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PERTURBATION OF GENERALIZED KUHN-TUCKER POINTS IN FINITE-DIMENSIONAL OPTIMIZATION

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Abstract. A general optimization model is set up that includes standard nonlinear programming but also allows for max functions, penalties and constraint-monitoring expressions. First-order necessary conditions are given in terms of a Lagrangian function. It is shown that when the data elements in the problem depend smoothly on parameters, the set-valued mapping that gives for each parameter vector the corresponding primal-dual vector pairs satisfying these first-order conditions is proto-differentiable. Moreover the derivatives can be calculated by solving an auxiliary problem in extended linear-quadratic programming.

Keywords: Sensitivity analysis of optimal solutions, proto-differentiability of set-valued mappings, nonsmooth analysis.

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1. Introduction

A very large and versatile class of optimization problems can be posed in the form

$$(\mathcal{P}) \qquad \text{minimize } f(x) + h(F(x)) \text{ over all } x \in X,$$

where X is a nonempty polyhedral (convex) set in $\mathbb{R}^n$, the mappings $f : \mathbb{R}^n \to \mathbb{R}$ and $F : \mathbb{R}^n \to \mathbb{R}^m$ are of class $\mathcal{C}^2$, and the function $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ is convex and possibly extended-real-valued, specifically of the form

$$h(u) = \sup_{y \in Y}\{y{\cdot}u - g(y)\} = (g + \delta_Y)^*(u) \tag{1.1}$$

for a nonempty polyhedral (convex) set $Y \subset \mathbb{R}^m$ and a convex function $g : \mathbb{R}^m \to \mathbb{R}$ of class $\mathcal{C}^2$.

Example 1. If $g \equiv 0$ and Y is a cone, $(\mathcal{P})$ is the classical problem of minimizing $f(x)$ subject to $x \in X$ and $F(x) \in K$, where K is the cone polar to Y. This is true because $h(u) = 0$ when $u \in K$ but $h(u) = \infty$ when $u \notin K$.

Example 2. If $g \equiv 0$ and Y is a box, consisting of the vectors $y = (y_1, \ldots, y_m)$ satisfying $a_i \leq y_i \leq b_i$ for given bounds a_i and b_i with $-\infty < a_i \leq 0 \leq b_i < \infty$, the expression $h(F(x))$ gives linear penalties relative to $F(x) = 0$. One has in terms of $F(x) = (f_1(x), \ldots, f_m(x))$ that

$$h(F(x)) = \Sigma_{i=1}^m (a_i \min\{f_i(x), 0\} + b_i \max\{f_i(x), 0\}).$$

Example 3. If $f \equiv 0$, $g \equiv 0$ and Y is the unit simplex consisting of the vectors $y = (y_1, \ldots, y_m)$ such that $y_i \geq 0$ and $y_1 + \cdots y_m = 1$, the function being minimized in $(\mathcal{P})$ is

$$\varphi(x) = \max\{f_1(x), \ldots, f_m(x)\}.$$

Example 4. When g does not necessarily vanish but is a quadratic function, $g(y) = \frac{1}{2}y{\cdot}Qy$ for a positive *semi*definite symmetric matrix Q, one obtains as h a general "monitoring" function of the form $\rho_{Y,Q}$ studied in Rockafellar [1]. This case subsumes the preceding three as well as various mixtures, and it also allows for augmented Lagrangian expressions.

Example 5. The case of Example 4 where f is quadratic convex and F is affine gives *extended linear-quadratic programming*, a subject developed in some detail in [1] for the sake of applications to optimal control and also to stochastic programming, cf. Rockafellar and Wets [2], [3], [4]. Then $(\mathcal{P})$ consists of minimizing over X a function of the form

$$p{\cdot}x + \tfrac{1}{2}x{\cdot}Px + \rho_{Y,Q}(q - Rx), \text{ where } \rho_{Y,Q}(u) := \sup_{y \in Y}\{y{\cdot}u - \tfrac{1}{2}y{\cdot}Qy\}.$$

Our main interest here is the development of sensitivity analysis for "quasi-optimal" solutions x to $(\mathcal{P})$ and their associated multiplier vectors y. We say "quasi-optimal" because instead of true optimality we shall be working with points that only satisfy first-order necessary conditions for optimality. Such conditions are unlikely to be sufficient for optimality unless $(\mathcal{P})$ happens to be a convex type of problem, but they are of importance nonetheless in the development of computational procedures. We must begin by formulating the conditions and establishing their validity. Afterward, we shall introduce parameterizations with respect to which a new form of sensitivity analysis will be carried out.

