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GENERALIZED DIRECTIONAL DERIVATIVES AND SUBGRADIENTS OF NONCONVEX FUNCTIONS

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1. Introduction. Studies of optimization problems and certain kinds of differential equations have led in recent years to the development of a generalized theory of differentiation quite distinct in spirit and range of application from the one based on L. Schwartz's "distributions." This theory associates with an extended-real-valued function f on a linear topological space E and a point $x \in E$ certain elements of the dual space E^* called *subgradients* or *generalized gradients* of f at x. These form a set $\partial f(x)$ that is always convex and weak*-closed (possibly empty). The multifunction $\partial f : x \rightarrow \partial f(x)$ is the *subdifferential* of f.

Rules that relate ∂f to generalized directional derivatives of f, or allow ∂f to be expressed or estimated in terms of the subdifferentials of other functions (when $f = f_1 + f_2, f = g \circ A$, etc.), comprise the *subdifferential calculus*. Such rules are used especially in analyzing the condition $0 \in \partial f(x)$, which typically means that x is some sort of "quasi-optimal" point for f. The extended-real-valued nature of f is essential in such a context as a device for representing constraints.

Subdifferential calculus began with convex functions on $\mathbf{R}^n$. Rockafellar [**43**] defined ∂f for such functions, showed how to characterize $\partial f(x)$ in terms of one-sided directional derivatives $f'(x; y)$, and proved that under mild restrictions rules such as

$$(1.1) \qquad \partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x)$$

are valid. This branch of convex analysis was developed further by Moreau, Rockafellar and others in the 1960's and applied to many kinds of optimization problems (cf. [**42**], [**44**], [**45**] for expositions). Besides convex functions, it covers concave functions and saddle functions (functions of two vector variables which are convex in one argument and concave in the other); for rules of type (1.1) for saddle functions, see McLinden [**38**].

The multifunctions ∂f in these cases yield, or are closely associated with, maximal monotone "operators" in the sense of G. Minty and L. E. Browder (cf. [**41**], [**4**], [**46**], [**47**]). The subdifferential calculus has served correspondingly as a model for results on when a sum of maximal monotone operators is again maximal monotone (cf. [**48**]).

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F. H. Clarke in his 1973 thesis [**6**] made a major contribution in showing how the definition of ∂f could be extended to arbitrary lower semicontinuous functions $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ in such a way that ∂f is the subdifferential of convex analysis when f is convex or concave or a finite saddle function, and ∂f reduces to the ordinary gradient mapping ∇f when f is continuously differentiable. He showed that

$$(1.2) \qquad \partial(f_1 + f_2)(x) \subset \partial f_1(x) + \partial f_2(x)$$

when f_1 and f_2 are Lipschitzian (Lipschitz continuous) in a neighborhood of x and proved another rule when f is the maximum of a collection of continuously differentiable functions. He also characterized $\partial f(x)$ by way of a generalized directional derivative expression $f^\circ(x; y)$ when f is locally Lipschitzian. These basic results were published in [**7**]. In [**8**], [**10**] Clarke broadened the definition of the subdifferential to functions on Banach spaces and extended the subdifferential calculus to continuous sums and pointwise maxima of locally Lipschitzian functions, as well as the composition of a locally Lipschitzian mapping and a differentiable mapping. The naturalness of this concept of ∂f has been underscored by a mean value theorem (Lebourg [**37**]) and an inverse mapping theorem (Clarke [**9**]). Aubin [**1**] has provided an account of the Lipschitzian case which also treats the operation of infimal convolution.

Clarke did not characterize $\partial f(x)$ in terms of any "directional derivatives" in the non-Lipschitzian case, but his definition is connected with a certain tangent cone to the epigraph of f at $(x, f(x))$. Hiriart-Urruty [**26**], [**27**], has observed that this cone is the epigraph of a sublinear function which must give the desired "derivatives," but until now no direct formula for this function, involving limits of difference quotients of some kind, has been discovered.

Indeed, ∂f itself has not yet been given a direct definition in the general case: Clarke's approach has been to define ∂f for *locally Lipschitzian* functions in terms of certain limits, use this to define generalized tangent and normal cones to closed sets, and finally apply the latter to the epigraphs of l.s.c. functions. The lack of a more straightforward characterization of ∂f is one of the chief reasons why the subdifferential calculus for nonconvex functions has so far been limited mainly to the Lipschitzian case.

Clarke's results do provide a direct formula for tangent cones in finite-dimensional spaces [**7**, Proposition 3.7]. Thibault [**55**] has adopted this formula in separable Banach spaces in order to bypass the first of the three stages of Clarke's definition of ∂f, apparently without realizing that the formula implies the convexity of the cones in question. For Banach spaces, the equivalence with the initial form of Clarke's definition of tangent cones (and hence the convexity of the cones described by this formula) had been demonstrated by Hiriart-Urruty [**27**]. More recently, the convexity has been proved by Rockafellar [**49**] by a direct argument in $\mathbf{R}^n$ that easily carries over to any linear topological space. This opens the way to a direct definition of ∂f along the lines proposed by Hiriart-Urruty, because the argument can be applied to epigraphs and translated into a statement about limits of certain difference quotients.

The aim of the present paper is to carry out this project. Subderivatives $f^{\uparrow}(x; y)$ are defined in terms of a "lim sup inf" which is a kind of minimax version of "lim sup" and "lim inf." It is shown that $f^{\uparrow}(x; y)$ is always lower semicontinuous and sublinear in y, in particular convex. If f is convex, $f^{\uparrow}(x; \cdot)$ is just the l.s.c. hull of the directional derivative function $f'(x; \cdot)$. If f is Lipschitzian around x, $f^{\uparrow}(x; y)$ reduces to Clarke's derivative $f^{\circ}(x; y)$. The latter fact is generalized to a large class of functions, said to be "directionally Lipschitzian" at x, through a broadened definition of $f^{\circ}(x; y)$; these functions have the important property that $\partial(-f)(x) = -\partial f(x)$.

The subderivatives $f^{\uparrow}(x; y)$ thus furnish, by the duality between sublinear functions and convex sets, a new approach to $\partial f(x)$ that covers the general nonconvex case in a more analytic manner and without appealing to the existence of a norm. This approach makes possible an extension of rules like (1.2) to cases where the functions are neither convex nor locally Lipschitzian, although the details will not be given here (see [**50**]).

The results in this paper are thus aimed ultimately at applications to diverse problems of optimization, finite-dimensional and infinite-dimensional, following the now-familiar pattern for subgradients in the convex case (cf. [**45**], [**51**], [**52**], [**53**], for example). As far as nonconvex (nonsmooth) problems are concerned, very important progress in this direction has been made by Clarke in mathematical programming [**10**] and optimal control and the calculus of variations [**6**], [**8**], [**11**], [**12**], [**13**], [**14**], [**15**], [**16**], [**17**], [**18**], [**19**] (for a recent survey of the subject see [**20**]). Hiriart-Urruty's substantial thesis [**26**] has included the first applications to nonconvex stochastic programming problems, as well as results in basic mathematical programming [**28**], [**29**], and the study of marginal functions [**30**] (an excellent term he has coined for functions f which express the optimal value in some optimization problem as a function of parameters on which the problem depends). Generalized gradients of certain nonconvex marginal functions have also been studied by Gauvin [**23**] and Aubin/Clarke [**2**]. Applications of Clarke's concepts to algorithms for nonconvex optimization have been explored by Feuer [**21**], [**22**], Goldstein [**25**], Chaney/Goldstein [**5**] and Mifflin [**39**], [**40**]. These concepts are also put to use in recent work of Ioffe on the stability of solution sets [**32**], [**33**], and general optimality conditions [**34**], [**35**], [**36**].

2. Limit concepts. Throughout this paper, the topology of the linear space E is assumed to be locally convex and separated (Hausdorff). The definition of upper and lower subderivatives of functions on E will depend on a new limit notion for functions $g(s, y)$ of $s \in S$ and $y \in E$, where S is any topological space. As geometric motivation for this notion, we begin by recalling Hausdorff's definition [**25**, p. 147] of the "lim inf" of a sequence of sets, or more generally of a multifunction in terms of its argument (see also Berg [3, Chapter VI]). We denote by $\mathcal{N}(s), \mathcal{N}(y)$ the collection of all neighborhoods of s, y, etc.

Let Γ be any multifunction from S to E. (Thus for each $s \in S$, $\Gamma(s)$ is some subset of E, possibly empty.) The set

$$(2.1) \qquad \Delta(s) = \liminf_{s' \to s} \Gamma(s')$$

consists by definition of all $y \in E$ such that

$$\forall Y \in \mathcal{N}(y), \exists\, U \in \mathcal{N}(s), \forall s' \in U : Y \cap \Gamma(s') \neq \emptyset.$$

Equivalently,

$$(2.2) \qquad \Delta(s) \bigcap_{V \in \mathcal{N}(0)} \bigcup_{U \in \mathcal{N}(s)} \bigcap_{s' \in U} [\Gamma(s') + V].$$

Note that $\Delta(s)$ would not be affected if $\Gamma(s')$ were replaced by $\operatorname{cl} \Gamma(s')$ for all s', in which event Γ would be closed-valued. A closed-valued multifunction Γ is said to be *lower semicontinuous* if $\Delta(s) = \Gamma(s)$ for all s.

