exponents). *Suppose a debt instrument specified in conformity with conditions (b)(c)(d) of Definition 2.1 has positive net return but accrual periods that are not all equal in length. Then the yield value offered by the method of fractional exponents is well defined and positive, but it is higher than the true yield obtained from the corresponding consistent, full specification provided by Theorem 2.2:*

$$\tilde{y} \ > \ y. \tag{5.17}$$

Although the total interest received over the life of the instrument will be the same in both cases, that is,

$$\widetilde{I}_1 + \cdots + \widetilde{I}_m \ = \ I_1 + \cdots + I_m, \tag{5.18}$$

the receipt of this interest income will necessarily be shifted in time. In other words, there will be some dates i_k beyond i_0 for which $\widetilde{I}_k < I_k$, but others for which $\widetilde{I}_k > I_k$.

Proof. First we make a comparison with the equation used for the method of extra dates in order to confirm that equation (5.10) does have a unique solution $\tilde{y} > 0$. For a fully compatible accrual period length $\overline{\theta}$ in the sense of Definition 5.1 (for instance $\overline{\theta} = 1/365$) let n denote the highest of the multiples $n_1, \ldots, n_m$, so that $\tilde{\theta} = n\overline{\theta}$ on the basis of (5.2) and (5.8). Then $\theta_k/\tilde{\theta} = n_k/n$, so that

$$\big(1 + \tilde{\theta}\tilde{y}\big)^{\theta_k/\tilde{\theta}} = \big[\,\big(1 + \tilde{\theta}\tilde{y}\big)^{1/n}\,\big]^{n_k}.$$

Let $\overline{y}$ be the unique value such that

$$\big(1+\tilde{\theta}\tilde{y}\big)^{1/n} = \big(1+\overline{\theta}\overline{y}\big),$$

or in other words,

$$\overline{y} = \big[\,\big(1+\tilde{\theta}\tilde{y}\big)^{1/n} - 1\,\big]/\overline{\theta}.$$

Equation (5.12) then reduces to equation (5.3), which we have seen to be the correct yield-to-maturity equation for the altered specification of the instrument in which all accrual periods have been reduced to length $\overline{\theta}$. At the same time the quantities $\widetilde{V}_k$ and $\widetilde{I}_k$ are identified with their counterparts $\overline{V}_k$ and $\overline{I}_k$ in (5.6) and (5.7). It was established in Theorem 5.1 that the income stream $\overline{I}_1,\ldots,\overline{I}_m$ is a distortion of $I_1,\ldots,I_m$, so the same must be true of $\widetilde{I}_1,\ldots,\widetilde{I}_m$.

We still must prove (5.14), and for this purpose we to turn to properties of the calculus relation

$$p\int_0^z (1+w)^{p-1}dw = (1+z)^p - 1 \;\text{ for }\; z>0. \tag{5.19}$$

When $p=1$, both sides of this relation degenerate to z. When $p<1$, the inequality $(1+w)^{p-1} < 1$ holds in the integrand for $w>0$, and the integral is then strictly bounded above by z. In other words, we have

$$pz > (1+z)^p - 1.$$

Applying this to $z=\tilde{\theta}\tilde{y}$ and $p=\theta_k/\tilde{\theta}$ we obtain

$$\big(1+\tilde{\theta}\tilde{y}\big)^{\theta_k/\tilde{\theta}} \le \big(1+\theta_k\tilde{y}\big), \;\text{ with strict inequality if }\; \theta_k<\theta. \tag{5.20}$$

It follows then from equation (5.3) that

$$V_0 > \frac{C_1}{(1+\theta_1\tilde{y})} + \frac{C_2}{(1+\theta_1\tilde{y})(1+\theta_2\tilde{y})} + \cdots + \frac{C_m}{(1+\theta_1\tilde{y})(1+\theta_2\tilde{y})\cdots(1+\theta_m\tilde{y})}. \tag{5.21}$$

For the function f utilized in the proof of Theorem 2.2, we have from (2.14) and (5.21) that $f(y) > f(\tilde{y})$, and since the function value decreases for larger values of its argument we conclude that $\tilde{y} > y$. □

As with the method of extra dates, the distortion of income that results from the method of fractional exponents could in general go in either time direction, but in the important case of standard bonds income to the holder is systematically shifted a bit into the future.[59] This will be established in Theorem 5.3 below.

[59] Actual income in the explicit payments C_k is not postponed, but merely the receipt some portion of the implicit interest payments. The effects are unlikely to be of much significance, however; see Example 5.1.

It should be noted carefully that the distortion issue addressed in Theorem 5.2 is separate from the question of whether one method of determining interest income might better reflect economic content than another. That issue is behind the introduction of the revised and particularized specifications in Section 3. It is separate also from questions of convenience. Instead, the issue goes to the heart of the theory of constant annual yield. *Is the direct description of how interest income is generated by a debt instrument with constant annual yield, as embodied in Definition 2.1, with simple interest earned over periods delimited by compounding dates, correct and admissible as a mathematical building block?* To reject this description would be to deny the validity of traditional financial arrangements of the simplest sort. On the other hand, if the description is accepted, the true yield-to-maturity equation has to be the one in Theorem 2.2. Then the method of fractional exponents comes off at best as a unnecessarily inaccurate approach to calculation.

If one were to start with a consistently specified instrument in the sense of Definition 2.1 with its interest income stream $I_1, \ldots, I_m$ correctly laid out, but such that the accrual periods are not all of the same length, and then compute the yield $\tilde{y}$ indicated by the method of fractional exponents and use that to get an interest income stream $\widetilde{I}_1, \ldots, \widetilde{I}_m$, one would always find a difference. According to Theorem 5.2 the payment decomposition $C_k = I_k + R_k$ provided by the instrument's specification would be overturned. On various dates i_k it would be replaced by a distinctly different decomposition $C_k = \widetilde{I}_k + \widetilde{R}_k$ in which sometimes $\widetilde{I}_k$ is larger than I_k and sometimes smaller.

This discrepancy deserves emphasis because some tax regulations actually refer to the method of fractional exponents as the "exact" approach to computing interest income, in contrast to approaches utilizing simple interest over irregular accrual periods, which are termed "approximate."[60] The analysis provided here shows that the two terms have somehow gotten reversed in their mathematical content.

Although the method of fractional exponents is kin to the method of extra dates to achieve equal spacing, there is an important distinction. The method of extra dates replaces the given specification of an instrument by an altered one, which is then treated in perfect accord with general theory. The inserted dates are not just an accounting fiction but occasions for the receipt of implicit interest payments. The method of fractional exponents is in contrast a hybrid procedure in which such intermediate dates end up having no role. Whatever the ideas from which the method is derived, interest is ultimately regarded as arriving just on the given dates i_k. This is not quite the way the matter comes out in practice, because the concept of the

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[60] See IRS Proposed Regs. of 1986, S1.1272–1(c)(2)(ii)(B) and (C), and Garlock [1, p. A–2].

"basis" of an instrument, which will be explained in Section 6, spreads the receipt of interest evenly over each period. Still, the results even in this respect will be different when the method of fractional exponents is used[61] instead of the method of extra dates or the original specification itself through the yield-to-maturity equation in Theorem 2.2.

From another direction, such inconsistency underscores once more the fact that the method of fractional exponents does not fit squarely with the meaning of constant annual yield. If it did, the growth factors $(1+\tilde{\theta}\tilde{y})^{\theta_k/\tilde{\theta}}$ in (5.11) for the specified accrual periods could be expressed in the form $(1+\theta_k y')$ for some single value $y' \geq 0$. But this is not possible without equal spacing of dates. Instead, one has

$$\big(1+\tilde{\theta}\tilde{y}\big)^{\theta_k/\tilde{\theta}} = \big(1+\theta_k\tilde{y}_k\big) \ \text{ for } \ k=1,\ldots,m,$$

where the value $\tilde{y}_k$ is defined by

$$\tilde{y}_k = \big[\,\big(1+\tilde{\theta}\tilde{y}\big)^{\theta_k/\tilde{\theta}} - 1\,\big]/\theta_k.$$

This means that *the method of fractional exponents effectively applies to each accrual period* $[i_{k-1}, i_k]$ *an annual yield rate* $\tilde{y}_k$ *which varies with the length of the period*. It is truly a method that comes up with a scheme of *variable* yields from which to compute taxable income or expense, instead of keeping to the goal of determining interest relative to a single, fixed yield rate as the law prescribes.