Definition 1. *The Lagrangian function for problem* $(\mathcal{P})$ *is*

$$L(x,y) = f(x) + y{\cdot}F(x) - g(y) \ \text{ for } x \in X \text{ and } y \in Y.$$

A generalized Kuhn-Tucker point is a pair $(x,y) \in X \times Y$ *such that*

$$-\nabla_x L(x,y) \in N_X(x) \quad \text{and} \quad \nabla_y L(x,y) \in N_Y(y), \tag{1.2}$$

where $N_X(x)$ *and* $N_Y(y)$ *are the normal cones to* X *and* Y *at* x *and* y *in the sense of convex analysis.*

Definition 2. *A point* x *is called a feasible solution to* $(\mathcal{P})$ *if* $x \in X$ *and* $F(x) \in U$, *where* $U = \operatorname{dom} h = \{u \,|\, h(u) \leq \infty\}$ *(a nonempty, convex set). The basic constraint qualification will be said to be satisfied at such a point* x *if the only vector* $y \in N_U(F(x))$ *with* $-y\nabla F(x) \in N_X(x)$ *is* $y = 0$.

In classical nonlinear programming, this basic constraint qualification reduces to the Mangasarian-Fromovitz constraint qualification. Indeed, in Example 1 the convex set U is a polyhedral cone K such as the set of $u = (u_1, \ldots, u_m)$ satisfying $u_i \leq 0$ for $i = 1, \ldots, s$ and $u_i = 0$ for $i = s+1, \ldots, m$. With $X = \mathbb{R}^n$ the condition says then that the only $y = (y_1, \ldots, y_m)$ in Y (the polar of K) for which $y_1 f_1(x) + \cdots y_m f_m(x) = 0$ and $y_1 \nabla f_1(x) + \cdots y_m \nabla f_m(x) = 0$ is $y = (0, \ldots, 0)$.

Theorem 1. *If* x *is a locally optimal solution to* $(\mathcal{P})$ *at which the basic constraint qualification is satisfied, there is a vector* y *such that* (x,y) *is a generalized Kuhn-Tucker point.*

Proof. We rely on methods of nonsmooth analysis and in particular the calculus of Clarke subgradients in Rockafellar [5]. Let $k_0(x) = f(x) + h(F(x))$ and $k = k_0 + \delta_X$ (with δ_X the indicator of x). The function h is lower semicontinuous, while f and F are smooth, so k_0 and k are lower semicontinuous. The feasible solutions to $(\mathcal{P})$ form the effective domain of k, which we are taking to be nonempty, and the locally optimal solutions to $(\mathcal{P})$ are the points at which k has a local minimum. Such a point must in particular satisfy $0 \in \partial k(x)$. We shall show from estimates of $\partial k(x)$ that this implies the existence of a vector y such that (x,y) is a generalized Kuhn-Tucker point.

A rule provided in [5, Corollary 8.1.2] tells us that for a point x in the effective domain of k one has

$$\partial k(x) \subset \partial k_0(x) + N_X(x) \ \text{ and } \ \partial^\infty k(x) \subset \partial^\infty k_0(x) + N_X(x) \tag{1.3}$$

when the only $v \in \partial^\infty k_0(x)$ satisfying $-v \in N_X(x)$ is $v = 0$. (Here ∂^∞ indicates so-called *singular* subgradients; see [5].) Further, from [5, Corollary 8.1.3] we have

$$\partial k_0(x) \subset \partial f(x) + \partial h(F(x))\nabla F(x) \ \text{ and } \ \partial^\infty k_0(x) \subset \partial^\infty f(x) + \partial^\infty h(F(x))\nabla F(x) \tag{1.4}$$

when the only $y \in \partial^\infty h(F(x))$ satisfying $0 \in \partial^\infty f(x) + y\nabla F(x)$ is $y = 0$. Inasmuch as f is smooth, the set $\partial^\infty f(x)$ is just $\{0\}$, while the set $\partial f(x)$ is just $\{\nabla f(x)\}$. Because h is convex with U as its effective domain, $\partial^\infty h(F(x))$ coincides with $N_U(F(x))$. If the basic constraint qualification holds at x, one has in particular that no $y \in N_U(F(x))$ gives $y\nabla F(x) = 0$, so the assumption required for (1.4) is fulfilled and in fact (1.4) takes the form

$$\partial k_0(x) \subset \nabla f(x) + \partial h(F(x))\nabla F(x) \text{ and } \partial^\infty k_0(x) \subset \mathcal{N}_U(F(x))\nabla F(x). \tag{1.5}$$