Recall next that the *epigraph* of a function $g : E \to R \cup \{\pm\infty\}$ is the set

$$\operatorname{epi} g = \{(y, \beta) \in E \times R \,|\, \beta \geqq g(y)\},$$

and this is closed if and only if g is lower semicontinuous on E:

$$g(y) = \liminf_{y' \to y} g(y') = \sup_{Y \in \mathcal{N}(y)} \inf_{y' \in Y} g(y') \quad \text{for all } y.$$

Consider now an arbitrary extended-real-valued function g on $S \times E$. We shall be interested in the expression "lim sup inf" defined as follows:

$$(2.3) \qquad \begin{aligned} h(s, y) &= \limsup_{s' \to s} \inf_{y' \to y} g(s', y') \\ &\triangleq \sup_{Y \in \mathcal{N}(y)} \inf_{U \in \mathcal{N}(s)} \sup_{s' \in S} \inf_{y' \in Y} g(s', y'). \end{aligned}$$

In terms of sequences (when the topologies on S and E can so be described), this expression can also be characterized as

$$\sup_{\{s_k\} \to s} \inf_{\{y_k\} \to y} \theta(\{s_k\}, \{y_k\}), \text{ where } \theta(\{s_k\}, \{y_k\}) = \limsup_{k \to \infty} g(s_k, y_k),$$

where the infimum is over the collection of all sequences converging to y, and the supremum is over the collection of all sequences converging to s.

PROPOSITION 1. *For each $s \in S$, let $\Gamma(s)$ denote the subset of $E \times \mathbf{R}$ which is the epigraph of the function $y \to g(s, y)$, and let $\Delta(s)$ be the limit set in (2.1). Then $\Delta(s)$ is the epigraph of the function $y \to h(s, y)$ in (2.3), and hence in particular $h(s, y)$ is lower semicontinuous in y.*

Proof. It suffices in $E \times \mathbf{R}$ to consider neighborhoods of product form. Thus, to say that $(y, \beta) \in \Delta(s)$ is to say that

$$\forall Y \in \mathcal{N}(y), \forall \epsilon > 0, \exists\, U \in \mathcal{N}(s), \forall s' \in U,$$
$$\exists\, (y', \beta') \in [Y \times (\beta - \epsilon, \beta + \epsilon)] \cap \Gamma(s'),$$

where the final condition means simply that there exists $y' \in Y$ such that

$g(s', y') < \beta + \epsilon$. Therefore $(y, \beta) \in \Delta(s)$ if and only if

$$\forall Y \in \mathcal{N}(y), \forall \epsilon > 0, \exists U \in \mathcal{N}(s) \quad \text{such that}$$
$$\sup_{s' \in U} \inf_{y' \in Y} g(s', y') \leqq \beta + \epsilon.$$

From this it is easy to see that $\Delta(s)$ consists of the pairs (y, β) such that $\beta \geqq h(s, y)$, which was the fact to be proved.

Remark 1. It follows from Proposition 1 that h would not be affected if g were replaced by its lower semicontinuous hull in the y argument. (This would be equivalent to replacing $\Gamma(s)$ by its closure.)

Remark 2. The reader should not fall into the trap of thinking that when Δ is defined by (2.1), then Δ is itself lower semicontinuous. A counterexample is furnished below. Correspondingly in view of Proposition 1, it does not follow from (2.3) that h has the semicontinuity property

(2.4) $$\lim \sup_{s' \to s} \inf_{y' \to y} h(s', y') = h(s, y).$$

One's first reaction to this state of affairs might be to reject the definition of lim inf Γ and to try to substitute for it another which does have the lacking property. As a matter of fact, it can be shown there does exist a maximal multifunction $\tilde{\Delta} \subset \Gamma$ that is lower semicontinuous (given by the union of the graphs of the l.s.c. multifunctions $\subset \Gamma$), and presumably this is what one really ought to define as lim inf Γ. The trouble is that $\tilde{\Delta}$ is hard to describe more concretely. In particular, no formula is known which expresses the graph of $\tilde{\Delta}$ in terms of limits (of some sort) involving elements of the graph of Γ. The implication for the epigraph setting is that, while there is indeed a natural function $\tilde{h}$ which could be substituted for h and would be the least function majorizing g and having property (2.4), one does not know how to express $\tilde{h}$ in some relatively simple fashion analogous to (2.3). Without such an expression, it would be difficult to work with $\tilde{h}$ in applications. We therefore turn our backs on the temptation of such an approach.

Counterexample. Let Γ be the multifunction from $\mathbf{R}$ to $\mathbf{R}^2$ defined by

$$\Gamma(s) = \begin{cases} [(0, 0), (1, 2)] \quad \text{(line segment)} \quad \text{if } |s| \geqq 1, \\ [(0, 0), (1, 1/k)] \quad \text{if} \quad 1/(k+1) \leqq |s| < 1/k \quad (k = 1, 2, \ldots), \\ [(0, 0), (1, 0)] \quad \text{if} \quad s = 0. \end{cases}$$

Then $\Gamma(s)$ is a nonempty closed convex set for each $s \in \mathbf{R}$. One has $\Delta(s) = \{(0, 0)\}$ if $s = 1/k$ $(k = 1, 2, \ldots)$, but $\Delta(s) = \Gamma(s)$ otherwise. Thus

$$\lim \inf_{s' \to 0} \Delta(s') = \{(0, 0)\} \neq \Delta(0),$$

and Δ is not lower semicontinuous at $s = 0$.

Remark 3. It is useful and natural to extend the mixed limit notation beyond the "lim sup inf" in (2.3). Thus

$$\text{(2.5)} \qquad \liminf_{s' \to s} \sup_{y' \to y} g(s', y) = \inf_{Y \in \mathcal{N}(y)} \sup_{U \in \mathcal{N}(s)} \inf_{s' \in S} \sup_{y' \in Y} g(s', y'),$$

$$\text{(2.6)} \qquad \limsup_{s' \to s} \sup_{y' \to y} g(s', y') = \inf_{Y \in \mathcal{N}(y)} \inf_{U \in \mathcal{N}(s)} \sup_{s' \in S} \sup_{y' \in Y} g(s', y'),$$

and similarly "lim inf inf." Obviously "lim sup sup" is the same as "lim sup" with respect to $(s', y') \to (s, y)$, but sometimes, as will be seen in § 5, there is advantage typographically in writing the limit in this way. In the context of Proposition 1, "lim inf inf" would correspond to taking the "lim sup" of the epigraph multifunction Γ, i.e., to closing the graph of Γ. (If $\Gamma(s)$ is taken instead to be the hypograph of $g(s,\cdot)$, the "lim inf" and "lim sup" operations for Γ correspond to "lim inf sup" and "lim sup sup" for g.) From this it is an easy and unambiguous step to mixed limits involving "inf" and "sup" any number of times in any order, for instance,

$$\limsup_{s' \to s} \sup_{r' \to r} \inf_{y' \to y} \sup_{z' \to z} g(s', r', y', z').$$

3. Tangent cones. For any set $C \subset E$ and any $x \in C$, the *tangent cone to C at x* is defined to be the set

$$\text{(3.1)} \qquad T_C(x) = \liminf_{\substack{x' \to_C x \\ t \downarrow 0}} t^{-1}(C - x'),$$

where the notation is used that

$$x' \to_C x \Leftrightarrow x' \to x \text{ with } x' \in C.$$

The multifunction $T_C : x \to T_C(x)$ is thus generated by the "lim inf" operation (2.1) from the multifunction Γ defined on the topological space $S = C \times [0, \infty)$ by

$$\Gamma(x', t) = \begin{cases} t^{-1}(C - x') & \text{if } \ t > 0, \\ E \ \text{ if } \ t = 0. \end{cases}$$

(Actually, T_C is the restriction of the corresponding Δ to the set of pairs (x, t) with $x \in C, t = 0$; for $t > 0$, Δ coincides with Γ, i.e., Γ is always lower semicontinuous.) In terms of neighborhoods, (3.1) takes the form

$$\text{(3.2)} \qquad T_C(x) = \bigcap_{V \in \mathcal{N}(0)} \bigcup_{\substack{X \in \mathcal{N}(x) \\ \lambda > 0}} \bigcap_{\substack{x' \in C \cap X \\ t \in (0, \lambda)}} [t^{-1}(C - x') + V];$$

in other words, $y \in T_C(x)$ if and only if for every symmetric $V \in \mathcal{N}(0)$ there exist $X \in \mathcal{N}(x)$ and $\lambda > 0$ such that

$$x' + t(y + V) \text{ meets } C \text{ for all } x' \in C \cap X, t \in (0, \lambda).$$

An equivalent description in terms of convergence of (generalized) sequences is that $T_C(x)$ consists of the vectors y such that whenever $x_k \to_C x$ and $t_k \downarrow 0$

there exists $y_k \to y$ with $y_k + t_k x_k \in C$. Incidentally, it is easy to see that the right side of (3.2) is unaffected if C is replaced by cl C, and therefore

$$T_{\mathrm{cl}\, C}(x) = T_C(x) \text{ for all } x \in \mathrm{cl}\, C. \tag{3.3}$$

In particular, there is no harm in speaking of $T_C(x)$ as defined by (3.1) even for points $x \in \mathrm{cl}\, C$ with $x \notin C$.

The tangent cone has not previously been recognized as coming from a "lim inf" or defined for general spaces E. Thibault [**55**] in the context of separable Banach spaces used as a starting point the sequential form of the present definition, which in the case of $\mathbf{R}^n$ had been known to be equivalent to Clarke's definition (cf. [**7**, Proposition 3.7]). Hiriart-Urruty has verified in [**27**, Theorem 1] that the equivalence extends to all cases when Clarke's approach is applicable, namely when E is normable. It follows that in such cases $T_C(x)$ is a closed convex cone containing the origin, since these properties are immediate from the original version of Clarke's definition. In [**49**, Theorem 1], we have demonstrated these properties of $T_C(x)$ by a direct argument based on (3.2); this argument is presented in terms of $E = \mathbf{R}^n$ but actually carries over a general E with little more than a broadening of notation.

