TAX BASIS AND NONLINEARITY IN CASH STREAM VALUATION

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Abstract. The value of a future cash stream is often taken to be its net present value with respect to some term structure. This means that a linear formula is used in which each future payment is discounted by a factor deemed appropriate for the date on which the payment will be made. In a money market with taxes and shorting costs, however, there is no theoretical support for the existence of a universal term structure for this purpose. What is worse, reliance on linear formulas can be seriously inaccurate relative to true worth and can lead to paradoxes of disequilibrium. A consistent no-arbitrage theory of valuation in such a market requires instead that taxed and untaxed investors be grouped in separate classes with different valuation operators. Such operators are linear to scale but nonlinear with respect to addition.

Here it is established that although these valuation operators provide general bounds applicable across an entire class, individual investors within a tax class can have more special operators because of the influence of existing holdings. These customized valuation operators have the feature of not even being linear to scale. In consequence of this nonlinearity, investors from the same or different tax classes can undertake advantageous trades even when the market is in a no-arbitrage state, but such trade opportunities are limited. Some degree of activity in financial markets can thereby be understood without appeal to differences in utility functions or temporary disequilibrium due to random disturbances.

Keywords: term structure, cash stream valuation, tax effects, no-arbitrage equilibrium, portfolio adjustment.

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1. INTRODUCTION

Central in much of finance has been the notion that cash streams can be valued in a simple and universal manner for all investors through the linear formulas of NPV (net present value), in which each future payment is discounted by a factor dependent only on the date of the payment. With the discovery of NPV paradoxes like the one of Schaefer [1982a], however, reliance on such linear formulas became untenable for markets involving more than one tax class of investors.

A serious question was thereby raised for both theory and practice, because real debt markets do involve very different tax classes. For instance, the taxes faced by corporate investors in the United States are not the same as those faced by trading houses, while pension funds incur no taxes at all. Taxes paid to foreign countries may raise distinctions as well. Major players in debt markets are therefore apt to be looking at different after-tax versions of the same before-tax cash stream. But there is no consistent method of assessing the values of these after-tax cash streams by linear discounting, as in standard NPV.

In previous work, in Dermody and Rockafellar [1991], we developed a way around this difficulty by valuing cash streams according to their possible replication at current prices by portfolios of financial instruments available in markets. Relative to any tax class of investors we imputed an after-tax "long" price and an after-tax "short" price to each after-tax cash stream. The "long" price is the lowest cost at which *every* investor in the class can put together a portfolio of long and short positions that will bring in the cash stream in question, whereas the "short" price is the highest amount of current cash that can be extracted through taking on a portfolio that in net effect obligates the investor to pay out this future cash stream. The difference between these long and short prices constitutes a natural extension of the current borrowing-lending spread, which a linear NPV formula would ignore.[1]

Both the "long" and the "short" valuation operators for each tax class are nonlinear, in that the value assigned to the sum of two cash streams is generally different from the sum of the values assigned to them separately, although there is linearity to scale. Through duality they correspond to optimization of NPV over a polyhedral convex set of term structures rather than application of a single term structure.

This no-arbitrage model counters the widespread notion among practitioners that investors, even when they belong to different tax classes, ought to be able to view a cash stream in the same way, for instance as equivalent to some sequence of zero-coupon bonds.

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[1] Examples with real market data, worked out by Dermody [1993], show that the errors of NPV relative to the long and short values imputed to a given cash stream can well be on the order of 1%. Such errors are very large by the standards of bond trading, where a discrepancy of a few basis points (.01% of face value) can be a reason for taking action.

Yet an important issue remains, which wee now address here. Without further analysis there could be an impression that investors belonging to the *same* tax class might nonetheless be obliged to use a common valuation, at least as dictated by a no-arbitrage approach that avoids invoking utility functions. Or, it could be imagined that the process of refining the theory down to the level of individual investors might merely correspond to looking at smaller and smaller classes of investors in the same mathematical framework that works for whole tax classes, with no real difference in the form taken by the valuation operators. We show here that neither is the case.