Then the basic constraint qualification validates the assumption underlying (1.3) as well, and we obtain

$$\partial k(x) \subset \nabla f(x) + \partial h(F(x))\nabla F(x) + N_X(x) \text{ and } \partial^\infty k(x) \subset N_U(F(x))\nabla F(x) + N_X(x).$$

Most importantly, the condition $0 \in \partial k(x)$ is seen to imply under our basic constraint qualification the existence of a vector y satisfying

$$y \in \partial h(F(x)) \text{ and } 0 \in \nabla f(x) + y\nabla F(x) + N_X(x) \tag{1.6}$$

The convexity of h and its conjugacy with $g + \delta_Y$ give us by [6, Theorems 23.5 and 23.8] the calculation

$$y \in \partial h(F(x)) \quad \Leftrightarrow \quad F(x) \in \partial(g + \delta_Y)(y) = \partial g(y) + \partial \delta_Y(y) = \nabla g(x) + N_Y(y).$$

Condition (1.6) is therefore equivalent to the generalized Kuhn-Tucker condition (1.2). □

Theorem 2. *Suppose $(\mathcal{P})$ is of convex type in the sense that f is convex and $y{\cdot}F(\cdot)$ is convex for every $y \in Y$. If x is a feasible solution for which there exists a y such that (x, y) is a generalized Kuhn-Tucker point, then x is a (globally) optimal solution to $(\mathcal{P})$. (The basic constraint qualification is not required.)*

Proof. The convexity hypothesis implies that $L(x, y)$ is convex in x for $y \in Y$ as well as concave in y for each x. In addition the sets X and Y are convex. The generalized Kuhn-Tucker condition (1.2) is then the same as the condition that (x, y) be a saddle point of L relative to $X \times Y$. Since

$$f(x) + h(F(x)) = \sup_{y \in Y} L(x, y),$$

the saddle point condition is sufficient for the global minimum of $f(x) + h(F(x))$ relative to $x \in X$, as is well known in convex optimization. □

Under our assumption that X and Y are not just convex but polyhedral, which has not yet been utilized really but will be important in our main result in the next section, it would be possible to derive general second-order necessary and sufficient conditions for local optimality in $(\mathcal{P})$. This could be accomplished by applying the theory of second-order epi-differentiability in Rockafellar [7] to the essential objective function k for $(\mathcal{P})$ (as introduced in the proof of Theorem 1). We shall not carry this out here, however, since it would sidetrack us from the main theme of analyzing the behavior of the first-order conditions under perturbations. A more general approach to second-order optimality conditions could be taken in the framework devised by Burke [8].

2. Parameterization and sensitivity.

Passing from a single problem to a whole family of problems, we consider

$$(\mathcal{P}(u,v,w)) \qquad \text{minimize } f(w,x) - x{\cdot}v + h(F(w,x) - u) \text{ over all } x \in X$$

for parameter vectors $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, and $w \in \mathbb{R}^d$. The assumptions are the same as before, except that f and F are now $\mathcal{C}^2$ in x and w jointly rather than just in x. The Lagrangian for $(\mathcal{P}(u,v,w))$ is obviously

$$L(w,x,y) - x{\cdot}v - y{\cdot}u, \text{ where } L(w,x,y) = f(w,x) + y{\cdot}F(w,x) - g(y), \tag{2.1}$$

and the condition for a generalized Kuhn-Tucker point is

$$-\nabla_x L(w,x,y) + v \in N_X(x) \text{ and } \nabla_y L(w,x,y) - u \in N_Y(y). \tag{2.2}$$

The basic constraint qualification is satisfied at a feasible point x if and only if the sole vector $y \in N_U(F(w,x) - u)$ such that $v - y{\cdot}\nabla_x F(w,x) \in N_X(x)$ is $y = 0$.