THEOREM 1. *For any set $C \subset E$ and any $x \in C$, $T_C(x)$ is a closed convex cone in E containing* 0. *If C is convex, $T_C(x)$ coincides with the closed tangent cone to C at x in the sense of convex analysis.*

Proof. The first assertion follows by the argument of [**49**, Theorem 1], as just explained. For the second assertion, recall first that $t^{-1}(C - x)$ is nonincreasing in $t > 0$ when C is convex, so that (3.2) reduces to

$$T_C(x) = \bigcap_{V \in \mathscr{N}(0)} \bigcup_{\lambda > 0} \left[\bigcup_{X \in \mathscr{N}(x)} \bigcap_{x' \in C \cap X} [\lambda^{-1}(C - x') + V]\right]. \tag{3.4}$$

On the other hand, the closed tangent cone in the sense of convex analysis is

$$\mathrm{cl} \bigcup_{\lambda > 0} \lambda^{-1}(C - x) = \bigcap_{V \in \mathscr{N}(0)} \bigcup_{\lambda > 0} [\lambda^{-1}(C - x) + V]. \tag{3.5}$$

To prove equality between (3.4) and (3.5), it is enough to demonstrate for fixed $V \in \mathscr{N}(0)$ and $\lambda > 0$ that

$$\bigcup_{X \in \mathscr{N}(x)} \bigcap_{x' \in C \cap X} [\lambda^{-1}(C - x') + V] = \lambda^{-1}(C - x) + V. \tag{3.6}$$

Since E is locally convex, it can be supposed that V is convex as well as open and symmetric. Trivially, the inclusion $\subset$ holds in (3.6), so we may concentrate on the opposite inequality. For arbitrary $\theta \in (0, 1)$, the set $X_\theta = (x + \lambda\theta V)$ is a neighborhood of x such that

$$x' \in X_\theta \Rightarrow \lambda^{-1}(x' - x) + (1 - \theta)V \subset V.$$

Then one has

$$\begin{aligned} &\bigcap_{x' \in C \cap X_\theta} [\lambda^{-1}(C - x') + V] \\ &\qquad = \bigcap_{x' \in C \cap X_\theta} [\lambda^{-1}(C - x) + \lambda^{-1}(x - x') + V] \\ &\qquad\qquad \supset \lambda^{-1}(C - x) + (1 - \theta)V. \end{aligned}$$

Since this is true for all $\theta \in (0, 1)$ and V is open, we obtain the desired conclusion that $\supset$ holds in (3.6).

4. Subderivatives. Let f be any extended-real-valued function on E, and let x be any point where f is finite. Using the notation

$$(4.1) \qquad (x', \alpha') \downarrow_f x \Leftrightarrow (x', \alpha') \to (x, f(x)) \text{ with } \alpha' \geqq f(x'),$$

we define the *upper subderivative* of f at x with respect to y by

$$(4.2) \quad f^{\uparrow}(x; y) = \limsup_{\substack{(x',\alpha') \downarrow_f x \\ t \downarrow 0}} \inf_{y' \to y} \frac{f(x' + ty') - \alpha'}{t},$$

or in other words by the "lim sup inf" operation (2.3) in the case of the space $S = (\operatorname{epi} f) \times [0, \infty)$ and the function

$$(4.3) \qquad g(x', \alpha', t, y') = \begin{cases} [f(x' + ty') - \alpha']/t & \text{if } t > 0, \\ -\infty & \text{if } t = 0, \end{cases}$$

at the point $(x, f(x), 0, y)$. If f happens to be lower semicontinuous at x, the definition can be expressed in the slightly simpler form

$$(4.4) \qquad f^{\uparrow}(x; y) = \limsup_{\substack{x' \to_f x \\ t \downarrow 0}} \inf_{y' \to y} \frac{f(x' + ty') - f(x')}{t},$$

where

$$(4.5) \qquad x' \to_f x \Leftrightarrow x' \to x \quad \text{and} \quad f(x') \to f(x).$$

Thus in the latter case one has the characterization

$$(4.6) \qquad f^{\uparrow}(x; y) \leqq \beta \Leftrightarrow \begin{cases} \forall Y \in \mathcal{N}(y), \beta' > \beta, \exists X \in \mathcal{N}(x), \delta > 0, \lambda > 0, \\ \forall t \in (0, \lambda), x' \in X \quad \text{with} \quad f(x') \leqq f(x) + \delta, \\ \exists y' \in Y \quad \text{with} \quad f(x' + ty') \leqq f(x') + t\beta'. \end{cases}$$

Of course if f is continuous at x, it suffices to have $x' \to x$ in (4.4), and the conditions in (4.6) involving δ can be dropped.

For the statement of the main result about $f^{\uparrow}(x; y)$, we recall that an extended-real-valued function l on E is *sublinear* if it is convex, positively homogeneous (satisfies $l(\lambda y) = \lambda l(y)$ for all $y \in E$ and $\lambda > 0$), and is not identically $+\infty$. These properties hold if and only if epi l is a nonempty convex cone in $E \times \mathbf{R}$.

We recall also that the one-sided directional derivative

$$(4.7) \qquad f'(x; y) = \lim_{t \downarrow 0} (f(x + ty) - f(x))/t$$

exists for all y when f is convex (although it may be infinite). Convexity implies that the difference quotient in (4.7) is nondecreasing in $t > 0$, and that $f'(x; y)$ is sublinear with respect to y.

THEOREM 2. *The function* $y \to f^\uparrow(x; y)$ *is sublinear and lower semicontinuous, and its epigraph is the tangent cone* $T_{\text{epi} f}(x, f(x))$. *If* f *is convex, then*

$$(4.8) \qquad f^\uparrow(x; y) = \liminf_{y' \to y} f'(x; y') \quad \textit{for all} \quad y \in E,$$

and in fact $f^\uparrow(x; y) = f'(x; y)$ *for any* y *such that* f *is bounded above in a neighborhood of* $x + \lambda y$ *for some* $\lambda > 0$.

Proof. For each (x', α', t) the epigraph of the function $g(x', \alpha', t, \cdot)$ in (4.3) is the set

$$\Gamma(x', \alpha', t) = \begin{cases} t^{-1}[(\text{epi} f) - (x', \alpha')] & \text{if} \quad t > 0, \\ E \times \mathbf{R} & \text{if } t = 0. \end{cases}$$

Since f is obtained from g by "lim sup inf," we have from Proposition 1 that $f^\uparrow(x; \cdot)$ is a lower semicontinuous function whose epigraph is

$$\liminf_{\substack{(x', \alpha') \downarrow_f x \\ t \downarrow 0}} \Gamma(x', \alpha', t) = T_{\text{epi} f}(x, f(x)).$$

The latter is a convex cone by Theorem 1, so $f^\uparrow(x; \cdot)$ is sublinear. In the case of f convex, $T_{\text{epi} f}(x, f(x))$ is by Theorem 1 the same as the closed tangent cone to epi f at $(x, f(x))$ in the sense of convex analysis, i.e., the set

$$\text{cl} \bigcup_{t>0} t^{-1}[(\text{epi} f) - (x, f(x))] = \text{cl epi } f'(x; \cdot),$$

and this equality is expressed by (4.8). If for some y there exist $Y \in \mathcal{N}(y)$, $\lambda > 0$ and $\alpha \in R$ such that $f(x + \lambda y') \leqq \alpha$ for all $y' \in Y$, then

$$f'(x; y') \leqq [f(x + \lambda y') - f(x)]/\lambda \leqq [\alpha - f(x)]/\lambda \quad \text{for all } y' \in Y.$$

Thus the convex function $f'(x; \cdot)$ is bounded above in a neighborhood of y, hence continuous at y, and the "lim inf" in (4.8) is superfluous. This finishes the proof of Theorem 2.

Theorem 1 can be recovered from Theorem 2 as the case where f is the indicator function

$$(4.9) \qquad \psi_C(x) = \begin{cases} 0 \text{ if } x \in C, \\ \infty \text{ if } x \notin C, \end{cases}$$

because then

$$(4.10) \quad \psi_C(x; y) = \begin{cases} 0 \text{ if } y \in T_C(x), \\ \infty \text{ if } y \notin T_C(x). \end{cases}$$

Another observation of some interest concerns the lower semicontinuous hull

$$(4.11) \quad \text{cl}^\downarrow f(x) = \liminf_{x' \to x} f(x').$$

Namely,

$$(4.12) \quad (\text{cl}^\downarrow f)^\uparrow(x; y) = f^\uparrow(x; y) \quad \text{if } f \text{ is l.s.c. at } x.$$

This is the version of (3.3) that holds for tangent cones to epigraphs.

Of course, the geometric proof of sublinearity in Theorem 2, based on Theorem 1, could be translated into a direct argument.

Parallel to the above, we define the *lower subderivative* of f at x with respect to y by

$$(4.13)\quad f^{\downarrow}(x; y) = \liminf_{\substack{(x', \alpha')\uparrow_f x \\ t \downarrow 0}} \sup_{y' \to y} \frac{f(x' + ty') - \alpha'}{t}$$

using the notation

$$(x', \alpha')\uparrow_f x \Leftrightarrow (x', \alpha') \to (x, f(x) \quad \text{with} \quad \alpha' \leqq f(x').$$

If f is upper semicontinuous at x, this reduces to

$$(4.14)\quad f^{\downarrow}(x; y) = \liminf_{\substack{x' \to_f x \\ t \downarrow 0}} \sup_{y' \to y} \frac{f(x' + ty') - f(x')}{t}.$$

The obvious analog of Theorem 1 holds for $f^{\downarrow}(x; y)$ and concave functions.