The key to the refinement comes in allowing investors to take advantage of their existing holdings when valuing cash streams. While the long and short valuation operators for a particular tax class provide general guidelines for all investors in that class, regardless of holdings, an individual investor may enjoy special trading opportunities. Instead of taking on a new long position in some security, it may be possible to achieve much the same result at more favorable terms by closing out an existing short position in that security, for instance.

We demonstrate that when the "trades" available to an investor are augmented by such possibilities, a customized pair of long and short valuation operators is obtained. These operators generally narrow the gap between the imputed long and short valuations appropriate to the entire tax class to which the investor belongs, but they deviate in not even being linear to scale. The extra nonlinearity comes from the fact that the special opportunities on which the customized operators are based are limited by the extent of the investor's existing holdings, whereas the underlying model for whole tax classes does not incorporate any bounds on how much of a given security can be taken on in a long or short position.

A major complication at this level results from the actual rules of taxation that must be followed, cf. Dermody and Rockafellar [1994]. For U. S. taxes in particular, the rules require attention not only to the nature of the instrument held in a long or short position, but also the date on which the position was assumed and the amount of money that then changed hands. A current "basis value" is not enough in itself. In essence, a holding of some quantity of a particular instrument has to be subdivided according to such data, which could have a significant effect in assessing financial advantages.

We insist on such detail for two reasons. First, it underscores the ability of no-arbitrage models of cash stream valuation to handle practical questions of portfolio management in an operations research mode, in contrast to utility models, which can maintain no such expectations. Second, by revealing that the opportunities for special trades can depend on relatively smaller quantities among existing holdings than might otherwise be imagined, even for large investors, it indicates that nonlinearity of valuation for individual investors is pervasive. In other words, the closer one gets to practical decision making, the farther one should be from relying on traditional NPV.

Despite the nonlinearity, we are able to show that linearity to scale, although not linearity with respect to addition, can be obtained by focusing on very small trades, provided that the investor has first adjusted holdings relative to the current market to extract as much present cash as possible without altering the cash stream associated with those holdings. This provides insights which help further in understanding how the customized valuation operators can be compared to the operators for a tax class, and how an individual investor can be associated transiently with a smaller "term structure packet" within the packet associated with the class. As a by-product, a mechanism is brought to light for trades to take place among various investors even when the market is in a state of no-arbitrage equilibrium.

To keep the mathematics manageable in line with our theoretical purpose, we focus on *nonspeculative* investment involving *essentially riskless* securities, such as noncallable U.S. government bonds. Of course, the properties we uncover have implications well beyond this limited context, which we believe must carefully be understood first. No inherent mathematical obstacle exists to extending the analysis to a state space model of risk, but the notation would be still more complicated. Also, one of ideas we want to bring out, that no-arbitrage equilibrium need not be static even in the absence of stochastics, would be obscured. For this reason we try to be as thorough as possible at the present stage in eliminating speculative aspects of investment even where these might otherwise appear to be relevant.

2. TRADES AND HOLDINGS

In building on the valuation theory in Dermody and Rockafellar [1991], we work in the background of a fixed sequence of future dates, indexed by $i = 1, \ldots, m$. A *cash stream* is a vector $w = (w_1, \ldots, w_m)$, where w_i is the income received on date i and may be positive, zero or negative. An investor wishing to acquire such a cash stream, to be added to whatever cash stream may already be provided by existing holdings, can do so by putting together, at current prices, a portfolio of long and short positions in various securities traded in financial markets. A fixed set of securities, indexed by $j = 1, \ldots, n$, is considered for this purpose.