Effects of the Method of Fractional Exponents. The consequences of using the method of fractional exponents to estimate interest income, instead of using the yield-to-maturity equation (2.14) provided by basic theory, stand out especially in the case of the particularized specification of a standard bond as in Example 3.9. The accrual length $\tilde{\theta}$ in the method of fractional exponents can be identified then with θ, so equation (3.33) is replaced by

$$\begin{aligned} P_a = &\frac{C}{\big(1+\theta\tilde{y}^{**}\big)^{\theta_a/\theta}} + \frac{C}{\big(1+\theta\tilde{y}^{**}\big)^{\theta_a/\theta}\big(1+\theta\tilde{y}^{**}\big)} \\ &+ \frac{C}{\big(1+\theta\tilde{y}^{**}\big)^{\theta_a/\theta}\big(1+\theta\tilde{y}^{**}\big)^2} + \cdots + \frac{C+V_0}{\big(1+\theta\tilde{y}^{**}\big)^{\theta_a/\theta}\big(1+\theta\tilde{y}^{**}\big)^{m-\bar{k}}}. \end{aligned} \tag{5.22}$$

[61] The method of fractional exponents is sometimes described as based on daily compounding as the "most accurate" way of keeping track of interest, cf. Garlock [1, p. A–2]. But if that viewpoint were *consistently* adopted the method would not revert to linear accrual over the original periods when it comes to "basis" and capital gain; accrual would become nonlinear for such purposes. Anyway, daily compounding is just one of the ways that interest may be specified in financial transactions in conformity with constant annual yield (as made precise Definition 2.1). All are equally "accurate."

The value $\tilde{y}^{**}$ obtained by solving this equation is used to calculate income amounts $\widetilde{I}_{\overline{k}}, \ldots, \widetilde{I}_m$ different from the ones in Example 3.9. This is done by first setting

$$\widetilde{V}_{\overline{k}}^{**} = \big(1 + \theta\tilde{y}^{**}\big)^{\theta_a/\theta} P_a - C \tag{5.23}$$

along with

$$\widetilde{I}_{\overline{k}}^{**} = \big[\,\big(1 + \theta\tilde{y}^{**}\big)^{\theta_a/\theta} - 1\,\big] P_a \tag{5.24}$$

and thereafter taking

$$\begin{aligned} \widetilde{V}_k^{**} &= \big(1 + \theta\tilde{y}^{**}\big)\widetilde{V}_{k-1}^{**} - C \ \text{ for } \ k = \overline{k} + 1, \ldots, m-1, \\ \widetilde{I}_k^{**} &= \theta\tilde{y}^{**} V_{k-1}^{**} \ \text{ for } \ k = \overline{k} + 1, \ldots, m, \end{aligned} \tag{5.25}$$

which will result in $\widetilde{V}_m^{**} = 0$.

Theorem 5.3 (income shift induced by method of fractional exponents)**.** *Suppose that a standard bond is acquired on an intermediate date within an accrual period* $[i_{\overline{k}-1}, i_{\overline{k}}]$ *that is not the final period. Then the yield rate* $\tilde{y}^{**}$ *determined from the method of fractional exponents will be higher than the particularized yield rate* y^{**}*:*

$$\tilde{y}^{**} \ > \ y^{**}.$$

The interest stream $\widetilde{I}_{\overline{k}}, \ldots, \widetilde{I}_m$ *developed by that method will exhibit a systematic shift of income into the future relative to the interest stream* $I_{\overline{k}}, \ldots, I_m$ *derived from the particularized specification in Example 3.9. In other words, one will have*

$$\widetilde{I}_{\overline{k}}^{**} + \cdots + \widetilde{I}_k^{**} \ < \ I_{\overline{k}}^{**} + \cdots + I_k^{**} \ \textit{ for } \ k = \overline{k}, \ldots, m-1, \tag{5.26}$$

although in the end the total imputed interest will be the same,

$$\widetilde{I}_{\overline{k}}^{**} + \cdots + \widetilde{I}_m^{**} \ = \ I_{\overline{k}}^{**} + \cdots + I_m^{**}. \tag{5.27}$$

Proof. We know from Theorem 5.2 that $\tilde{y}^{**} > y^{**}$. Comparing the formula

$$V_{k-1}^{**} = \big(1 + \theta y^{**}\big)^{-1}\big[V_k^{**} + C\big]$$

(cf. (3.36)) with the corresponding formula

$$\widetilde{V}_{k-1}^{**} = \big(1 + \theta\tilde{y}^{**}\big)^{-1}\big[V_k^{**} + C\big]$$

(cf. (5.25)), where $\widetilde{V}_m^{**} = V_m^{**} = 0$, we find then that

$$\widetilde{V}_k^{**} < V_k^{**} \text{ for } k = \overline{k}, \ldots, m-1. \tag{5.28}$$

But also $I_k^{**} + \cdots + I_m^{**} = (m-k+1)C - V_k^{**}$ (as follows from Theorem 2.1, there being $m-k+1$ payments of the coupon amount C on the dates $i_k, \ldots, m$), whereas $\widetilde{I}_k^{**} + \cdots + \widetilde{I}_m^{**} = C_k + \cdots + C_m - \widetilde{V}_k^{**}$. The inequalities in (5.28) therefore imply

$$\widetilde{I}_k^{**} + \cdots + \widetilde{I}_m^{**} \;>\; I_{\overline{k}}^{**} + \cdots + I_k^{**} \text{ for } k = \overline{k}+1, \ldots, m. \tag{5.29}$$

On the other hand, (5.27) is correct because both sides give the net return $(m-\overline{k}+1)C - P_a$ to the holder who acquired the bond on date a. The combination of (5.27) and (5.29) produces the inequalities claimed in (5.26). □

The shift identified in Theorem 5.3 is not likely to be of much significance, as the next example illustrates. Thus, there is little incentive for employing the method of fractional exponents rather than the consistent yield-to-maturity calculation in Example 3.9 in order to gain some tax advantage. On the other hand, its usage obscures the mathematical picture.

Example 5.1 (numerical comparisons for a zero-coupon bond). *For a detailed example, which will show the differences between the interest stream computed under three different approaches, consider a zero-coupon bond which was purchased with 1.25 years left in its life. Let the date of acquisition be September 30, 1990, the date of maturity be December 31, 1991, and the intervening dates in the specification (perhaps as a result of forced respecification to semiannual compounding) be December 31, 1990, and June 30, 1991. These dates will be designated by a, i_1, i_2, and $i_3 = i_m$. (For notational simplicity only, we are supposing that $\overline{k} = 1$, i.e., that the instrument was acquired in its first accrual period.) We have $\theta_1 = .25$ but $\theta_2 = \theta_3 = .5$. Also, since a zero-coupon bond is involved, we have $C_1 = C_2 = 0$ but $C_3 = V_0$. The yield-to-maturity equation provided by Theorem 2.2 therefore takes the form*

$$P_a = \frac{V_0}{(1+.25\,y)(1+.5\,y)^2}. \tag{5.30}$$

In contrast, the equation used by the method of fractional exponents from (5.22) would be

$$P_a = \frac{V_0}{(1+.5\,\tilde{y})^{2.5}}. \tag{5.31}$$

A third approach would be to respecify the bond with five quarterly accrual periods by introducing the dates i_2' = March 31, 1991, and i_3' = September 30, 1990. Then, in application of the method of extra dates in (5.3), the equation would be

$$P_a = \frac{V_0}{(1+.25\,\overline{y})^5}. \tag{5.32}$$

Let us suppose that the face value is $V_0 = \$1,000,000$ *while the acquisition price is* $P_a = \$906,428$. *This choice of* P_a *has the property that the direct yield-to-maturity equation (5.30) is satisfied with* $y = .8$. *(For any choice of* $P_a > 0$ *the unique annual yield value* y *could be determined from (5.31) by a numerical method programmed on a computer.) The values of* $\tilde{y}$ *and* $\overline{y}$ *can be calculated from (5.31) and (5.32) by rewriting the first equation as*

$$\left(1 + .5\,\tilde{y}\right) = \left[V_0/P_a\right]^{1/2.5}$$

and the second as

$$\left(1 + .25\,\overline{y}\right) = \left[V_0/P_a\right]^{1/5}.$$

All three values can then be used to compute their associated streams of interest income. The numerical results are displayed in Table 5.1.

Approach	Direct	Fractional	Extra Dates
Yield Value	$y = 8.000\%$	$\tilde{y} = 8.015\%$	$\overline{y} = 7.937\%$
Interest on i_1	$I_1 = \$18,129$	$\widetilde{I}_1 = \$17,986$	$\overline{I}_1 = \$17,986$
Interest on $i_{2'}$			$\overline{I}_{2'} = \$18,343$
Interest on i_2	$I_2 = \$36,982$	$\widetilde{I}_2 = \$37,051$	$\overline{I}_2 = \$18,708$
Interest on $i_{3'}$			$\overline{I}_{3'} = \$19,078$
Interest on i_3	$I_3 = \$38,461$	$\widetilde{I}_3 = \$38,535$	$\overline{I}_3 = \$19,457$
Total Interest	$I = \$93,572$	$\widetilde{I} = \$93,572$	$\overline{I} = \$93,572$

Table 5.1. A comparison of interest calculations in Example 5.1.