The relationship between $f^{\downarrow}$ and $f^{\uparrow}$ is not trivial and will be addressed in the next section.

5. Lipschitzian and directionally Lipschitzian functions. It will now be shown that in many of the most important cases both upper and lower subderivatives can be reduced essentially to the simpler expression

$$(5.1)\quad f^{\circ}(x; y) = \limsup_{\substack{(x', \alpha')\downarrow_f x \\ t \downarrow 0}} \frac{f(x' + ty) - \alpha'}{t},$$

which will be called the *generalized Clarke derivative of f at x with respect to y*. (It is still assumed, of course, that f is an extended-real-valued function on E which is finite at x.) If f is l.s.c. at x, the formula becomes

$$(5.2)\quad f^{\circ}(x; y) = \limsup_{\substack{x' \to_f x \\ t \downarrow 0}} \frac{f(x' + ty) - f(x')}{t},$$

and if f is actually continuous at x the convergence $x' \to_f x$ can be simplified to $x' \to x$. The latter version of the formula is the one introduced by Clarke [**6**], [**7**], [**8**], who employed it only for locally Lipschitzian functions on normed spaces.

In the general case, f is said to be *Lipschitzian around* x if there is a neighborhood of x on which f is finite and satisfies, for some continuous seminorm p on E and constant $\mu \geqq 0$, the inequality

$$(5.3)\quad |f(x'') - f(x')| \leqq \mu p(x'' - x') \quad \text{for all} \quad x', x''.$$

(If E is a normed space, p can always be taken to be the norm in question.) If f is Lipschitzian around each point of a set C, it is said to be *locally Lipschitzian on C*. Obviously f is continuous on a neighborhood of x if it is Lipschitzian around x, so that formula (5.2) is applicable (with $x' \to x$) and yields via (5.3) the inequality

$$f^\circ(x; y) \leqq \mu p(y) \quad \text{for all} \quad y \in E.$$

Locally Lipschitzian functions on general spaces have been treated in these terms by Lebourg [**37**].

We shall say that f is *directionally Lipschitzian at x with respect to a vector y* if (f is finite at x and)

$$(5.4) \qquad \limsup_{\substack{(x',\alpha')\downarrow_f x \\ t\downarrow 0}} \sup_{y' \to y} \frac{f(x'+ty') - \alpha'}{t} < \infty,$$

a condition which can be simplified when f is l.s.c. at x to

$$(5.5) \qquad \limsup_{\substack{x' \to_f x \\ t\downarrow 0}} \sup_{y' \to y} \frac{f(x'+ty') - f(x')}{t} < \infty.$$

An easy fact to verify, and which gives rise to this terminology, is that f is Lipschitzian around x if and only if it is directionally Lipschitzian at x with respect to $y = 0$. Accordingly, we shall say f is *directionally Lipschitzian* at x if there is at least one y, not necessarily 0, such that f is directionally Lipschitzian at x with respect to y.

The geometric approach to this concept and its implications for the derivatives $f^\circ(x; y)$ lies with the *hypertangent cone* $H_C(x)$ to a set C at a point $x \in C$. This consists of the vectors y' such that there exist $X \in \mathcal{N}(x)$ and $\lambda > 0$ with

$$(5.7) \qquad x' + ty' \in C \quad \text{for all} \quad x' \in C \cap X \quad \text{and} \quad t \in (0, \lambda).$$

Expanding somewhat on the terminology in [**49**], we shall say that C is *epi-Lipschitzian at x with respect to y* if property (5.7) holds simultaneously for all y' in some neighborhood $Y \in \mathcal{N}(y)$. When C is closed and $y \neq 0$, this means that C can be represented locally as the epigraph of a Lipschitzian function (cf. [**49**, § 4]).

THEOREM 3. *For any extended-real-valued function f on E and any point x where f is finite, the function $y \to f^\circ(x; y)$ is sublinear. If f is directionally Lipschitzian at x, then so is $-f$, and one has*

$$(5.8) \qquad f^\uparrow(x; y) = -f^\downarrow(x; -y) = \liminf_{y' \to y} f^\circ(x; y') \text{ for every } y \in E.$$

In this case the vectors y with respect to which f is directionally Lipschitzian at x are those belonging to

$$(5.9) \qquad \operatorname{int}\{y \mid f^\uparrow(x; y) < \infty\},$$

and at each such y the function $f^\circ(x; \cdot)$ is continuous with

$$(5.10) \quad f^\uparrow(x; y) = -f^\downarrow(x; -y) = f^\circ(x; y) = \limsup_{\substack{(x',\alpha')\downarrow_f x \\ t\downarrow 0}} \sup_{y' \to y} \frac{f(x'+ty') - \alpha'}{t}.$$

Before proving Theorem 3, we state two consequences.

COROLLARY 1. *If f is Lipschitzian around x, then the function $y \to f^\circ(x; y)$ is finite, sublinear and continuous with*

$$(5.11) \quad f^\uparrow(x; y) = -f^\downarrow(x; -y) = f^\circ(x; y) \text{ for all } y \in E.$$

COROLLARY 2. *For any set $C \subset E$ and any $x \in C$, the hypertangent cone $H_C(x)$ to C at x is a convex cone containing O. If C is epi-Lipschitzian at x with respect to some y, then the vectors y with this property are precisely those belonging to* int $H_C(x)$, *and one has* $T_C(x) = \text{cl}\, H_C(x)$. *If in addition x is a boundary point of C, then the set $C' = (E \backslash C) \cup \{x\}$ is likewise epi-Lipschitzian at x, and $T_{C'}(x) = -T_{C'}(x)$.*

Corollary 1 is the case where f is directionally Lipschitzian at x with respect to every $y \in E$. The assertion in Corollary 1 about $f^\circ(x; \cdot)$ being finite, sublinear and continuous when f is Lipschitzian is not new; cf. [**6**], [**7**], [**8**] and [**37**]. Corollary 2 is obtained by taking f to be the indicator ψ_C in (4.9).

Proof of Theorem 3. We begin by demonstrating that the theorem can be derived in turn by applying Corollary 2 to $C = \text{epi}\, f$ at $(x, f(x))$, so that a direct proof of Corollary 2 will suffice.

The hypertangent cone $H_{\text{epi}\, f}(x, f(x))$ consists of the vectors (y, β) in $E \times \mathbf{R}$ such that there exist $X \in \mathcal{N}(x)$, $\delta > 0$ and $\lambda > 0$ with

$$(5.12) \quad (x', \alpha') + t(y, \beta) \in \text{epi}\, f \quad \text{for all } (x', \alpha') \in \text{epi}\, f, \quad t \in (0, \lambda), \quad \text{with} \quad x' \in X, \quad |\alpha' - f(x)| \leqq \delta.$$

From this and the definition (5.1) of f° it is readily seen that

$$(5.13) \quad f^\circ(x; y) = \inf\{\beta \in \mathbf{R} \mid (y, \beta) \in H_{\text{epi}\, f}(x, f(x))\}, \quad H_{\text{epi}\, f}(x, f(x)) \supset \{(y, \beta) \in E \times \mathbf{R} \mid f^\circ(x; y) < \beta\}.$$

According to Corollary 2, $H_{\text{epi}\, f}(x, f(x))$ is a nonempty convex cone, hence so is epi $f^\circ(x; \cdot)$ by (5.13), and this means that $f^\circ(x; \cdot)$ is a sublinear function.

In similar fashion, one sees that the vectors (y, β) with respect to which epi f is epi-Lipschitzian are those satisfying

$$(5.14) \quad \limsup_{\substack{(x', \alpha') \downarrow_f x \\ t \downarrow 0}} \sup_{y' \to y} \frac{f(x' + ty') - \alpha'}{t} < \beta.$$

Corollary 2 tells us that when the set of such vectors (y, β) is nonempty, it coincides with the interior of $H_{\text{epi}\, f}(x, f(x))$. This is by (5.13) the same as the interior of epi $f^\circ(x; \cdot)$, i.e., the set of $(y, \beta) \in E \times \mathbf{R}$ such that $f^\circ(x; \cdot)$ is bounded above by β on a neighborhood of y. Since $f^\circ(x; \cdot)$ is a convex function, it is continuous at every point of the set

$$(5.15) \quad \text{int}\{y' \mid f^\circ(x; y') < \infty\},$$

if it is bounded above in a neighborhood of one point. Of course, y satisfies (5.14) for some β if and only if f is directionally Lipschitzian at x with respect to y. Therefore, such vectors (if any exist) are the ones belonging to (5.15), and at each such y one has

$$\limsup_{\substack{(x',\alpha')\downarrow_f x \\ t\downarrow 0}} \sup_{y'\to y} \frac{f(x'+ty)-\alpha'}{t} = f^\circ(x;y) = \lim_{y'\to y} f^\circ(x;y').$$

We turn now to (5.8), which must be derived from Corollary 2 under the assumption that $\operatorname{epi} f$ is epi-Lipschitzian at $(x, f(x))$. The epigraph of the function

$$y \to \liminf_{y'\to y} f^\circ(x;y')$$

is by the above the closure of $H_{\operatorname{epi} f}(x, f(x))$. We already know from Theorem 1 that $\operatorname{epi} f^\uparrow(x;\cdot)$ coincides with $T_{\operatorname{epi} f}(x, f(x))$, and by the same token the hypograph $\operatorname{hyp} f^\downarrow(x;\cdot)$ coincides with $T_{\operatorname{hyp} f}(x, f(x))$. Thus the epigraph of the function $y \to -f^\downarrow(x;-y)$ is $-T_F(x, f(x))$, where $F = (E \times \mathbf{R})\backslash(\operatorname{epi} f)$ (recall (3.3)), and (5.8) is the equation

$$T_{\operatorname{epi} f}(x, f(x)) = -T_F(x, f(x)) = \operatorname{cl} H_{\operatorname{epi} f}(x, f(x)),$$

which follows from Corollary 2. Obviously (5.8) and the continuity of $f^\circ(x;\cdot)$ on the set (5.15) imply that the latter set is the same as the one in (5.9), and with this observation we have covered all the assertions of Theorem 3.