Security j pays the before-tax cash stream $(a_{1j}, \ldots, a_{mj})$. It can be purchased at the market ask price $P_j > 0$, which we also call the current *long price* because it corresponds to acquiring a new unit long position. Besides a long position there is the possibility of acquiring a unit short position in security j at the current *short price* $p_j \geq 0$. This refers to a deal having the end result that the investor receives the net amount p_j in the present in return for taking on the obligation of paying out the before-tax cash stream $(a_{1j}, \ldots, a_{mj})$. Generally, p_j is the market bid price minus the short-borrowing cost, but cases where the security is not available for shorting in an actual market sense are accounted for as ones having $p_j = 0$.

The systematic treatment of both long and short positions at different prices, $P_j > p_j$, is a crucial point where we diverge from traditional NPV as summarized for instance in

Ruback [1986], since that theory is invalid unless $P_j = p_j > 0$ for all j (and in particular, real short-borrowing is available for *every* security). It is what distinguishes our approach from the portfolio adjustment work of Hodges and Schaefer [1977] and the econometric models of Schaefer [1981] [1982b] and Ronn [1987], which despite some concern with shorting or transaction costs aim at determining a single "best" term structure for a given situation. The fact that the existence of infinitely many consistent term structures is unavoidable in any no-arbitrage model of cash stream valuation involving transaction costs was first demonstrated by Dermody and Prisman [1988].

Another distinguishing feature of our model is the treatment of taxes. We will be concentrating here on a single investor as a member of just one tax class among many. We suppose that for all investors in this tax class the holder of a unit long position in security j is obliged to pay the tax amount $t_{ij}(P_j)$ (possibly 0) on date i, whereas the holder of a unit short position is credited with $s_{ij}(p_j)$.

Every investor in the fixed tax class under consideration can put together what we call a *trade* (X, x), in which $X = (X_1, \ldots, X_n)$ gives the amounts $X_j \geq 0$ of new long positions in the various securities j, while $x = (x_1, \ldots, x_n)$ gives the amounts $x_j \geq 0$ of new short positions.[2] Relative to the further choice of a *holdover schedule* $h = (h_1, \ldots, h_m)$, in which the quantity $h_i \geq 0$ represents a credit or reserve of money available on date i from being held over from the preceding date (the initial amount h_1 coming from the present), the consequences of the trade (X, x) are as follows. The investor pays the amount

$$\sum\nolimits_{j=1}^{n} P_j X_j - \sum\nolimits_{j=1}^{n} p_j x_j + h_1 \tag{2.1}$$

in the present in return for receiving on future date i, for $i = 1, \ldots, m$, the after-tax amount

$$\sum\nolimits_{j=1}^{n} \big[a_{ij} - t_{ij}(P_j)\big] X_j - \sum\nolimits_{j=1}^{n} \big[a_{ij} - s_{ij}(p_j)\big] x_j + h_i - h_{i+1} \tag{2.2}$$

(with h_{m+1} interpreted always as 0). We assume throughout that the *weak no-arbitrage* condition, WNA, holds:

$$\left\{ \begin{array}{l} \text{there is no combination of a trade } (X,x) \text{ and a holdover schedule } h \text{ for which} \\ \text{the amounts (2.2) are all nonnegative, and yet the amount (2.1) is negative.} \end{array} \right. \tag{2.3}$$

This ensures for any *after-tax* cash stream w the existence of an *imputed long price* $V(w)$ and an *imputed short price* $v(w)$. The value $V(w)$ is the minimum of (2.1) over all combinations of a trade (X, x) and a holdover schedule h such that for $i = 1, \ldots, m$ the amount (2.2) is

[2] For any one security it may be expected that either $x_j = 0$ or $X_j = 0$ in a sensible trade, but this can be left to the economics of investment decisions rather than introduced as a direct requirement, which would unduly *complicate* the mathematics.

at least w_i. The value $v(w)$ is instead the maximum of the amount (2.1), with sign reversed, over all choices of a trade (X, x) and a holdover schedule h such that for $i = 1, \ldots, m$ the amount (2.2), likewise with sign reversed, does not exceed w_i. These values apply to *every* investor in the tax class under consideration, regardless of individual circumstances.