Our focus is on the set-valued mapping $S : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^d \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ defined by

$$S(u,v,w) = \{(x,y) \mid (x,y) \text{ is a Kuhn-Tucker point for } (\mathcal{P}(u,v,w))\}.$$

The set $S(u,v,w)$ could, of course, be empty for some choices of (u,v,w). The reason for introducing perturbations in the format of (u,v,w) rather than merely w, which in principle would suffice notationally to cover all the types of perturbations under consideration, is that the perturbations must be sufficiently "rich" to allow us to obtain our strongest result in Theorem 3 below.

Proposition 1. *The set-valued mapping S is upper semicontinuous in the sense that its graph*

$$\operatorname{gph} S = \{(u,v,w,x,y) \mid (x,y) \in S(u,v,w)\} \tag{2.3}$$

is a closed set.

Proof. This follows from the assumed continuity of the derivatives of L in (2.2) and the fact that the graphs of the set-valued mappings $x \mapsto N_X(x) = \partial\delta_X(x)$ and $y \mapsto N_Y(y) = \partial\delta_Y(y)$ are closed. The latter is known from convex analysis [6, Theorem 24.4]. □

The upper semicontinuity of S provides an underlying property of interest in the analysis of the sensitivity of the set of generalized Kuhn-Tucker points (x, y) in $(\mathcal{P})$ with respect to perturbations in the elements (u, v, w). If (x^ν, y^ν) is a generalized Kuhn-Tucker point for $(\mathcal{P}(u^\nu, v^\nu, w^\nu))$ and $(x^\nu, y^\nu) \to (x, y)$ and $(u^\nu, v^\nu, w^\nu) \to (u, v, w)$, then, by Proposition 1, (x, y) is a generalized Kuhn-Tucker point for $(\mathcal{P}(u, v, w))$ We wish to go much farther than such semicontinuity, however, and establish a form of generalized differentiability—a quantitative estimate for *directions and rates of change* of (x, y) with respect to perturbations in (u, v, w). For this we shall draw on concepts and results in Rockafellar [8], as specialized to S.

Definition 3. *The set-valued mapping S will be called proto-differentiable if for every (u, v, w) and choice of $(x, y) \in S(u, v, w)$ the following holds: the graph of the set-valued difference quotient mapping*

$$\Delta_t S(u', v', w') = [S(u + tu', v + tv', w + tw') - (x, y)]/t \text{ for } t > 0$$

converges as $t \downarrow 0$ (in the topology of set convergence) to the graph of another set-valued mapping from $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^d$ to $\mathbb{R}^n \times \mathbb{R}^m$. This limit mapping is then called the proto-derivative of S at (u, v, w) for the pair $(x, y) \in S(u, v, w)$ and is denoted by $S'_{(u,v,w),(x,y)}$.

Various characterizations of such generalized differentiability have been furnished in [8], and we shall not review them here. It is worth mentioning one fact, however.

Proposition 2. *In the case where S is proto-differentiable, one has that a pair $(\bar{x}', \bar{y}')$ belongs to $S'_{(\bar{u},\bar{v},\bar{w}),(\bar{x},\bar{y})}(\bar{u}', \bar{v}', \bar{w}')$ if and only if for all t in some interval $(0, \delta)$ there exist $(x(t), y(t)) \in S(u(t), v(t), w(t))$ with $(u(0), v(0), w(0)) = (\bar{u}, \bar{v}, \bar{w})$, $(x(0), y(0)) = (\bar{x}, \bar{y})$, such that*

$$(u'_+(0), v'_+(0), w'_+(0)) = (\bar{u}', \bar{v}', \bar{w}') \text{ and } (x'_+(0), y'_+(0)) = (\bar{x}', \bar{y}') \text{ (right derivatives).}$$

Proof. This specializes a property of proto-differentiability in [9, Prop. 2.3]. □