The tabulated results confirm various properties of the three approaches that we have developed theoretically. The relation $\overline{y} < y < \tilde{y}$ (Theorems 5.1 and 5.2) is exhibited along with the fact that the total interest received is always the same. Also evident is the slight shift of interest income to the future when the method of fractional exponents is used. The sum of $\overline{I}_{2'}$ and $\overline{I}_2$ in the method of extra dates equals $\widetilde{I}_2$, and likewise sum of $\overline{I}_{3'}$ and $\overline{I}_3$ equals $\widetilde{I}_3$. But the two portions in each case are not equal.

The method of extra dates is strictly based on constant yield to maturity over five quarterly periods instead of the given irregular periods, and the larger amounts assigned to the second halves of each of the semiannual periods reflect compounding on this finer scale. In particular, straight-line accrual of the interest amounts for each quarter under the method of extra dates will not be consonant with straight-line accrual of the interest amounts for the corresponding semiannual periods under the method of fractional exponents. This signals again the fact that the method of fractional exponents does not square with the theory of constant yield to maturity and suffers from a mixture of concepts. The direct method gives the unique interest values that come from using a constant annual yield rate with straight-line accrual (simple interest) over the original accrual periods.

6. BASIS AND GAIN

Various dates can be crucial to the taxation of a debt instrument besides the dates $i_0, \ldots, i_k, \ldots, i_m$ in its specification. The acquisition date a has already been seen as an example. If the instrument is sold before maturity, the date of disposal will be important similarly. Other dates of significance are the ones marking the ends of tax years.

It is essential therefore to keep track of the income earned from an instrument in a day-by-day manner. This need underlies the concept of the *basis* of the instrument on any date i relative to a given specification. We begin by discussing the basis relative to the nominal specification, where the explanation is the simplest. Then we go on to the revised and particularized specifications. Each shift in the interpretation of basis causes a parallel shift in the determination of *capital gain or loss*. The mathematical details are therefore unavoidable in moving toward our goal of being able to translate tax rules into precise mathematical formulas of wide applicability.

Basis Under the Nominal Specification. In tracking the amount of principal considered to be outstanding at any time under the nominal specification, any principal repayment amount R_k (when nonzero) is regarded as being received precisely on the compounding date i_k, but the interest payments I_k are generally viewed differently.[62] Each amount I_k is *accrued*, i.e., treated as arriving in equal increments distributed over the dates i (each day) in the preceding period, namely the ones satisfying $i_{k-1} < i \leq i_k$. The fraction of I_k earned on such a date is taken to be $1/(i_k - i_{k-1})$. (Note that the fraction can vary slightly from one accrual period to the next because of the calendar effects described in Footnote 10.) Equivalently, *the simple interest*

[62] This is the case for an *accrual-basis* taxpayer. For a *cash-basis* taxpayer, the nominal interest amounts I_k would be interpreted as earned in single amounts on the dates i_k, like the principal repayment amounts R_k. But even for a cash-basis taxpayer, this would only be valid for *nominal* interest. Other forms of interest developed through tax law, e.g. in connection with OID, are always accrued. (If a cash-basis taxpayer disposes of a debt instrument between interest payment dates, then straight-line accrual is used to determine the interest income during the last period before disposal.)

developed by the instrument over each accrual period is regarded as being paid as it is earned.

To capture the accrual mechanism conveniently for use later in mathematical formulas, we introduce the symbol

$$I(i', i) = \text{ nominal interest earned from date } i' \text{ through date } i \tag{6.1}$$

for any dates i' and i satisfying $i_0 \le i' < i \le i_m$. (This is to be interpreted as 0 in general formulas where the case $i' = i$ might arise.) Of particular importance will be amounts earned after the date a on which the instrument was acquired, which can be calculated through[63]

$$I(a, i) = \begin{cases} \dfrac{i - a}{i_k - i_{k-1}}\, I_k & \text{when } i_{k-1} \le a < i \le i_k, \text{ but} \\ I(a, i_{k-1}) + \dfrac{i - i_{k-1}}{i_k - i_{k-1}}\, I_k & \text{when } a < i_{k-1} < i \le i_k. \end{cases} \tag{6.2}$$

Here the amounts $I(a, i_{k-1})$, in cases where $a < i_{k-1}$, are given by the following formula, in which i_l denotes the first compounding date after a:

$$I(a, i_{k-1}) = \left(\frac{i_l - a}{i_l - i_{l-1}} \right) I_l + I_{l+1} + \cdots + I_{k-1} \text{ when } i_{l-1} \le a < i_l \le i_{k-1}. \tag{6.3}$$

(The terms $I_{l+1} + \cdots + I_{k-1}$ drop out when $l = k - 1$.) Obviously

$$I(i', i) = I(a, i) - I(a, i') \text{ when } a \le i' < i. \tag{6.4}$$

As developed out of this notion of accrual, the *nominal basis* in the instrument on date i is defined to be the quantity

$$\begin{gathered} B_i = V_{k-1} + I(i_{k-1}, i) = \Big(1 + t_i \theta_k y\Big) V_{k-1} \text{ with } t_i = \frac{i - i_{k-1}}{i_k - i_{k-1}} \\ \text{when } i_{k-1} \le i < i_k \quad (k = 1, \ldots, m). \end{gathered} \tag{6.5}$$

Observe that $t_i = 0$ when $i = i_{k-1}$, so the formula gives $B_i = V_{k-1}$ when $i = i_{k-1}$. Thus, *the nominal basis value agrees with the nominal principal value on all compounding dates*, but of course the nominal principal value is only defined on such dates, whereas the nominal basis is defined for every date i. An equivalent definition of B_i, in view of the relationship between V_0 and later values V_k, is

$$B_i \;=\; V_0 + I(i_0, i) - \Big[\text{ sum of } C_k \text{ payments for dates } i_k \le i \; \Big]. \tag{6.6}$$

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[63] For a cash-basis taxpayer, $I(i', i)$ would instead be the sum of the quantities I_k for the dates i_k such that $i' < i_k \le i$.

On any date i during the life of the instrument, the nominal basis is therefore the nominal value plus all the nominal interest accrued so far, minus the explicit payments received so far.

The nominal basis treats the accruing interest as if it were temporarily being added to the outstanding principal. The rationale is that this is the usual pattern when interest income is considered to have been earned but not yet made available to the holder. We have seen in Section 2 that reinvested interest appears as negative repayment of principal. From this perspective, the nominal basis can be identified with the amount of principal outstanding in the instrument when this is tracked not just on the dates i_k, but in an day-to-day manner relative to accrual. An important distinction, though, is that these additions to principal do not lead to compounding. Simple interest continues to be computed relative to the amount of principal present at the beginning of the accrual period.

The nominal basis must be distinguished from the *holder's nominal basis*, which is defined instead to be the quantity

$$H_i = B_i + (P_a - B_a) = P_a - (B_a - B_i), \tag{6.7}$$

with P_a the cost of acquisition on date a as earlier. The nominal basis could, though, be interpreted as the basis of a holder who acquired the instrument on its issue date by paying the nominal initial value V_0, a hypothetical *nominal holder*. If the price P_a, paid on date a by the general holder we have been considering, happens to have agreed with B_a, that holder's nominal basis on each future date i would agree with the nominal basis B_i.

The holder's nominal basis on date i represents the holder's nominal stake in the investment then. The final expression in (6.7) explains this amount as the initial investment on date a minus the net of any principal repayments provided by the instrument from date a to date i. The equivalent expression in the middle characterizes it as running exactly parallel to the basis of a "nominal holder" at all times, the constant difference being equal to the initial difference on date a.

The holder's nominal basis enters the computation of capital gain or loss at the time of disposal of an instrument, at least relative to its nominal specification. Alongside the acquisition date a and acquisition cost P_a in (3.16)–(3.17) let us now consider

$$d = \text{ date of date of disposal,} \qquad i_0 < a \le i_m, \tag{6.8}$$

which like a may or may not be one of the compounding dates i_k, and the amount

$$P_d = \text{ proceeds of disposal to the holder, with } P_d \ge 0 \text{ (but } P_d = 0 \text{ if } d = i_m). \tag{6.9}$$

The disposal may be considered to be through sale when $d < i_m$, and then P_d is the sale price minus transaction costs. The provision that $P_d = 0$ when d is the date of maturity, *and all* *payments from the instrument have already been received*, is only a mathematical convenience to avoid having to state special cases in formulas. “Disposal” at that time just refers to the end of the mathematical bookkeeping on the instrument.

The *nominal capital gain* for the holder on the disposal date d is the difference between the proceeds P_d and the holder's basis on date d relative to the nominal specification, namely from (6.7) the amount

$$P_d - H_d = \big(P_d - B_d\big) - \big(P_a - B_a\big) = \big(P_d - P_a\big) - \big(B_d - B_a\big). \tag{6.10}$$

A negative capital gain translates to a *nominal capital loss*.