We wish to explore, relative to this background, the situation faced by an investor already having various positions in the given securities. The tax effects of such positions depend on both the date and price of acquisition, cf. Dermody and Rockafellar [1994]. Therefore, the possibility of more than one long or short position in each security j must be allowed for the sake of dealing optimally with taxes when questions arise of closing out some old position as a substitute, in part, for taking on a new one.

Accordingly, we suppose in general that the individual investor has long positions in j in the amounts $\overline{X}_{jl} > 0$ in categories indexed by $l = 1, \ldots, L_j$, with $\bar{t}_{ijl}$ the tax to be paid on date i for each unit in category l. The possibility of no long position at all in j is handled notationally as the case where $L_j = 0$. Likewise we allow for short positions in j in the amounts $\overline{x}_{jl}$ in categories indexed by $l = 1, \ldots, l_j$, with $\bar{s}_{ijl}$ the tax credit received per unit on date i. Again, the possibility of no short position at all in j corresponds to $l_j = 0$.[3] The current holdings represented in this manner provide the investor with the after-tax cash stream $\overline{w} = (\overline{w}_1, \ldots, \overline{w}_m)$, where

$$\overline{w}_i = \sum\nolimits_{j=1}^{n} \left[\sum\nolimits_{l=1}^{L_j} \left[a_{ij} - \bar{t}_{ijl}\right] \overline{X}_{jl} - \sum\nolimits_{l=1}^{l_j} \left[a_{ij} - \bar{s}_{ijl}\right] \overline{x}_{jl} \right]. \tag{2.4}$$

The reasons for the investor having this previously acquired portfolio and its associated cash stream are not the subject here. Rather, we address two fundamental issues:

1. *Is it possible at current prices to adjust the holdings in such a way that the resulting portfolio furnishes an after-tax cash stream at least as good as $\overline{w}$, and yet the cost of the adjustment is negative, i.e., there is positive income obtained in the present? If so, what is the most present cash that can be obtained this way, and how?*
2. *At what cost, relative to the savings potentially available through adjustment in the manner just described, could the investor move instead to a new cash stream at least as good as $\overline{w} + w$, where w is some specified cash stream? Can a consistent theory of valuation on the level of the individual investor be derived in this way?*

These two issues are closely interrelated, because it is important for purposes of valuation theory to separate out from the financial consequences of moving optimally from $\bar{w}$ to $\bar{w} + w$ the opportunities inherent merely in remaining at $\bar{w}$. It is equally important in our context

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[3] There is no reason to exclude the possibility of long and short positions being held simultaneously, especially since that would make the mathematics harder.

to separate the assessment of present value from speculation on how prices may behave in the future, except to the extent that this may already be reflected in current prices for the essentially riskless securities j. We are careful therefore *in setting up valuation formulas* to treat all long and short positions only for what they would offer if held to maturity.[4] But obviously, *this restriction does not preclude later adjustments in the positions*, since that is precisely what we are occupied with in the questions posed above. It simply means that from the theoretical perspective adopted here, *the possible profits from such adjustments are not to be counted on in the present, even in expectation.*

3. EXTENDED TRADES AND PORTFOLIO ADJUSTMENT

Trades (X, x), which add new long or short positions to the investor's portfolio, need to be extended to include decisions to close out some existing long or short positions, at least in part. Questions of capital gain or loss then come up. We denote by B_{jl} the current basis values for the unit long positions of j in categories $l = 1, \dots, L_j$, and by b_{jl} the ones for the short positions in categories $l = 1, \dots, l_j$.

Consider now the effect of selling a unit of j from a long position in category l. The investor receives the amount P'_j, the bid price in the current market, but loses the before-tax income amount a_{ij} on date i for $i = 1, \dots, m$. On the other hand, the taxes due on date i are reduced by $\bar{t}_{ijl}$. The difference $P'_j - B_{jl}$ is a *capital gain* (or, if negative, a capital loss) leading to a different tax obligation, which may be recorded as requiring the payment of the amount $g_i\big[P'_j - B_{jl}\big]$ on date i for $i = 1, \dots, m$ (this being negative in the case of a loss). Here the g_i's are known coefficients satisfying $g_i \geq 0$ and $\sum_{i=1}^m g_i < 1$.[5] The future consequence of closing out a unit long position of j in category l would therefore be to subtract from the investor's after-tax income on each date i the net amount $a_{ij} - \bar{t}_{ijl} + g_i\big[P'_j - B_{jl}\big]$.