The principal result of this paper will involve the following auxiliary problem, symbolized in a suggestive manner for reasons soon to be apparent. This problem, which falls into the same category as the problems $(\mathcal{P})$ we have been occupied with, depends on a choice of (u, v, w) and $(x, y) \in S(u, v, w)$, although for simplicity we have not tried to reflect this fully in the notation (the subscript $_*$ stands for all the missing parameters):

$$(\mathcal{P}'_{(u,v,w),(x,y)}(u', v', w')) : \text{ minimize } f_*(w', x') - v' \cdot x' + h_*(F_*(w', x') - u') \text{ over all } x' \in X_*,$$

where

$$\begin{aligned}
f_*(w',x') &= x'\cdot\nabla^2_{xw}L(w,x,y)w' + \tfrac{1}{2}x'\cdot\nabla^2_{xx}L(w,x,y)x',\\
F_*(w',x') &= \nabla^2_{yw}L(w,x,y)w' + \nabla^2_{yx}L(w,x,y)x' = \nabla_w F(w,x)w' + \nabla_x F(w,x)x',\\
h_*(u') &= \sup\nolimits_{y'\in Y_*}\{y'\cdot u' - g_*(y')\} = (g_* + \delta_{Y_*})(y') \ \text{ for}\\
g_*(y') &= -\tfrac{1}{2}y'\cdot\nabla^2_{yy}L(w,x,y)y' = \tfrac{1}{2}y'\cdot\nabla^2 g(y)y',\\
Y_* &= \{y'\in T_Y(y) \,|\, y' \perp \nabla_y L(w,x,y) - u\},\\
X_* &= \{x'\in T_X(x) \,|\, x' \perp \nabla_x L(w,x,y) - v\}.
\end{aligned}$$

The notation $T_X(x)$ and $T_Y(y)$ gives the *tangent cones* to X at x and Y at y, which are the polars of the cones $N_X(x)$ and $N_Y(y)$.

Theorem 3. *The set-valued mapping S is proto-differentiable. Furthermore, its derivative set $S'_{(u,v,w),(x,y)}(u',v',w')$ at (u,v,w) for any $(x,y)\in S(u,v,w)$ is the set of generalized Kuhn-Tucker points (x',y') for the auxiliary problem $(\mathcal{P}'_{(u,v,w),(x,y)}(u',v',w'))$.*

Proof. In terms of a change of notation to $s=(v,-u)$, $z=(x,y)$, $Z=X\times Y$ and $G(w,z)=(\nabla_x L(w,x,y), -\nabla_y L(w,x,y))$, we can express the generalized Kuhn-Tucker conditions (2.2) as the variational inequality

$$-G(w,z) + s \in N_Z(z), \quad z\in Z. \tag{2.4}$$

In studying S we are studying the set-valued mapping T that associates with each pair (w,s) the corresponding set of solutions z to this variational inequality. Here G is a mapping of class $\mathcal{C}^1$ (because L is of class $\mathcal{C}^2$), and Z is a polyhedral set (because X and Y are polyhedral). We have proved in [9, Theorem 5.6] that in the presence of this degree of regularity the mapping T is proto-differentiable. Furthermore its derivative set $T'_{(w,s),z}(w',s')$ at (w,s) for any $z\in T(w,s)$ consists of the solutions z' to the auxiliary variational inequality

$$-G_*(w',z') + s' \in N_{Z_*}(z'), \quad z'\in Z_*, \tag{2.5}$$

where

$$\begin{aligned}
G_*(w',z') &= \nabla_w G(w,z)w' + \nabla_z G(w,z)z',\\
Z_* &= \{z'\in T_Z(z) \,|\, z'\perp G(w,z) - s\}.
\end{aligned} \tag{2.6}$$

Referring to the definition of G, we see that

$$\begin{aligned}
G_*(w',x',y') = (&\nabla^2_{xw}L(w,x,y)w' + \nabla^2_{xx}L(w,x,y)x' + \nabla^2_{xy}L(w,x,y)y',\\
&-\nabla^2_{yw}L(w,x,y)w' - \nabla^2_{yx}L(w,x,y)x' - \nabla^2_{yy}L(w,x,y)y').
\end{aligned}$$

This can be written in terms of the function

$$\begin{aligned}
L_*(w',x',y') &= x'\cdot\nabla^2_{xw}L(w,x,y)w' + y'\cdot\nabla^2_{yw}L(w,x,y)w'\\
&\quad + \tfrac{1}{2}x'\cdot\nabla^2_{xx}L(w,x,y)x' + y'\cdot\nabla^2_{yx}L(w,x,y)x' + \tfrac{1}{2}y'\cdot\nabla^2_{yy}L(w,x,y)y'
\end{aligned} \tag{2.7}$$

as the mapping

$$G_*(w',x',y') = (\nabla_{x'}L_*(w',x',y'), -\nabla_{y'}L_*(w',x',y')). \tag{2.8}$$