The idea behind the nominal basis in its handling of intermediate dates can now be understood from another angle. Suppose the disposal date d falls between compounding dates $i_{\tilde{k}-1}$ and $i_{\tilde{k}}$. The quantity P_d should anticipate the fact that d, which is also the acquisition date a' for some new holder, is nearer to the remaining payments $C_{\tilde{k}}, \ldots, C_m$, than is the date $i_{\tilde{k}-1}$. To this extent P_d should be elevated above the market value that the instrument had on date $i_{\tilde{k}-1}$, the amount being roughly the time value of the investment for the number of days from $i_{\tilde{k}-1}$ to d. Since a taxpayer who is selling the instrument has already placed the accrued interest for this period in the category of ordinary income, it makes sense that the amount in question should not appear also as capital gain due to a higher price P_d. Therefore B_d, rather than the nominal principal value $V_{\tilde{k}-1}$, is subtracted rather from P_d in determining capital gain. Similar thinking applies to P_a and B_a when a is an intermediate date, except that then the taxpayer under consideration is in the role of buyer instead of seller.

Basis Under the Revised Specification. It will be necessary to have notation for handling the effects of the revised specification on a day-to-day basis, just as for the nominal specification. We let

$$I^*(i', i) = \text{ revised interest earned from date } i' \text{ through date } i \tag{6.11}$$

for any dates i' and i satisfying $i_0 \le i' < i \le i_m$, and observe in parallel with (6.2) and (6.3) that

$$I^*(a, i) = \begin{cases} \dfrac{i - a}{i_k - i_{k-1}}\, I_k^* & \text{when } i_{k-1} \le a < i \le i_k, \text{ but} \\[2ex] I^*(a, i_{k-1}) + \dfrac{i - i_{k-1}}{i_k - i_{k-1}}\, I_k^* & \text{when } a < i_{k-1} < i \le i_k, \end{cases} \tag{6.12}$$

where the amounts $I^*(a, i_{k-1})$, in cases having $a < i_k$, are given by

$$I^*(a, i_{k-1}) = \left(\frac{i_l - a}{i_l - i_{l-1}}\right) I_l^* + I_{l+1}^* + \cdots + I_{k-1}^* \ \text{ when } \ i_{l-1} \le a < i_l \le i_{k-1} \tag{6.13}$$

and satisfy

$$I^*(i', i) = I^*(a, i) - I^*(a, i') \ \text{ when } \ a \le i' < i \le s. \tag{6.14}$$

The difference

$$I^*(i',i) - I(i',i), \text{ where } i' < i,$$

gives the amount of interest implicitly received from date i' through date i beyond the nominal amount, due to discount at original issue, if present.

The *revised basis* in the instrument on date i is defined to be the quantity

$$B_i^* = V_{k-1}^* + I^*(i_{k-1}, i) = \Big(1 + t_i\theta_k y^*\Big)V_{k-1}^* \text{ with } t_i = \frac{i - i_{k-1}}{i_k - i_{k-1}} \qquad \text{when } i_{k-1} \le i < i_k \quad (k = 1, \ldots, m). \tag{6.15}$$

The revised basis agrees with the revised principal value on the dates $i_0, i_1, \ldots, i_m$, but in contrast to the latter it is defined for all intermediate dates as well. Just as in equation (6.5) for the nominal basis, we have

$$B_i^* = V_0^* + I^*(i_0, i) - \big[\text{ sum of } C_k \text{ payments for dates } i_k \le i \ \big]. \tag{6.16}$$

From this equation and the earlier one in (6.5) the relation

$$(B_i^* - B_{i'}^*) - (B_i - B_{i'}) = I^*(i',i) - I(i',i) \tag{6.17}$$

can be deduced as a generalization of (3.13) by noting that (6.4) and (6.14) hold equally well with the date a replaced by i_0. Here $B_i^* - B_{i'}^*$ is the net increase in principal from date i' to date i relative to the revised specification, and $B_i - B_{i'}$ is the same thing relative to the nominal specification.

The *holder's revised basis* on date i is the quantity

$$H_i^* = B_i^* + (P_a - B_a^*) = P_a - (B_a^* - B_i^*), \tag{6.18}$$

which parallels the nominal version in (6.7). The *revised capital gain* for the holder on the disposal date d is obtained by using this version of the holder's basis in place of the nominal one in (6.10):

$$P_d - H_d^* = \big(P_d - B_d^*\big) - \big(P_a - B_a^*\big) = \big(P_d - P_a\big) - \big(B_d^* - B_a^*\big). \tag{6.19}$$

Basis Under a Particularized Specification. We look next at the particularized specification, as introduced in Section 3. In this case the assumption that the holder acquired the instrument on date a for the price P_a enters from the beginning.

The day-by-day consequences of the particularized specification involve

$$I^{**}(i',i) = \text{ particularized interest earned from date } i' \text{ through date } i \tag{6.20}$$

for any dates i' and i satisfying $i_{\overline{k}-1} \le i' < i \le i_m$. We observe that

$$I^{**}(a,i) = \begin{cases} \dfrac{i-a}{i_{\overline{k}} - i_{\overline{k}-1}}\, I^{**}_{\overline{k}} & \text{when } a < i \le i_{\overline{k}}, \text{ but} \\ I^{**}(a, i_{k-1}) + \dfrac{i - i_{k-1}}{i_k - i_{k-1}}\, I^{**}_k & \text{when } i_{k-1} < i \le i_k,\ k \ge \overline{k}. \end{cases} \tag{6.21}$$

The amounts $I^{**}(a, i_{k-1})$, in cases where $a < i_{k-1}$, are given by

$$I^{**}(a, i_{k-1}) = \left(\frac{i_l - a}{i_l - i_{l-1}}\right) I^{**}_l + I^{**}_{l+1} + \cdots + I^{**}_{k-1} \text{ when } i_{l-1} \le a < i_l \le i_{k-1}. \tag{6.22}$$

If $P_a < B_a$, the difference

$$I^{**}(i', i) - I(i', i), \text{ where } i' < i, \tag{6.23}$$

is positive and gives the amount of interest viewed as implicitly received from date i' through date i above the nominal amount, when the particularized specification is used in place of the nominal specification. If $P_a > B_a$, the difference is negative and reflects an implicit interest cost to the holder which offsets some of the nominal income.

The *particularized basis* in the instrument on a date $i > a$ is defined to be

$$\begin{gathered} B^{**}_i = V^{**}_{k-1} + I^{**}(i_{k-1}, i) = \Big(1 + t_i \theta_k y^{**}\Big) V^{**}_{k-1} \text{ with } t_i = \frac{i - i_{k-1}}{i_k - i_{k-1}} \\ \text{when } i_{k-1} \le i < i_k \quad (k = \overline{k}, \ldots, m). \end{gathered} \tag{6.24}$$

When i is a compounding date i_k, B^{**}_i agrees with the particularized value V^{**}_k on that date. One has

$$B^{**}_a = P_a, \tag{6.25}$$

and moreover

$$B^{**}_i = V^{**}_0 + I^{**}(i_0, i) - \Big[\text{ sum of } C_k \text{ payments for dates } i_k \le i \ \Big]. \tag{6.26}$$

As in (6.14) we will have for any dates i and i' with $a \le i' < i$ that

$$\begin{aligned} (B_i - B_{i'}) - (B^{**}_i - B^{**}_{i'}) &= I^{**}(i', i) - I(i', i), \\ (B^*_i - B^*_{i'}) - (B^{**}_i - B^{**}_{i'}) &= I^{**}(i', i) - I^*(i', i). \end{aligned} \tag{6.27}$$

The *holder's particularized basis* on any date i simply coincides with the particularized basis:

$$H^{**}_i = B^{**}_i. \tag{6.28}$$

This fits with the other versions of the holder's basis in (6.7) and (6.15) because the quantity

$$B^{**}_i + (P_a - B^{**}_a) = P_a - (B^{**}_a - B^{**}_i)$$

reduces to B_i^{**} through the definition of B_a^{**} as P_a. The *particularized capital gain* for the holder on the disposal date d comes out therefore as just

$$P_d - H_d^{**} = P_d - B_d^{**}. \tag{6.29}$$

As final note to this theme, we record that the revised basis for an instrument can be interpreted as the holder's particularized basis for an original holder of the instrument.

7. TAX RULES FOR LONG-TERM OBLIGATIONS

In broad outline, tax law has evolved since the 1950's from relying on the nominal specification of a debt instrument[64] toward insisting on the particularized specification for each holder as the key to how much interest income is really provided to that holder.[65]

The evolution has not been complete even in the treatment of recently issued instruments, however, and many "fossils" are embedded in older strata. The main departures from taxation based directly on the particularized specification are seen in the fact that tax law at present

(a) exhibits asymmetry between "discount" and "premium" cases,
(b) provides "simplifications" for investors unable to cope with algebraic formulas,
(c) maintains earlier compromises for instruments issued in the past,
(d) gives different stipulations of the lengths of acceptable accrual periods for various instruments, and
(e) demands separate handling and accounting distinctions in situations which mathematically could be regarded as identical.