Similarly, in closing out a unit short position of j in category l the investor would have to pay an amount p'_j in the present, where p'_j is calculated by subtracting from the ask price P_j (which is paid to buy back a unit to cover the one that was sold after being short-borrowed) the sale value of the unused short-borrowing rights to maturity. (In effect, p_j and p'_j are the current bid and ask prices for shorting.) On the other hand, the amount $a_{ij} - \bar{s}_{ijl} - g_i\big[b_{jl} - p'_j\big]$

[4] For short positions this means we focus on "term short borrowing," in contrast to nonterm short borrowing which can be terminated by either party at any time.

[5] Under the rules in effect in the U. S., the fractions g_i would be 0 except for four dates during the current accounting year when estimated taxes are due, as well as the final tax-due date for the year. Corporate capital losses cannot now offset ordinary income. This extra complication could be handled mathematically, but we prefer here to assume that the investor's circumstances will be adequate in each year so as not to trigger this provision.

would be added to the after-tax income on each date i. While it will be true that

$$p_j < p'_j < P'_j < P_j, \tag{3.1}$$

these prices are not necessarily larger or smaller than any of the basis amounts B_{jl} or b_{jl}.

We wish now to consider the effects of an *extended trade* (X, x, X', x') in combination with a holdover schedule h, which will mean the following. As in the model for a tax class rather than an individual investor, X and x are vectors constituting a trade (X, x) with

$$X_j \geq 0 \text{ and } x_j \geq 0, \tag{3.2}$$

and h is still a vector with components

$$h_i \geq 0, \tag{3.3}$$

but X' and x' are variable arrays with components X'_{jl} and x'_{jl} satisfying

$$\begin{aligned} 0 \leq X'_{jl} \leq \overline{X}_{jl} \ \text{ for } l = 1, \ldots, L_j, \\ 0 \leq x'_{jl} \leq \overline{x}_{jl} \ \text{ for } l = 1, \ldots, l_j. \end{aligned} \tag{3.4}$$

The quantity X'_{jl} refers to the number of units sold from the existing long position in j in category l, while x'_{jl} refers to removing some of the portion of the existing short position in j in category l. In making such an extended trade the investor pays the amount

$$F_0(X, x, X', x') = \sum_{j=1}^{n} \left[P_j X_j + p'_j \sum_{l=1}^{l_j} x'_{jl} \right] - \sum_{j=1}^{n} \left[p_j x_j + P'_j \sum_{l=1}^{L_j} X'_{jl} \right] \tag{3.5}$$

in the present, but adds to after-tax income on each date i the (possibly negative) amount

$$\begin{aligned} &F_i(X, x, X', x') = \\ &\qquad \sum_{j=1}^{n} \left[\big[a_{ij} - t_{ij}(P_j)\big] X_j + \sum_{l=1}^{l_j} \Big[a_{ij} - \bar{s}_{ijl} - g_i \big[b_{jl} - p'_j\big] \Big] x'_{jl} \right] \\ &\qquad - \sum_{j=1}^{n} \left[\big[a_{ij} - s_{ij}(p_j)\big] x_j + \sum_{l=1}^{L_j} \Big[a_{ij} - \bar{t}_{ijl} + g_i \big[P'_j - B_{jl}\big] \Big] X'_{jl} \right]. \end{aligned} \tag{3.6}$$

Our interest lies in examining the costs the investor would have to incur in moving by such an extended trade from the cash stream $\overline{w}$ already secured to a new cash stream $\overline{w} + w$.