We are left now with the task of proving Corollary 2 directly. Clearly $H_C(x)$ contains O and is closed under multiplication by positive scalars. If y_1 and y_2 belong to $H_C(x)$ there exist $X_i \in \mathcal{N}(x)$ and $\lambda_i > 0$ such that

(5.16) $\quad x' + ty_i \in C \quad$ for all $\quad x' \in C \cap X_i, \quad t \in (0, \lambda_i) \quad (i = 1, 2).$

Choose $X \in \mathcal{N}(x)$ and $\lambda > 0$ small enough that $X \subset X_1$, $\lambda < \lambda_1$ and

(5.17) $\quad X + ty_1 \subset X_2 \quad$ for all $\quad t \in (0, \lambda).$

Then for $x' \in C \cap X$ and $t \in (0, \lambda)$ one has $x' + ty_1 \in C \cap X_2$ by (5.16) and (5.17), so that $(x' + ty_1) + ty_2 \in C$ by (5.16). Thus

(5.18) $\quad x' + t(y_1 + y_2) \in C \quad$ for all $\quad x' \in C \cap X, \quad t \in (0, \lambda),$

and it follows that $H_C(x)$ is a convex cone.

Let K denote the set of all y with respect to which C is epi-Lipschitzian at x, and assume $K \neq \emptyset$. It is trivial that K is an open set containing all positive multiples of its elements, and that $K \subset H_C(x) \subset T_C(x)$. To prove that

$$K = \operatorname{int} H_C(x) \quad \text{and} \quad T_C(x) = \operatorname{cl} H_C(x),$$

it will be enough to prove $\operatorname{int} T_C(x) \subset K$ (because $T_C(x)$ is convex by Theorem 1).

The inclusion $\operatorname{int} T_C(x) \subset K$ can be established by verifying

(5.19) $\quad K + T_C(x) \subset K,$

for if the latter holds and $y \in \operatorname{int} T_C(x)$, then for arbitrary $y_0 \in K$ there exists $\lambda > 0$ with $y - \lambda y_0 \in \operatorname{int} T_C(x)$; since also $\lambda y_0 \in K$ and $\lambda y_0 + (y - \lambda y_0) = y$, it follows from (5.19) that $y \in K$.

Let $y_1 \in K$ and $y_2 \in T_C(x)$. In order to verify (5.19), we must demonstrate that $y_1 + y_2 \in K$, i.e., that there exist $V \in \mathscr{N}(0)$, $X \in \mathscr{N}(x)$ and $\lambda > 0$ such that

(5.20) $\quad (C \cap X) + t(y + y_2 + V) \subset C \quad \text{for all} \quad t \in (0, \lambda).$

Since $y_1 \in K$, we know there exist $V_1 \in \mathscr{N}(0)$, $X \in \mathscr{N}(x)$ and $\lambda_1 > 0$ such that

(5.21) $\quad (C \cap X_1) + t(y_1 + V_1) \subset C \quad \text{for all} \quad t \in (0, \lambda_1).$

Select $V \in \mathscr{N}(0)$ small enough that

(5.22) $\quad V + V \subset V_1 \quad \text{and} \quad x + V + t(y_2 - V) \subset X_1 \quad \text{for all} \quad t \in [0, 1].$

Since $y_2 \in T_C(x)$, we have from (3.2) the existence of $X_2 \in \mathscr{N}(x)$ and $\lambda_2 > 0$ such that

(5.23) $\quad y_2 \in [t^{-1}(C - x') + V] \quad \text{for all} \quad x' \in C \cap X_2, \quad t \in (0, \lambda_2).$

Now let

(5.24) $\quad X = X_2 \cap (x + V), \quad \lambda = \min\{\lambda_1, \lambda_2, 1\}.$

We claim that (5.20) holds, as desired.

Indeed, suppose $x' \in C \cap X$, $t \in (0, \lambda)$. Then (5.23) is applicable, so there exists $v \in V$ such that $y_2 - v \in t^{-1}(C - x')$, or in other words

$$x' + t(y_2 - v) \in C.$$

Furthermore,

$$x' + t(y_2 - v) \in (x + V) + t(y_2 - V) \subset X_1$$

by (5.24) and (5.22). Then (5.21) and the first condition in (5.22) imply

$$C \supset x' + t(y_2 - v) + t(y_1 + V_1) \supset x' + t(y_1 + y_2 + V),$$

and this verifies (5.20).

The only thing remaining is the assertion of Corollary 2 about C' when x is a boundary point of C. Suppose y is a vector with respect to which C is epi-Lipschitzian at x: there exist $Y \in \mathscr{N}(y)$, $X \in \mathscr{N}(x)$, $\lambda > 0$ such that

(5.25) $\quad x' + tY \subset C \quad \text{for all} \quad x \in C \cap X, \quad t \in (0, \lambda).$

Choose open $Y' \in \mathscr{N}(y)$, $X' \in \mathscr{N}(x)$ and $\lambda' \in (0, \lambda)$ such that

(5.26) $\quad Y' \subset Y \quad \text{and} \quad X' - tY' \subset X \quad \text{for all} \quad t \in (0, \lambda').$

Then

(5.27) $x' - tY' \subset C'$ for all $x' \in C' \cap X'$, $t \in (0, \lambda')$,

for if not there would exist $x' \in C' \cap X'$ and $t \in (0, \lambda')$ such that $x' - tY'$ contains a point $x'' \notin C'$. In this event we would have $x'' \in X$ by (5.26) and $x'' \in C$; furthermore $x' \in (x'' + tY') \cap C'$. The latter implies $x'' + tY' \not\subset C$, because the set $x'' + tY'$ is open and the only point of C' not in C is x, which is a boundary point of C. This contradicts (5.25), since $x'' \in C \cap X$, $Y' \subset Y$ and $t \in (0, \lambda') \subset (0, \lambda)$. Thus (5.27) is true, and since $-Y' \in \mathcal{N}(-y)$ we conclude that C' is epi-Lipschitzian at x with respect to $-y$.

This argument can be reversed to show that in fact C' is epi-Lipschitzian at x with respect to $-y$ if and only if C is epi-Lipschitzian at x with respect to y. Applying to C' the part of Corollary 2 already proved for C, we obtain $\operatorname{int} H_{C'}(x) = -\operatorname{int} H_C(x) \neq \emptyset$ and $T_{C'}(x) = \operatorname{cl} H_{C'}(x)$. Moreover $H_{C'}(x)$ is convex, like $H_C(x)$, so

$$\operatorname{cl} H_{C'}(x) = \operatorname{cl}(\operatorname{int} H_{C'}(x)) = -\operatorname{cl}(\operatorname{int} H_C(x)) = -\operatorname{cl} H_C(x).$$

Therefore $-T_{C'}(x) = \operatorname{cl} H_C(x) = T_C(x)$, and the proof of Theorem 3 is finished.

6. Criteria for directionally Lipschitzian behavior. As a complement to Theorem 3, we now furnish several conditions guaranteeing that f is Lipschitzian or directionally Lipschitzian at x.

PROPOSITION 2. *Suppose that E is finite-dimensional, and that f is lower semicontinuous in a neighborhood of x and finite at x. Then f is directionally Lipschitzian at x if and only if the set*

(6.1) $D(x) = \{y \in E \mid f^\uparrow(x; y) < \infty\}$

is not included in some hyperplane of E. Furthermore, f is Lipschitzian around x if and only if $D(x) = E$.

Proof. Since the conclusion involves only local properties of f at x, we can replace f by its l.s.c. hull $\operatorname{cl}^\downarrow f$ if necessary and thereby reduce to the case where f is l.s.c. on all of E. Then $\operatorname{epi} f$ is a closed set in $E \times \mathbf{R}$. We have shown in [**49**, Theorems 2, 3] that a closed set C in a finite-dimensional space is epi-Lipschitzian with respect to y at a point x if and only if $y \in \operatorname{int} T_C(x)$. According to the proof of Theorem 3, f is directionally Lipschitzian at x with respect to y if and only if $\operatorname{epi} f$ is epi-Lipschitzian at $(x, f(x))$ with respect to (y, β) for some $\beta \in \mathbf{R}$, and the latter condition is therefore equivalent to

(6.2) $(y, \beta) \in \operatorname{int} \operatorname{epi} f^\uparrow(x; \cdot)$,

because $\operatorname{epi} f^\uparrow(x; \cdot)$ is the cone $T_{\operatorname{epi} f}(x, f(x))$ (Theorem 2). Since $f^\uparrow(x; \cdot)$ is a sublinear function (Theorem 2), the set $D(x)$ is a convex cone. Since E is finite-dimensional, (6.2) is equivalent to $y \in \operatorname{int} D(x)$ and $f^\uparrow(x; y) < \beta$

[44, § 6]. A convex set in a finite-dimensional space has nonempty interior if and only if it is not included in some hyperplane.