In order to determine the taxable income associated with a given instrument, it is generally necessary to know not only the instrument's nominal specification but the issue value (original issue price) and also the holder's data on acquisition and disposition as in (3.16)–(3.17) and (6.8)–(6.9). From this information the instrument's revised specification and particularized specification can be filled out. However, this is still not enough. Also critical is the instrument's classification according to type and date of issue. For *long-term* instruments, which are the

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[64] Internal Revenue Code SS61(a), 163(a), 451(a), and 461(a).

[65] Internal Revenue Code SS1271–1278.

Table 7.1 Date Ranges for Tax Rules

Set 0. Instruments issued before 1955; nongovernmental noncorporate instruments issued from 1 January 1955 through 1 July 1982.

Set 1. Governmental instruments issued from 1 January 1955 through 1 July 1982; corporate instruments issued from 1 January 1955 through 27 May 1969.

Set 2. Corporate instruments issued from 28 May 1969 through 1 July 1982.

Set 3. Instruments issued from 2 July 1982 through 18 July 1984 and acquired through 18 July 1984.

Set 4. Instruments issued from 2 July 1982 through 18 July 1984 but acquired from 19 July 1984 onward.

Set 5. Instruments issued from 19 July 1984 through 31 December 1984.

Set 6. Instruments issued from 1 January 1985 through 27 September 1985.

Set 7. Instruments issued from 28 September 1985 through 21 December 1992.

Set 8. Instruments issued from 22 December 1992 on.

topic of this section, there are eight separate categories (Sets 0 to 8) to consider.[66] [67] [68] [69]

The following distinction with respect to original issue discount is also important. Here we see asymmetry in operation, since there is nothing comparable in the tax code with respect to original issue premium. (No threshold is needed for the latter.) *It is essential here for the reader to be aware of the conclusions reached at the end of Section 4, because they are the source of a number of differences between the way we state things and they way they seem to appear in various regulations.* The following definition is a case in point, because in this we are identifying "stated redemption price at maturity" with V_0.

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[66] Here for Set 0 see S1232 of the 1954 Code and S117(f) of the 1939 Code. For Sets 3–7 see SS1271–1273 and 1275–1278 of the 1984 Code, and SS1.1271–1 to 1.1275-5 of the IRS Proposed Regulations of 1986.

[67] Excluded from this classification are obligations issued by a natural person before 2 March 1984 as well as, beyond that date, all nonbusiness loans from one natural person to another (as long as the total between the parties does not exceed $10,000, and tax avoidance is not the primary purpose).

[68] U.S. Savings Bonds, which are in small denominations and not part of mainstream finance, are handled as belonging to Set 0 regardless of the date of issue.

[69] Annuities based on the life of an individual are left out, because they require different treatment.

Definition 7.1 (*de minimis* rule for original issue discount). *A debt instrument is called an OID instrument when it is not in Set 0, its issue value V_0^* is less than its nominal value V_0, and the difference exceeds a certain minimal amount derived as follows. Generally the threshold is*

$$V_0 - V_0^* > (N/400)V_0, \tag{7.1}$$

where N denotes the number of full years included in the time span from issue to maturity.[70] *The test is different, however, if the instrument is an installment obligation (meaning that there is a positive principal repayment amount at some intermediate time, i.e., $R_k > 0$ for at least one k besides $k = m$). Then the threshold is that*[71]

$$V_0 - V_0^* \geq \begin{cases} (N/600)V_0 \text{ and} \\ (1/400)\big[\theta_1 R_1 + (\theta_1 + \theta_2)R_2 + \cdots + (\theta_1 + \cdots + \theta_m)R_m\big]. \end{cases} \tag{7.2}$$

Basically, it will be seen that any debt instrument which is not an OID instrument is handled relative to its nominal and particularized specifications only, and the revised specification is set aside. Otherwise, all three specifications may enter the picture from one side or another. Moreover rules concerning appropriate accrual periods then come up, which may require these specifications to be redone with respect to an altered date sequence–forced respecification as explained in Section 5.

The tax treatments of the different sets of instruments diverge mainly in the way that several kinds of implicit interest are regarded as received (or in the case of negative interest, expensed). The key is the relationship between the holder's acquisition cost P_a on date a and the nominal basis value B_a and revised basis value B_a^* on that date.

In overview, if $P_a < B_a$, the particularized specification of the instrument indicates that the holder theoretically receives positive implicit interest payments beyond any nominal interest

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[70] The years are measured from the date of issue to the same calendar date in each successive year (identifying February 29 with February 28), and any fraction of a year at the end is dropped.

[71] The rationale for these amounts seems to stem from rough estimates of interest at the annual rate y. The quantity yNV_0 is the simple interest earned by the initial principal V_0 over the N full years in the instrument. On the other hand, $y(\theta_1 + \cdots + \theta_k)R_k$ is the simple interest earned by the principal amount R_k in the k accrual periods before it is repaid, so that the sum of such amounts for $k = 1, \ldots, m$ constitutes all such interest earned by the various portions of the initial principal (since $R_1 + \cdots + R_m = V_0$). The rule requires the interest $y(V_0 - V_0^*)$ earned by the OID in one year to be more than (1/600)th of the first quantity and more than (1/400) of the second quantity before the rules for OID taxation are applied. The formula for the second threshold quantity has been extrapolated from the description provided in the IRS regulations, which in its wording only covers the case of equal accrual periods.

payments. On the other hand, if $P_a > B_a$, the holder theoretically has implicit interest expenses which may be eligible for offset of interest income from other sources; mathematically these can be viewed as negative implicit interest payments to be added in with other payments. In the case of an OID instrument with $P_a < B_a$ but $P_a \neq B_a^*$, the implicit interest payments are not viewed directly but seen as having two parts. One part is the amount that would apply to an original holder, or in other words, if $P_a = B_a^*$. The other part is a correction which is positive if $P_a < B_a^*$ (additional interest amounts become taxable) but negative if $P_a > B_a^*$ (the interest implicitly received from OID is offset by certain costs). Positive and negative interest amounts or corrections are generally not treated with symmetry.

For a systematic presentation more or less in parallel with the organization of current tax code, we shall set up the calculation of taxes for a given instrument in terms of one or more adjustments relative to the nominal rule of taxation that would simply follow the nominal specification. Afterward, we shall explain how the net effect of these adjustments can be viewed for recently issued instruments.

There are four types of adjustments to ordinary income which arise in the manner sketched, although not all simultaneously. The designations used in present tax literature are somewhat inconsistent and apt to be confusing,[72] so we generally hold back from them and refer simply to Adjustments 1–4. The precise nature of the adjustment types will be explained presently, but as a mnemonic and "glossary" we list them as follows.

Adjustment 1	"OID"	added
Adjustment 2	"acquisition premium"	subtracted
Adjustment 3	"market discount"	added
Adjustment 4	"amortizable premium"	subtracted

Table 7.2. Types of income adjustment for taxation of debt obligations.

[72] For instance, "market discount" can refer to discount relative to either the nominal basis or the revised basis, depending on the circumstances. The same quantity for a short-term obligation is customarily referred to instead as "acquisition discount." On the other hand, "acquisition premium" refers only to a truncated amount of premium relative to the revised basis. If it were not for the habit of using "acquisition premium" only in the context of long-term OID obligations and "acquisition discount" only for short-term non-OID, a bond purchased for a price between the nominal basis and the revised basis would simultaneously have both "acquisition premium" and "acquisition discount."

The notation we adopt for dealing with these adjustments quantitatively is

$$A_j(i', i) = \text{ total adjustment of type } j \text{ from date } i' \text{ through date } i. \tag{7.3}$$

Postponing temporarily the recitation of formulas for these quantities, which will vary according to the classification of the instrument, we describe how they are to be used. "Stripped" instruments and short positions are reserved for separate discussion in Section 9.

General Rule of Taxation: Long-Term Instruments. *The ordinary income deemed to be earned by the holder from date i' through date i (for any dates satisfying $i_0 \le a \le i' < i \le d \le i_m$, with a the date of acquisition by the holder and d the date of disposal (through sale or redemption) is, with a minor exception, the amount*

$$O(i', i) = \begin{cases} I(i', i) + A_1(i', i) - A_2(i', i) - A_4(i', i) & \textit{when } i' < i < d, \textit{ but} \\ I(i', d) + A_1(i', d) - A_2(i', d) + A_3(i', d) - A_4(a, d) & \textit{when } i' < d \le i, \end{cases} \tag{7.4}$$

whereas the capital gain on date d is

$$G = (P_d - P_a) - (B_d - B_a) - A_1(a, d) + A_2(a, d) - A_3(a, d) + A_4(a, d). \tag{7.5}$$

The exception concerns A_1 and A_2 in the case of instruments in Set 1; they are handled then just like A_3.[73] *For a tax-exempt instrument, the ordinary income amount in this formula is not taxable (regardless of whether the interest involved is explicit or implicit), but capital gain is taxable; it is given by the simpler formula*

$$G = (P_d - P_a) - (B_d - B_a) - A_1(a, d) \tag{7.6}$$

In the case of the issuer rather than a holder, the interest regarded as paid out from date i' through date i is

$$O(i', i) = I(i', i) + A_1(i', i) - A_2(i', i). \tag{7.7}$$

[73] Investors have an (irrevocable) option of treating A_3 just like A_1, A_2, and A_4 (for all instruments currently held, and all those acquired from then on). The positive implicit interest in question would then be included in each tax year as accrued. This would amount to volunteering to pay taxes earlier than necessary.