Note that L_* is closely tied to the auxiliary problem $(\mathcal{P}'_{(u,v,w),(x,y)}(u',v',w'))$. In fact the Lagrangian for this problem, as determined from Definition 1 with the obvious twist of notation, is

$$L_*(w',x',y') - x'\cdot v' - y'\cdot u' \ \text{ for } x' \in X_* \text{ and } y' \in Y_*. \tag{2.9}$$

Next we determine Z_*, using the fact that $Z = X \times Y$ and therefore

$$T_Z(x,y) = T_X(x) \times T_Y(y) \quad \text{and} \quad N_Z(x,y) = N_X(x) \times N_Y(y).$$

Since $G(w,z) - s = (\nabla_x L(w,x,y) - v, -\nabla_y L(w,x,y) + u)$ with

$$x'\cdot[-\nabla_x L(w,x,y) + v] \leq 0 \text{ for all } x' \in X \text{ and } y'\cdot[\nabla_y L(w,x,y) - u] \leq 0 \text{ for all } y' \in Y$$

(by the definition of the normality relations in (2.2) that underlie the meaning of (x,y) being a generalized Kuhn-Tucker point at (u,v,w)), a pair $z' = (x',y')$ in $T_Z(z)$ satisfies the condition defining Z_* in (2.6) if and only if $x' \in T_X(x)$ with $x' \perp [-\nabla_x L(w,x,y) + v]$ and $y' \in T_Y(y)$ with $y' \perp [\nabla_y L(w,x,y) - v]$. In other words,

$$Z_* = X_* \times Y_* \quad \text{and} \quad N_{Z_*}(z') = N_{X_*}(x') \times N_{Y_*}(y'). \tag{2.10}$$

It follows from this and (2.8) that the auxiliary variational inequality (2.5) takes the form

$$-\nabla_{x'}L_*(w',x',y') + v' \in N_{X_*}(x') \text{ and } \nabla_{y'}L_*(w',x',y') - u' \in N_{Y_*}(y'). \tag{2.11}$$

We now observe from the Lagrangian expression (2.9) for $(\mathcal{P}'_{(u,v,w),(x,y)}(u',v',w'))$ that this is the condition for (x',y') to be a generalized Kuhn-Tucker point in that problem. Thus the set $S'_{(u,v,w),(x,y)}(u',v',w')$ consists of just such points, as claimed. □

It is worth recording that the auxiliary problem $(\mathcal{P}'_{(u,v,w),(x,y)}(u',v',w'))$ is one of *extended linear-quadratic programming* as mentioned in Example 5. Moreover it is of convex type when $f_*(w',x')$ is convex in x', which of course is equivalent to the matrix $\nabla^2_{xx}L(w,x,y)$ being positive semidefinite. In that case solutions x' and multiplier vectors y' satisfying the generalized Kuhn-Tucker conditions for $(\mathcal{P}'_{(u,v,w),(x,y)}(u',v',w'))$ can be found numerically, after a reformulation, by applying algorithms for standard quadratic programming problems or linear complementarity problems.

Theorem 3 may be contrasted with results of Robinson and Shapiro. In Robinson [10] and [11], variational inequalities (generalized equations) are considered that could include the parameterized Kuhn-Tucker conditions (2.2) as a special case. The setting is broader in some important respects (the sets X and Y would only need to be convex, and the spaces in which they lie could be infinite-dimensional). The focus, however, is on assumptions under which the mapping that corresponds to S turns out not only to be single-valued in a localized sense but also has a Lipschitz property. Theorem 3 yields no such conclusions but requires no such assumptions, either. The form of differentiability is more general.

In Shapiro [12] the concern is not with Kuhn-Tucker points but with perturbations of optimal solutions alone (i.e. not in combination with multiplier vectors). Again the aim is to obtain single-valuedness and a Lipschitz property. To this end, certain second-order sufficient conditions for optimality are assumed to hold. No abstract constraint $x \in X$ is admitted, and the treatment of the other constraints is conventional: Y is a cone as in Example 1. The results are thus complementary to ours.

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