Of course, f is actually Lipschitzian around x if and only if f is directionally Lipschitzian with respect to $y = 0$. Since $D(x)$ is a cone, the condition $0 \in \operatorname{int} D(x)$ is equivalent to $D(x) = E$.

PROPOSITION 3. *Suppose f is convex on E and finite at x. Then f is directionally Lipschitzian at x if and only if there is an open subset of E on which f is bounded above.*

More specifically, f is directionally Lipschitzian at x with respect to y if and only if f is bounded above on a neighborhood of $x + \lambda y$ for some $\lambda > 0$. In particular, f is Lipschitzian around x if and only if f is bounded above on a neighborhood of x.

Proof. This is just the epigraph version of the assertion that a convex set C is epi-Lipschitzian at a point $x \in C$ relative to y if and only if $x + \lambda y \in \operatorname{int} C$ for some $\lambda > 0$. In this geometric assertion, the necessity of the condition is trivial. For the sufficiency, suppose $x + \lambda(y + V) \subset C$, where $\lambda > 0$ and $V \in \mathcal{N}(0)$. Choose $U \in \mathcal{N}(0)$ such that $\lambda^{-1}U + U \subset V$. Then $x' \in (x + U)$ implies

$$x' + \lambda(y + U) \subset x + \lambda(y + \lambda^{-1}U + U) \subset x + \lambda(y + V) \subset C.$$

Since C is convex, it follows that

$$x' + t(y + U) \subset C \quad \text{for all} \quad x' \in C \cap (x + U), \quad t \in (0, \lambda).$$

Therefore C is epi-Lipschitzian at x with respect to y.

COROLLARY. *The assertions in Proposition 3 hold for f concave, instead of convex, if "bounded above" is replaced by "bounded below," and y is replaced by $-y$.*

Proof. Apply Proposition 3 to $-f$ and invoke Theorem 3.

PROPOSITION 4. *Suppose f is nondecreasing with respect to the partial ordering induced on E by a nonempty closed convex cone $K : f(x') \leqq f(x'')$ when $x' \leqq_K x''$. If $\operatorname{int} K \neq \emptyset$ and f is finite at x, then f is directionally Lipschitzian at x with*

(6.3) $\quad f^{\uparrow}(x; y) \leqq 0$ *for all* $y \leqq_K 0$.

Proof. Suppose $-y \in \operatorname{int} K$. Then there exists $Y \in \mathcal{N}(y)$ such that $-Y \subset K$. For all $y' \in Y$ and $t \geqq 0$, one has $-ty' \in K$, so that $x' + ty' \leqq_K x'$ for all x'. Therefore

$$(f(x' + ty') - f(x'))/t \leqq 0 \quad \text{for all} \quad y' \in Y, \quad x' \in E, \quad \lambda \geqq 0,$$

and consequently

$$\limsup_{\substack{(x', \alpha') \downarrow_f x \\ t \downarrow 0}} \sup_{y' \to y} \frac{f(x' + ty) - \alpha'}{t} \leqq 0.$$

In particular, f is directionally Lipschitzian at x with respect to y, and $f^{\uparrow}(x; y) \leqq 0$. Since this holds for arbitrary y belonging to $-\operatorname{int} K$, and since the function $f^{\uparrow}(x; \cdot)$ is l.s.c. (Theorem 2), we conclude that $f^{\uparrow}(x; y) \leqq 0$ for every y belonging to the closure of $-\operatorname{int} K$, which is just $-K$ because K is convex and closed.

PROPOSITION 5. *Suppose there is a neighborhood X of x such that, for all $x' \in X$ and $y \in E$, $f(x')$ is finite and the one-sided derivative $f'(x'; y)$ exists. Suppose also that the function $(x', y) \mapsto f'(x'; y)$ is bounded on a neighborhood of $(x, 0)$. Then f is Lipschitzian around x. If in addition the function $x' \to f'(x'; y)$ is continuous at x for each $y \in E$ (and finite at x), then the function $y \to f'(x; y)$ is linear and continuous and*

$$(6.4) \qquad \lim_{\substack{x' \to x, y' \to y \\ t \downarrow 0}} \frac{f(x' + ty') - f(x')}{t} = f'(x; y) \quad \text{for all} \quad y \in E.$$

In this event $f'(x; y) = f^{\circ}(x; y) = f^{\uparrow}(x; y) = f^{\downarrow}(x; y)$ for all $y \in E$.

Proof. By hypothesis there exist convex $X' \in \mathcal{N}(x)$, $Y \in \mathcal{N}(y)$ and $\mu > 0$ such that f is finite on X' and

$$(6.5) \qquad f'(x'; y) \leqq \mu \quad \text{for all} \quad x' \in X', \quad y \in Y.$$

Choose $X_0 \in \mathcal{N}(x)$, $Y_0 \in \mathcal{N}(y)$, such that $X_0 \subset X'$, $Y_0 \subset Y$, and $X_0 + Y_0 \subset X'$. Then for $x' \in X_0$, $y \in Y_0$, we have $x' + ty \in X'$ for all $t \in [0, 1]$ (by the convexity of X'), so that the function $\phi(t) = f(x' + ty)$ is right differentiable on $[0, 1]$ with right derivative $f'(x' + ty; y) \leqq \mu$. Since a right differentiable function is the integral of its right derivative (cf. [**54**, p. 271]), it follows that $\phi(t) - \phi(0) \leqq t$ for all $t \in (0, 1)$. Thus

$$[f(x' + ty) - f(x')]/t \leqq \mu \quad \text{for all} \quad x' \in X_0, \quad y' \in Y_0, \quad t \in (0, 1).$$

Therefore f is Lipschitzian around x.

If in addition $f'(\cdot\,; y)$ is continuous at x, there exists for any $\epsilon > 0$ a neighborhood $X(y, \epsilon)$ of x on which f is finite and satisfies

$$f'(x; y) - \epsilon \leqq f'(x'; y) \leqq f'(x; y) + \epsilon \quad \text{for all} \quad x' \in X(y, \epsilon).$$

By an integration argument like the one just given, we obtain that for x' sufficiently near x and for $t > 0$ sufficiently small,

$$t[f'(x; y) - \epsilon] \leqq f(x' + ty) - f(x') \leqq t[f'(x; y) + \epsilon].$$

This being true for arbitrary $\epsilon > 0$, we have (6.4), at least in terms of the limit in $x' \to x$ and $t \downarrow 0$, with $y' \equiv y$. The limit in $y' \to y$ can be added harmlessly, because f is Lipschitzian around x. In particular, (6.4) implies

$$f^{\prime}(x;y) = \limsup_{\substack{x'\to x\\ t\downarrow 0}} \frac{f(x'+ty')-f(x')}{t} = f^{\circ}(x;y),$$

$$f'(x;y) = \liminf_{\substack{x''\to x\\ t\downarrow 0}} \frac{f(x'')-f(x''-ty)}{t} = -f^{\circ}(x;-y).$$

But $f^{\circ}(x;\cdot)$ is a continuous sublinear function by Corollary 1 of Theorem 3, so these two equations imply $f'(x;\cdot)$ is continuous and linear. They also yield the last equation in Proposition 5 by way of the same corollary.

7. Normal vectors and subgradients. Since the tangent cone $T_C(x)$ to a set $C \subset E$ at a point $x \in C$ is a nonempty closed convex cone (Theorem 1), it is polar to a certain nonempty weak*-closed convex cone $N_C(x)$ in the dual space E^*:

$$\begin{aligned} (7.1)\qquad N_C(x) &= \{z \in E^* \mid \langle y, z\rangle \leqq 0 \quad \text{for all} \quad y \in T_C(x)\},\\ T_C(x) &= \{y \in E \mid \langle y, z\rangle \leqq 0 \quad \text{for all} \quad z \in N_C(x)\}. \end{aligned}$$

The set $N_C(x)$ is defined to be the *normal cone to C at x*.

If C is convex, this definition of normal cone agrees with the one in convex analysis, because $T_C(x)$ is the same as the tangent cone in convex analysis (Theorem 1); then

$$(7.2)\qquad N_C(x) = \{z \in E^* \mid \langle z, x'-x\rangle \leqq 0 \quad \text{for all} \quad x' \in C\}.$$

If E is a normed space, $N_C(x)$ is identical to Clarke's normal cone, even though he defined it quite differently, because $T_C(x)$ is identical to the tangent cone he introduced in such spaces (as cited in § 3), and his normal and tangent cones were polar to each other.

The duality between tangents and normals can be extended by Theorem 2 into a duality between subderivatives and subgradients. There are only two possibilities for a sublinear function l on E which is lower semicontinuous:

(i) (proper case): $l(y) > -\infty$ for all y, $l(0) = 0$;
(ii) (improper case): $l(y) = \pm\infty$ for all y, $l(0) = -\infty$.

In the improper case there is little to be said, except that the set $\{y \mid l(y) = -\infty\}$ is a closed convex cone. The proper case, however, characterizes the support functions of the nonempty weak*-closed convex sets in E^* (cf. [**31**]): l is of this type if and only if there is a nonempty weak*-closed convex set $G \subset E^*$ such that

$$(7.3)\qquad l(y) = \sup\{\langle y, z\rangle \mid z \in G\}.$$

This set is unique and is determined from l by

$$(7.4)\qquad G = \{z \in E^* \mid \langle y, z\rangle \leqq l(y) \quad \text{for all} \quad y \in E\}.$$

Obviously G consists of a single element z if and only if l is linear, i.e., has the additional property that $l(y) = -l(-y)$ for all y; then $l(y) \equiv \langle y, z\rangle$. Furthermore, G is a bounded nonempty set if and only if l is finite on all of E; then G is weak*-compact and the "sup" in (7.3) can be strengthened to "max."