The several different rules that can come into play in calculating the adjustments, depending on the situation, are laid out below in Table 7.3. This table can be used to see which rules apply in a given case. (When an alternative rule is given in parentheses, this means that the holder can optionally use it as well, although it would generally be less advantageous.) In all cases involving symbols derived from the nominal, revised and particularized specifications of an instrument in which the accrual periods are not all equal, the quantities in question could be replaced by the approximate ones developed through application of the method of fractional exponents (although this would be no saving of mathematical effort).

Rules for Adjustment 1: Interest Added for OID.

Rule 0. Disregard; take $A_1(i', i) = 0$ always.

Rule 1. Let M_{ad} denote the number of *full* months from date a to date d, and let M be the number of *full* months from date of issue to date of maturity. (Full months are measured relative to the day in each month that corresponds to the day of issue; any fractional month is ignored.) Let D denote the effective amount of OID in the instrument on date a, namely

$$D = \begin{cases} B_a - B_a^* = I^*(a, i_m) - I(a, i_m) & \text{for an OID instrument,} \\ 0 & \text{for a non-OID instrument} \end{cases} \tag{7.8}$$

(where the equation is obtained from (6.13) with $i' = a$, $i = m$). Then take

$$A_1(a, d) = \begin{cases} (P_d - P_a) - (B_d - B_a) & \text{if } 0 < (P_d - P_a) - (B_d - B_a) \le \frac{M_{ad}}{M} D, \\ \frac{M_{ad}}{M} D & \text{if } 0 \le \frac{M_{ad}}{M} D < (P_d - P_a) - (B_d - B_a), \\ 0 & \text{if } (P_d - P_a) - (B_d - B_a) \le 0, \end{cases}$$
$$A_1(i', i) = \begin{cases} A_1(a, d) & \text{if } a \le i' < i = d, \\ 0 & \text{if } a \le i' < i < d. \end{cases} \tag{7.9}$$

Rule 2. Let $M^*_{i',i}$ denote the number of *exact* months from date i' to date i, and let M^* be the number of *exact* months from date of issue to date of maturity. (Exact months are measured relative to the day in each month that corresponds to the day of issue, and fractions are included. For instance, if date i falls on the 13th day of a 31-day month,while i' falls before the beginning of the month, then the fraction 13/31 would be included in $M^*(i', i)$ for the partial month at the end of the period from i' to i. Another such fraction would be included for a partial month at the beginning of the period, if any.) Then, with D the OID amount in (7.8), take

$$A_1(i', i) = \frac{M^*_{i',i}}{M^*} D. \tag{7.10}$$

<table>
<tr><th rowspan="2">ISSUE DATES</th><th colspan="4">A D J U S T M E N T S</th><th rowspan="2">Accrual Periods</th></tr>
<tr><th>1
OID</th><th>2
Acqui-
sition
Premium</th><th>3
Market
Discount</th><th>4
Amorti-
zable
Premium</th></tr>
<tr><td>Set 0: GC* $\leq$ 1954, and
NGNC* $\leq$ 1/Jul/82</td><td>Rule 0</td><td rowspan="2">Rule 0</td><td rowspan="5">Rule 0</td><td rowspan="7">Rule 0</td><td rowspan="3">Rule 0</td></tr>
<tr><td>Set 1: G* 1/Jan/55–1/Jul/82,
C* 1/Jan/55–27/May/69</td><td>Rule 1</td></tr>
<tr><td>Set 2: C* 28/May/69–1/Jul/82</td><td>Rule 2</td><td>Rule 1</td></tr>
<tr><td>Set 3: 2/Jul/82–18/Jul/84,
acquired $\leq$ 18/Jul/84</td><td rowspan="6">Rule 3</td><td>Rule2</td><td rowspan="3">Rule 1</td></tr>
<tr><td>Set 4: 2/Jul/82–18/Jul/84,
acquired $\geq$ 19/Jul/84</td><td rowspan="5">Rule 3</td></tr>
<tr><td>Set 5: 19/Jul/84–31/Dec/84</td><td rowspan="4">Rule 1
or 2</td></tr>
<tr><td>Set 6: 1/Jan/85–27/Sep/85</td><td rowspan="2">Rule 2</td></tr>
<tr><td>Set 7: 28/Sep/85–21/Dec/92 **</td><td rowspan="2">Rule 1</td></tr>
<tr><td>Set 8: 22/Dec/92–current **</td><td>Rule 3</td></tr>
</table>

* G=governmental, C=corporate, GC=G or C, NGNC=neither G nor C.

** For taxation of issuers, see a potential exception under the AHYDO Rules below.

Table 7.3 Applicable rules for income and expense from long-term obligations.

Rule 3. Take

$$A_1(i', i) = I^*(i', i) - I(i', i) \tag{7.11}$$

Rules for Adjustment 2: Interest Subtracted for Acquisition Premium.

Rule 0. Disregard; take $A_2(i', i) = 0$.

Rule 1. In terms of the exact month notation in Rule 2 for Adjustment 1, take

$$A_2(i', i) = \frac{M^*_{i',i}}{M^*}\,(P_a - B^*_a). \tag{7.12}$$

Rule 2. With D denoting the OID amount in (7.8), take

$$A_2(i',i) = \Big(\frac{i-i'}{i_m - i_0}\Big)D. \tag{7.13}$$

Rule 3. Define the fraction α by

$$\alpha = \begin{cases} \dfrac{P_a - B_a^*}{B_a - B_a^*} & \text{when } B_a^* < P_a \le B_a, \text{ but} \\ 0 & \text{otherwise.} \end{cases} \tag{7.14}$$

Then take

$$A_2(i',i) = \alpha\big[I(i',i) - I^*(i',i)\big]. \tag{7.15}$$

Rules for Adjustment 3: Interest Added for Market Discount.

Rule 0. Disregard; take $A_3(i',i) = 0$.

Rule 1. Consider first the quantity

$$D' = \begin{cases} B_a^* - P_a & \text{for an OID instrument with } B_a^* - P_a > (N_a/400)B_a^*, \\ B_a - P_a & \text{for a non-OID instrument with } B_a - P_a > (N_a/400)B_a, \\ 0 & \text{otherwise,} \end{cases} \tag{7.16}$$

where N_a is the number of full years from acquisition to maturity. (Full years are counted from the issue date to the corresponding calendar day in each successive year, and a fractional remainder (if any) is dropped.) If $D' \neq 0$, take

$$A_3(i',i) = \begin{cases} I^{**}(i',i) - I^*(i',i) & \text{for an OID instrument,} \\ I^{**}(i',i) - I(i',i) & \text{for a non-OID instrument.} \end{cases} \tag{7.17}$$

Rule 2. With $D' \neq 0$ as in (7.16), take[74]

$$A_3(i',i) = \Big(\frac{i-i'}{i_m - i_0}\Big)D'. \tag{7.18}$$

[74] Under this rule, in comparison to the preceding one, the holder of an instrument could end up paying more tax (never less), and paying it earlier. Incentive for such a choice it is therefore lacking, except that the rule is "simpler."

Rules for Adjustment 4: Interest Subtracted for Amortizable Premium.

Rule 0. Disregard (take $A_4(i', i) = 0$ for all intervals) or: open. The method can be one "regularly employed by the holder of the bond, if such method is reasonable," or it can be one designated as reasonable by tax regulations. In particular, it can be the following rule of straight-line amortization:

$$A_4(i', i) = \frac{i - i'}{m - a}(P_a - B_a). \tag{7.19}$$

Rule 1. Disregard (take $A_4(i', i) = 0$ for all intervals) or: take

$$A_4(i', i) = \begin{cases} I^{**}(i', i) - I(i', i) & \text{when } P_a > B_a, \text{ but} \\ 0 & \text{when } P_a \leq B_a. \end{cases} \tag{7.20}$$

Rules for Accrual Periods.

Rule 0. Use the dates i_k in the nominal specification.

Rule 1. Use the dates i_k in the nominal specification if the instrument is not an OID instrument. Otherwise respecify for taxation purposes with accrual periods exactly one year in length starting from the issue date except for the possibility of a fractional period at the end; see the discussion of forced respecification in Section 5.[75]

Rule 2. Use the dates i_k in the nominal specification if the number m of accrual periods is more than 2, the periods do not exceed one year, and they are equal in length except perhaps for a shorter period at the beginning or the end, or both. Otherwise respecify according to the longest internally compatible accrual period length $\overline{\theta}$ of length not exceeding one year (in the manner explained in Section 5; see Definition 5.1 in particular), unless $m \leq 2$. If $m = 2$, use a fully compatible accrual period length $\overline{\theta}$. If $m = 1$ (the case where there are no compounding dates intervening between the date of issue and the date of maturity), respecify with equal accrual periods six months in length, except for a possible shorter period at the beginning.

Rule 3. Use the dates i_k in the nominal specification if the accrual periods do not exceed one year. Otherwise respecify through extra dates so as to shorten the accrual periods to meet this requirement. The accrual periods do not have to be of equal length.

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[75] This is the most troublesome case of forced respecification. For some instruments is hard to make sense out of, although a prescription that would pass muster was provided in Section 2. It is likely that the more reasonable (although still inadequately unmotivated) Rule 2 that follows would be accepted for taxation in practice in cases where implementation seemed difficult. See IRS Proposed Regulations of 1986, 1.1272–1(d)(1)(i)(ii).

Net Effect for Recent Instruments. The relationship between present tax rules and taxation relative to the particularized specification of an instrument is clearest in the case of instruments in Set 7. Consider first the case of a non-OID instrument. If the implicit interest in Adjustment 3 were included in current income, instead of just being declared at the end (cf. Footnote 72), and if the rule in (7.14) were dropped (so that no threshold would be invoked before taxes started to apply), the result would be the particularized rule precisely. Thus, no matter where P_a lies in relation to B_a, the ordinary income declared as accruing from date i' to date i would be the particularized interest $I^{**}(i', i)$. The capital gain on date d would be $P_d - B_d^{**}$.

For an OID instrument almost the same is true, but a perturbation occurs because of use of the fraction α in Rule 3 for Adjustment 2. The ordinary income comes out as

$$O(i', i) = \begin{cases} I^{**}(i', i) & \text{when } P_a \geq B_a \text{ or } P_a \leq B_a^*, \text{ but} \\ (1-\alpha)I^*(i', i) + \alpha I(i', i) & \text{when } P_a = (1-\alpha)B_a^* + \alpha B_a,\ 0 < \alpha < 1. \end{cases} \tag{7.21}$$

where it may be recalled that the particularized interest is

$$I^{**}(i', i) = \begin{cases} I(i', i) & \text{when } P_a = B_a, \\ I^*(i', i) & \text{when } P_a = B_a^*. \end{cases} \tag{7.22}$$

Similarly, the capital gain comes out as

$$\begin{aligned} &G = P_d - \widetilde{B}_d^{**}, \text{ where} \\ &\widetilde{B}_d^{**} = \begin{cases} B_d^{**} & \text{when } P_a \geq B_a \text{ or } P_a \leq B_a^*, \text{ but} \\ (1-\alpha)B_d^* + \alpha B_d & \text{when } P_a = (1-\alpha)B_a^* + \alpha B_a,\ 0 < \alpha < 1, \end{cases} \end{aligned} \tag{7.23}$$

where

$$B_i^{**} = \begin{cases} B_i & \text{when } P_a = B_a, \\ B_i^* & \text{when } P_a = B_a^*. \end{cases} \tag{7.24}$$

Exception in Taxing Recent Issuers for High Interest Rates: AHYDO Rules.

For issuers, the adjustments listed in Table 7.2 as "added" would actually be subtracted from income, and vice versa. In particular, the amount $A_1(i, i')$ would generally furnish a deduction. In certain circumstances intended to be covered by the "Applicable High Yield Obligation" (AHYDO) rules in S163 (e)(5) and (i), in which the issuer is a corporation, it would seem that the deduction may be delayed or reduced. These rules are mathematically ambiguous, however, and no regulations have ever been written to provide guidance in implementing them. The consequence has nonetheless been to stop the issuance of any instruments that might be imagined as covered.

The affected instruments, called AHYDO instruments, are supposed to be the ones meeting all three of the following tests:

Test 1. Issued by a corporation after 10 July 1989 with a maturity of more than five years, i.e., with the time from date i_0 to date i_m exceeding 5 years.

Test 2. Having $y^* \le F + .05$, where F is the Applicable Federal Rate of interest on the issue date i_0 relative to the instrument's compounding period. (Such rates, explained in general in S1.1274(e)(1)(i), are specified by the IRS for every month. In the case of an instrument not fitting the compounding periods for which a rate is published, the rate for the published compounding period closest to the average period of the instrument is the one that presumably could be used.)

Test 3. Has "significant original issue discount," which is taken to mean that on at least one date i_k more than five years after i_0 the portion of $I^*(i_0, i_k)$ "not yet paid" exceeds I_1^*.

Test 3 causes a serious problem. Perhaps the portion mentioned in Test 3 could be identified with the quantity D_k^* of deferred interest as computed by (2.20) from the revised specification instead of the nominal specification. But the whole idea that imputed interest might not yet have been paid is fraught with serious inconsistencies, as we have demonstrated in Section 4; we have taken the position that all of $I^*(i_0, i_k)$ must be regarded always as having been paid. This test is inconsistent with the concept behind OID taxation itself, which insists that imputed payments have really been received by the holder.

Furthermore, the words "significant original issue discount" in Test 3 have little relation to the suggested criterion. For instance, an instrument having no OID at all—because $V_0^* = V_0$ (and therefore $I^*(i_0, i_k) = I(i_0, i_k)$ and $I_1^* = I_1$)—might be deemed as having significant original issue discount.

Anyway, an effort to give mathematical interpretation to the modification of $A_1(i', i)$ desired in the code runs into other difficulties. Not only amounts associated with OID under the revised specification, but those under the nominal specification and any particularized specification would have to be reconsidered in connection with what amount of interest has "really" been paid already between two dates i' or i or has not. This route leads straight to the dangers illuminated in Section 4.

8. TAX RULES FOR SHORT-TERM OBLIGATIONS

For short-term debt instruments the picture is much simpler, since for example there is no necessity of dealing with obligations from past eras in tax law. Furthermore, there are only two sets to keep straight: governmental and nongovernmental.

The taxation for each of these could be described independently as a unit, but with the patterns in Section 7 available it is convenient to indicate the treatment in the same manner. This will also make comparisons easier.

As before, we consider four types of adjustment to taxation by the nominal rule. These are identical to the ones in (7.1), except that for some reason the term "acquisition discount" is customary for Adjustments 1 and 3 in the short-term case instead of "market discount." Adjustments 1 and 2 in fact fall by the wayside (so the terms "original issue discount" and "acquisition premium" have no role).

General Rule of Taxation: Short-Term Instruments. *The ordinary income deemed to be earned by the holder*[76] *from date i' through date i (for any dates satisfying $a \le i' < i \le d$) is the amount*

$$O(i', i) = I(i', i) + A_1(i', i) + A_3(i', i) - A_4(i', i), \tag{8.1}$$

whereas the capital gain on date d is

$$G = (P_d - P_a) - (B_d - B_a) - A_1(a, d) - A_3(a, d) + A_4(a, d). \tag{8.2}$$

No adjustment of type A_2 is ever made. In the case of a tax-exempt instrument, $O(i', i)$ is instead 0 and the capital gain is

$$G = (P_d - P_a) - (B_d - B_a) - A_1(a, d) \tag{8.3}$$

For the issuer of an instrument, the amount of interest considered to be paid out from date i' to date i is

$$O(i', i) = I(i', i) + A_1(i', i). \tag{8.4}$$

The key to computing the adjustment amounts is provided in the following table.

Rules for Adjustment 1: Interest Added for Acquisition Discount, Type 1.

Rule $0'$. Disregard this type of interest income; take $A_1(i', i) = 0$.

Rule $1'$. Use daily compounding, i.e., pass to the altered specification (as in the method of extra dates) in which all dates from a to d are included. In that sense take[77]

$$A_1(i', i) = \begin{cases} I^*(i', i) - I(i', i) & \text{if } B_a^* < B_a, \text{ but} \\ 0 & \text{if } B_a^* \ge B_a. \end{cases} \tag{8.5}$$

[76] This refers to accrual-basis taxpayers only. For a cash-basis taxpayer, short-term obligations are taxed according to the nominal specification only.

[77] Once this rule has been adopted for an instrument, the holder must continue with it and is not permitted to switch later to Rule $2'$.

<table>
<tr><th></th><th colspan="3">A D J U S T M E N T S</th></tr>
<tr><th>INSTRUMENTS</th><th>1
OID</th><th>3
Market
Discount</th><th>4
Amortizable
Premium</th></tr>
<tr><td>Governmental</td><td>Rule 0’</td><td>Rule 1’ (or 2’)</td><td rowspan="2">Rule 1’</td></tr>
<tr><td>Nongovernmental</td><td>Rule 1’ (or 2’)</td><td>Rule 0’</td></tr>
</table>

Table 8.1 Applicable Rules for Income from Short-Term Obligations.