Bearing this in mind, we define the set of *subgradients* of a function f at a point x (where f is finite) by

$$\partial f(x) = \{z \in E^* \mid \langle y, z\rangle \leqq f^\uparrow(x; y) \quad \text{for all} \quad y \in E\}. \tag{7.5}$$

Theorem 2 then gives the following result.

THEOREM 4. *Let f be any extended-real-valued function on E, and let x be any point where f is finite. Then $\partial f(x)$ is a weak*-closed convex subset of E^* and*

$$\partial f(x) = \{z \in E^* \mid (z, -1) \in N_{\mathrm{epi}\, f}(x, f(x))\}. \tag{7.6}$$

If $f^\uparrow(x; 0) = -\infty$, then $\partial f(x)$ is empty, but otherwise $\partial f(x)$ is nonempty and

$$f^\uparrow(x; y) = \sup\{\langle y, z\rangle \mid z \in \partial f(x)\} \quad \textit{for all} \quad y \in E. \tag{7.7}$$

Proof. Since the function $l(y) = f^\uparrow(x; y)$ is l.c.s. and sublinear by Theorem 2, all these facts except (7.6) are immediate from the cited properties of such functions. As for (7.6), we recall from Theorem 2 that the epigraph of $f^\uparrow(x; \cdot)$ is $T_{\mathrm{epi}\, f}(x, f(x))$, and hence the normal cone $N_{\mathrm{epi}\, f}(x, f(x))$ consists of the pairs $(z, \gamma) \in E^* \times \mathbf{R}$ such that

$$\langle y, z\rangle + \beta\gamma \leqq 0 \quad \text{for all} \quad (y, \beta) \in \mathrm{epi}\, f^\uparrow(x; \cdot).$$

In particular, $(z, -1)$ belongs to $N_{\mathrm{epi}\, f}(x, f(x))$ if and only if

$$\langle y, z\rangle \leqq \beta \quad \text{for all} \quad (y, \beta) \in E \times \mathbf{R} \quad \text{with} \quad \beta \geqq f^\uparrow(x; y),$$

and this means by definition (7.5) that $z \in \partial f(x)$.

COROLLARY 1. *The subgradient set $\partial f(x)$ consists of a single element z if and only if $-f^\uparrow(x; -y) = f^\uparrow(x; y)$ for all y; then $f^\uparrow(x; y) \equiv \langle y, z\rangle$. (This is true in particular if f satisfies the assumptions in Proposition* 5.)

COROLLARY 2. *The subgradient set $\partial f(x)$ is nonempty weak*-compact if and only if $f^\uparrow(x; y)$ is finite for all y, in which event* "sup" *can be replaced by* "max" *in* (7.7). (*This is true in particular if f is Lipschitzian around x.*)

The parenthetical comment in Corollary 2 is based on Corollary 1 of Theorem 3. More will be said about the Lipschitzian case in the next section.

THEOREM 5. *If f is a convex function on E, and x is a point where f is finite, then $\partial f(x)$ agrees with the subgradient set in the sense of convex analysis:*

$$\begin{aligned} \partial f(x) &= \{z \in E^* \mid \langle y, z\rangle \leqq f'(x; y) \textit{ for all } y \in E\} \\ &= \{z \in E^* \mid f(x') \geqq f(x) + \langle x' - x, z\rangle \textit{ for all } x' \in E\}. \end{aligned} \tag{7.8}$$

Proof. The first equality is immediate from definition (7.5) and the assertion in Theorem 2 that (4.8) holds in the convex case. The second equality is well known in convex analysis and corresponds to the fact that the difference quotient in (4.8) is nondecreasing in $t > 0$.

While Theorem 5 shows that the definition of $\partial f(x)$ by (7.5) is in harmony with the well established definition in the case where f is convex, Theorem 4 has the same effect relative to Clarke's definition in the case where E is a normed space (cf. [**6**], [**7**], [**8**]). Clarke's approach involves first defining subgradients in a special way for locally Lipschitzian functions, applying this to the distance function associated with a set C to get a concept of the normal cone $N_C(x)$, and finally using (7.6) as the definition of $\partial f(x)$; the tangent cone $T_C(x)$ is defined by Clarke as the polar of $T_C(x)$ (so that (7.1) holds). The net result for $\partial f(x)$ must be the same, thanks to Hiriart-Urruty's proof [**33**] that Clarke's cone $T_C(x)$ can be described directly (when E is normed) by the sequential form of the present definition.

Of course, in the case of an indicator function ψ_C one has

$$\partial\psi_C(x) = N_C(x) \quad \text{for all} \quad x \in C. \tag{7.9}$$

8. Lower versus upper subgradients. The set $\partial f(x)$ has been defined in terms of $f^\uparrow(x; y)$, but the subderivatives $f^\downarrow(x; y)$ are capable of an equal role. When f is finite, we define

$$\tilde{\partial} f(x) = \{z \in E^* \mid \langle y, z\rangle \geqq f^\downarrow(x; y) \quad \text{for all} \quad y \in E\}. \tag{8.1}$$

Parallel to Theorem 4 we then have the fact that $\tilde{\partial} f(x)$ is a weak*-closed convex subset of E^*. If $f^\downarrow(x; y) = +\infty$, then $\tilde{\partial} f(x)$ is empty; otherwise $\tilde{\partial} f(x)$ is nonempty and

$$f^\downarrow(x; y) = \inf\{\langle y, z\rangle \mid z \in \tilde{\partial} f(x)\} \quad \text{for all} \quad y \in E. \tag{8.2}$$

The elements of $\tilde{\partial} f(x)$ could be called "upper" subgradients, to distinguish them from the "lower" subgradients in $\partial f(x)$. But it is clear that

$$\tilde{\partial} f(x) = -\partial(-f)(x), \tag{8.3}$$

so no really new concept is involved and a systematic insistence on "lower" and "upper" would be tedious. Comparing the sets in geometric terms by way of (7.6) with

$$F = \operatorname{epi} f \quad \text{and} \quad F' = [(E \times \mathbf{R})\backslash(\operatorname{epi} f)] \cup \{(x, f(x))\}, \tag{8.4}$$

one has

$$\begin{aligned} \partial f(x) &= \{z \mid (z, -1) \in N_F(x, f(x))\}, \\ \tilde{\partial} f(x) &= \{z \mid (z, -1) \in -N_{F'}(x, f(x))\}. \end{aligned} \tag{8.5}$$

It is not necessarily true that $\partial(-f)(x) = -\partial f(x)$, and therefore $\tilde{\partial} f(x)$

and $\partial f(x)$ can sometimes be different sets, neither included in the other. This somewhat unsatisfactory state of affairs has led Hiriart-Urruty ([**26**], [**27**]) to introduce the *symmetrized* subgradient set $Df(x)$ corresponding to the lower semicontinuous sublinear function l on E for which

$$\text{(8.6)} \qquad \operatorname{epi} l = T_F(x, f(x)) \cap -T_{F'}(x, f(x))$$

(with F, F' as in (8.5)). This means that

$$\text{(8.7)} \qquad Df(x) = \{z \mid \langle y, z\rangle \leqq l(y) \quad \text{for all} \quad y \in E\}, \quad \text{where}$$
$$l(y) = \max\{f^{\uparrow}(x; y), -f^{\downarrow}(x; -y)\}.$$

While l (and by implication $Df(x)$) does have a description of sorts in terms of limits of "difference quotients" of f, it is a rather complicated one and hard to work with, and this disadvantage must be weighed against the good effects wrought by the property $D(-f)(x) = -Df(x)$. Note from (8.5), (8.6), (8.7) that if both $\partial f(x)$ and $\tilde{\partial} f(x)$ are nonempty, then

$$Df(x) = \operatorname{cl}\operatorname{co}[\partial f(x) \cup \tilde{\partial} f(x)].$$

It will now be demonstrated that the cases where $\partial f(x)$ and $\tilde{\partial} f(x)$ are both nonempty, but different, must be regarded as somewhat pathological. For most purposes, therefore, a single concept of the subgradient set will suffice.

To state the result, we say that a nonempty weak*-closed convex set $Z \subset E^*$ is *nonasymptotic relative to a vector* $y \in E$ if for some $\beta \in R$ the set $\{z \in Z \mid \langle y, z\rangle \geqq \beta\}$ is nonempty and weak*-compact.

THEOREM 6. *Let x be a point where f is not only finite but directionally Lipschitzian. Then*

$$\text{(8.8)} \qquad \partial f(x) = \tilde{\partial} f(x) = Df(x)$$
$$= \{z \in E^* \mid \langle y, z\rangle \leqq f^0(x; y) \textit{ for all } y \in E\},$$

$$\text{(8.9)} \qquad \sup\{\langle y, z\rangle \mid z \in \partial f(x)\} = \liminf_{y' \to y} f^0(x; y').$$

If $\partial f(x) \neq \emptyset$, then the vectors y with respect to which $\partial f(x)$ is nonasymptotic are those with respect to which f is directionally Lipschitzian at x, and for each such y

$$\text{(8.13)} \qquad f^\circ(x; y) = \max\{\langle y, z\rangle \mid z \in \partial f(x)\}.$$

Proof. Formulas (8.8) and (8.9) are immediate from Theorem 3 and definitions (8.1) and (8.7). For a nonempty weak*-closed convex set $Z \subset E^*$ and its support function

$$\text{(8.11)} \qquad l(y) = \sup\{\langle y, z\rangle \mid z \in Z\},$$

it is known that Z is nonasymptotic with respect to y (and hence "sup" can be replaced by "max" in (8.11)) if and only if l is finite and continuous at y. In the present case of $Z = \partial f(x)$ and $l(y) = f^{\uparrow}(x; y)$, these vectors y are by Theorem 3 the ones with respect to which f is directionally Lipschitzian at x, and for each such one has $f^{\uparrow}(x; y) = f^\circ(x; y)$.

Corollary 1. *If E is finite-dimensional and $\partial f(x)$ is nonempty and nonasymptotic with respect to some vector y, then f is directionally Lipschitzian at x and the properties in the theorem hold.*

Proof. This follows via Proposition 2 and the fact about the "nonasymptotic" property that is cited in the preceding proof.

Corollary 2. *Suppose f is a concave function on E, and let x be a point where f is finite. In general one has*

$$\tilde{\partial} f(x) = \{z \in E^* \mid f(x') \leq f(x) + \langle x' - x, z\rangle \text{ for all } x' \in E\},$$

but if f is bounded below on some nonempty open set, it is also true that $\tilde{\partial} f(x) = \partial f(x)$ and

$$f^{\uparrow}(x; y) = \liminf_{y' \to y} -f'(x; -y') \text{ for all } y \in E.$$

(If f is not bounded below on any nonempty open set, then $\partial f(x) = \emptyset$ and $f^{\uparrow}(x; y) = -\infty$ for all y.)

Proof. Apply Proposition 3 and Theorem 5 to $-f$. The validity of the final assertion is seen from the fact that $\operatorname{cl}(\operatorname{epi} f) = E \times \mathbf{R}$ under this assumption, so that

$$\operatorname{epi} f^{\uparrow}(x; \cdot) = T_{\operatorname{epi} f}(x, f(x)) = E \times \mathbf{R}$$

by (3.3).

Corollary 3. *Suppose f is nondecreasing with respect to the partial ordering on E induced by a closed convex cone K with nonempty interior. Let*

$$K^* = \{z \in E^* \mid \langle y, z\rangle \geq 0 \text{ for all } y \in K\} \quad (\text{dual nonnegative cone}).$$

Then at each point x where f is finite, one has

$$\partial f(x) = \tilde{\partial} f(x) \subset K^*.$$

Proof. This follows from Proposition 4.

References

1. J.-P. Aubin, *Gradients generalisés de Clarke*, lecture notes, Centre de Recherches Mathématiques, Université de Montréal, 1977.
2. J.-P. Aubin and F. H. Clarke, *Multiplicateur de Lagrange en optimisation non convexe et applications*, C.R. Acad. Sci. Paris *285* (1977), 451–454.
3. C. Berge, *Espaces topologiques, fonctions multivoques* (Dunod, Paris, 1959).
4. F. E. Browder, *Multivalued monotone nonlinear mappings and duality mappings in Banach spaces*, Trans. A.M.S. *118* (1965), 338–351.
5. R. Chaney and A. A. Goldstein, *On directions of steepest descent* (to appear).
6. F. H. Clarke, *Necessary conditions for nonsmooth problems in optimal control and the calculus of variations*, Thesis, University of Washington, Seattle (1973).
7. ——— *Generalized gradients and applications*, Trans. M.M.S. *205* (1975), 247–262.
8. ——— *Generalized gradients of Lipschitz functionals*, Advances in Math. (to appear).

(Available as Tech. Report 1687, Math. Research Center, University of Wisconsin, Madison, 1976.)

9. ——— *On the inverse function theorem*, Pacific J. Math. *64* (1976), 97–102.

10. ——— *A new approach to Lagrange multipliers*, Math. of Op. Research *1* (1976), 165–174.

11. ——— *La condition hamiltonienne d'optimalité*, C. R. Acad. Sci. Paris *280* (1975), 1205–1207.

12. ——— *The Euler-Lagrange differential inclusion*, J. Diff. Eq. *19* (1975), 80–90.

13. ——— *Optimal solutions to differential inclusions*, J. Opt. Theory Appl. *19* (1976), 469–478.

14. ——— *The generalized problem of Bolza*, SIAM J. Control. Opt. *14* (1976), 682–699.

15. ——— *The maximum principle under minimal hypotheses*, SIAM J. Control Opt. *14* (1976), 1078–1091.

16. ——— *Necessary conditions for a general control problem*, *Calculus of variations and control theory* (Academic Press, 1976), 257–278.

17. ——— *Inequality constraints in the calculus of variations*, Can. J. Math. *39* (1977), 528–540.

18. ——— *Multiple integrals of Lipschitz functions in the calculus of variations*, Proc. A.M.S. *64* (1977), 260–264.

19. ——— *The Erdmann condition and Hamiltonian inclusions in optimal control and the calculus of variations*, SIAM J. Control Opt.

20. ——— *Optimal control and the true Hamiltonian*, SIAM Review (to appear).

21. A. Feuer, *An implementable mathematical programming algorithm for admissible fundamental functions*, Thesis, Columbia University (1974).

22. ——— *Minimizing well-behaved functions*, Proceedings 12th Annual Alberton Conference on Circuit and System Theory (University of Illinois Press, 1974), 25–33.

23. J. Gauvin, *The generalized gradient of a marginal function in mathematical programming*, Math. of Op. Research (to appear).

24. A. A. Goldstein, *Optimization of Lipschitz continuous functions*, Math. Prog. *13* (1977), 14–22.

25. F. Hausdorff, *Mengenlehre*, 3rd edition (Springer, Berlin, 1927).

26. J.-B. Hiriart-Urruty, *Contributions á la programmation mathématique: Cas deterministe et stochastique*, Thèse, Université de Clermont-Ferrand II (1977).

27. ——— *Tangent cones, generalized gradients and mathematical programming in Banach spaces*, Math. of Op. Research (to appear).

28. ——— *Conditions necessaires d'optimalite en programmation non différentiable*, C. R. Acad. Sci. Paris *283* (1976), 843–845.

29. ——— *On optimality conditions in nondifferentiable programming*, Math. Prog. (to appear).

30. ——— *Gradients généralisés de fonctions marginales*, SIAM J. Control Opt. (to appear).

31. L. Hörmander, *Sur la fonction d'appui des ensembles convexes dans un espace localement convexe*, Arkiv för Math. *3* (1954), 181–186.

32. A. D. Ioffe, *Estimates of the distance to level sets*, SIAM J. Control Opt. (to appear).

33. ——— *Regular points of Lipschitz mappings*, Trans. A.M.S. (to appear).

34. ——— *Necessary and sufficient conditions for a local minimum. 1. A reduction theorem and first-order conditions*, SIAM J. Control Opt. *17* (1979), 245–250.

35. ——— *Necessary and sufficient conditions for a local minimum. 2. Conditions of Levitin-Miljutin-Osmolovskii-type*, SIAM J. Control Opt. (to appear).

36. ——— *Necessary and sufficient conditions for a local minimum. 3. Second-order conditions and augmented duality*, SIAM J. Control Opt. (to appear).

37. G. Lebourg, *Valeur moyenne pour gradient généralisé*, C. R. Acad. Aci. Paris, Ser. A. *281* (1975), 795–797.

38. L. McLinden, *Dual operations on saddle functions*, Trans. A.M.S. *179* (1973), 363–381.

39. R. Mifflin, *Semismooth and semiconvex functions in constrained optimization*, SIAM J. Control Opt. *15* (1977), 959–972.

40. ——— *An algorithm for constrained optimization with semismooth functions*, Math. of Op. Research *2* (1977), 191–207.

41. G. J. Minty, *Monotone (nonlinear) operators in Hilbert space*, Duke Math. J. *29* (1962), 341–346.

42. J. J. Moreau, *Fonctionelles convexes*, lecture notes, Séminaire "Equations aux Derivées Partielles," Collège de France, 1966.

43. R. T. Rockafellar, *Convex functions and dual extremum problems*, thesis, Harvard (1963).

44. ——— *Convex analysis* (Princeton University Press, 1970).

45. ——— *Conjugate duality and optimization*, Regional Conference Series in Applied Math. *16* (S.I.A.M. Publications, Philadelphia, 1974).

46. ——— *On the maximal monotonicity of subdifferential mappings*, Pacific J. Math. *33* (1970), 209–216.

47. ——— *Monotone operators associated with saddle functions and minimax problems*, *Nonlinear functional analysis*, Part 1, Symposia in Pure Math. *18* (Amer. Math. Soc., Providence, R.I., 1970), 397–407.

48. ——— *On the maximality of sums of nonlinear monotone operators*, Trans. A.M.S. *149* (1970), 75–88.

49. ——— *Clarke's tangent cones and the boundaries of closed sets in R^3*, Nonlin. Analysis. Th. Meth. Appl. *3* (1979), 145–154.

50. ——— *Directionally Lipschitzian functions and subdifferential calculus*, Proc. London Math. Soc. *39* (1979), 331–355.

51. ——— *Dual problems of Lagrange for arcs of bounded variation*, *Calculus of variations and control theory* (Academic Press, 1976), 155–192.

52. ——— *Conjugate convex functions in optimal control and the calculus of variations*, J. Math. Anal. Appl. *32* (1970), 174–222.

53. R. T. Rockafellar and R. J.-B. Wets, *The optimal recourse problem in discrete time: L^1-multipliers for inequality constraints*, SIAM J. Control Opt. *16* (1978).

54. S. Saks, *Theory of the integral* (Hafner Publishing Co., New York, second revised edition, 1937).

55. L. Thibault, *Propriétés des sous-differentiels de fonctions localement Lipschitziennes définies sur un espace de Banach séparable. Applications*, Thèse, Université des Sciences et Techniques du Languedoc, Monpellier (1976).

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