Rule $2'$. Take

$$A_1(i',i) = \begin{cases} \dfrac{i-i'}{i_m - i_0}\,(B_a - B_a^*) & \text{if } B_a^* < B_a, \text{ but} \\ 0 & \text{if } B_a^* \geq B_a. \end{cases} \tag{8.6}$$

Rules for Adjustment 3: Interest Added for Acquisition Discount, Type 2.

Rule $0'$. Disregard this type of interest income; take $A_3(i',i) = 0$.

Rule $1'$. Use daily compounding i.e., pass to the altered specification (as in the method of extra dates) in which all dates from a to d are included. In that sense take

$$A_3(i',i) = \begin{cases} I(i',i) - I^{**}(i',i) & \text{if } P_a < B_a, \text{ but} \\ 0 & \text{if } P_a \geq B_a. \end{cases} \tag{8.7}$$

Rule $2'$. Take

$$A_3(i',i) = \begin{cases} \dfrac{i-i'}{i_m - i_0}\,(B_a - P_a) & \text{if } P_a < B_a, \text{ but} \\ 0 & \text{if } P_a \geq B_a. \end{cases} \tag{8.8}$$

Rules for Adjustment 4: Interest Subtracted for Amortizable Premium.

Rule $1'$. Use daily compounding, and in that sense take

$$A_4(i',i) = \begin{cases} I(i',i) - I^{**}(i',i) & \text{if } P_a > B_a, \text{ but} \\ 0 & \text{if } P_a \leq B_a. \end{cases} \tag{8.9}$$

Net Effect for Governmental Instruments. In comparing the rules for Adjustments 3 and 4 in the case of a short-term governmental obligation,it becomes clear that the particularized specification is precisely what is involved. The ordinary income is always $O(i', i) = I^{**}(i', i)$, whereas the capital gain is always $P_d - B_d^{**}$. But the particularized specification is the one obtained by first reinterpreting the accrual periods in the instrument as consisting of single days—daily compounding.

9. STRIPPED INSTRUMENTS AND SHORT POSITIONS

Two ways that new instruments are often created out of existing ones will now be examined.

Taxation of Stripped Instruments. The holder of any debt instrument may sell the rights to some of the payments, or portions of them, to other parties. This process is called *stripping*. Basically, stripping transforms an existing instrument into two or more instruments, which are regarded as newly issued. The old instrument disappears, leaving only the memory of whether it was "governmental" or "tax-exempt," which is passed on to the new instruments. These are in turn taxed by the rules already covered, once their specifications are fully understood. The only complications in arriving at the specifications are in the treatment of the accrual period and the basis of the holder who does the splitting.

Suppose that an existing instrument with payments C_k on dates i_k is divided on a certain date b into n new instruments indexed by $j = 1, \ldots, n$, where the jth instrument pays the amount C_{jk} on date i_k,

$$C_k = C_{1k} + \cdots + C_{nk} \text{ for } i_k > b, \text{ with } C_{jk} \geq 0 \text{ for } j = 1, \ldots, n. \tag{9.1}$$

For each j there must of course be at least one payment $C_{jk} > 0$. The latest of the dates i_k for which this is true becomes the maturity date for the jth new instrument, while the date b becomes the issue date.

A particularized specification for the jth new instrument, with issue date b replacing acquisition date a in the formulas laid out in Section 3, will be uniquely determined as soon as the value to replace P_a in these formulas has been fixed, which will be taken care of shortly. The nominal specification and the revised specification will be identified with this particularized specification. It will be governmental, or tax-exempt, according to the classification of the parent instrument, except that the yield on any stripped instrument is tax exempt only to the extent that it does not exceed the original yield on the underlying instrument.

For a purchaser of the jth new instrument at the time of splitting, the value on the acquisition date b is the price actually paid then for the instrument. Taxation proceeds accordingly, just as with the acquisition of any other debt obligation.

For the holder who did the splitting, the main issue is how to allocate the current basis of the parent instrument to the new instruments in order to determine capital gain or loss if a new instrument is sold, or the initial value on date b if the new instrument is kept. Let[78]

$$P_{jd} = \begin{cases} \text{sale price} & \text{if the } j\text{th new instrument is sold,} \\ \text{fair market value} & \text{if the } j\text{th new instrument is kept.} \end{cases} \tag{9.2}$$

The holder's current basis in the old instrument as of the splitting date b, which for purposes of capital gains has been seen in Sections 7 and 8 to be

$$H = \begin{cases} P_a + (B_b - B_a) + A_1(a,b) - A_2(a,b) + A_3(a,b) - A_4(a,b) \\ \qquad\qquad \text{if long-term, not tax-exempt,} \\ P_a + (B_b - B_a) + A_1(a,b) + A_3(a,b) - A_4(a,b) \\ \qquad\qquad \text{if short-term, not tax-exempt,} \\ P_a + (B_b - B_a) + A_1(a,b) \\ \qquad\qquad \text{if tax-exempt, either long- or short-term,} \end{cases} \tag{9.3}$$

must by law be allocated to the new instruments in the same proportions as the values P_{jd}. In other words, the portion of the basis assigned to the jth new instrument must be

$$H_j = \frac{P_{jd}}{P_{1d} + P_{2d} + \cdots + P_{nd}} H. \tag{9.4}$$

Therefore, if the jth instrument is sold the corresponding capital gain (or loss) to the holder will be

$$G_j = P_{jd} - H_j,$$

whereas if it is kept, its value on date b, for working out the particularized specification, is regarded as H_j.

Taxation of Short Positions. In assuming a *short position* in a bond (or conceivably some other kind of debt instrument), an investor receives a sum of money in the present for taking on the obligation of meeting the stream of payments associated with that instrument. The mechanics of the transaction need not concern us here, just the tax consequences. These come from the view that the investor has in effect *issued* a new instrument in which the dates i_k and payments C_k are those in the remaining life of the shorted bond, and the amount received is the issue price. From these elements the associated yield y can be derived through Theorem 2.2. One then has a consistent, full specification in the sense of Definition 2.1. This specification fills the roles of the nominal, revised and particularized specifications simultaneously—the standpoint being that of an issuer.

[78] There is no cut-and-dried method of determining the fair market value of a debt obligation unless it closely resembles some instrument that is commonly traded, but various mathematical approaches to comparison can be devised. Tax law is vague on this point. The approach would of course have to be "reasonable."

10. SUMMARY

This study has had two main goals: the elucidation of the mathematical principles underlying the taxation of debt instruments, and the development of formulas for such taxation that will facilitate computerization in financial decision making. The basic problem is that of determining for a given instrument, acquired for a certain price at a certain time, how much interest income should be deemed to have been received by a holder in any period, and how much repayment of principle. In the United States, this matter has increasingly been approached in terms of constructing from the instrument's data an appropriate constant annual yield rate. The prescriptions for doing this have been incomplete and to some degree inconsistent, however.

The mathematical theory of constant yield has here been developed and laid out in a form able to cope with the many complications that come up in practice. A number of misconceptions have been cleared up, and a rigorous methodology has been furnished for resolving ambiguities such as have caused serious difficulties in the past. This methodology reveals how debt instruments, even with payment patterns different from those commonly marketed today, must be taxed if dangerous inconsistencies with current rules are to be avoided. Issuers contemplating new instruments have thereby been furnished with a powerful tool for tracing tax consequences.

In the course of developing the new methodology, a number of examples have been put together which demonstrate how far astray the effects of tax regulations can be from their original intent. The source of the trouble has been shown to be a lack of appreciation of how much can go wrong when rules are devised as reasonable-seeming extrapolations from simple, familiar cases, instead of being formulated and tested within a broad and consistent mathematical framework. The public policy implications are evident: in the complicated financial world of today, where markets can move quickly but the regulatory process is slow, it is essential to subject proposed tax rules to careful mathematical analysis from the start, and this analysis needs to be revolve around a sound theory of how debt instruments—conceived quite generally—provide income to a holder.

An important part of the approach taken in this paper has been the notion of a full, consistent specification of a debt instrument. This notion sets a new standard for resolving conflicts and uncertainties about taxation. As many as three such specifications may be necessary in computing taxes according to current rules, and these have been explained and contrasted in detail. Technicalities over the spacing of payment dates and the accrual of interest between such dates, which cannot be avoided in determining capital gain or loss, have been addressed rigorously as well.

As a result of this care, it has been possible to provide formulas which will enable computers to keep track of the tax consequences of debt holdings at a level previously not attainable. These formulas apply to both long-term and short-term instruments, which are divided into a number of categories according to their type and their date of issue, as is required in determining which era of tax rules should govern the calculations. They have been organized to start from a basic representation of income to the holder and then adjust it in response to various circumstances, much in the manner that tax law itself is organized. Tables have been furnished to indicate the particular rules of adjustment to be invoked, and these rules have been cast in mathematical form. The taxation of stripped instruments and short positions has been covered as well.

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