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Vol. 12, No. 1, February 1976
Printed in Belgium

Saddle Points of Hamiltonian Systems in Convex Lagrange Problems Having a Nonzero Discount Rate

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Received November 8, 1974

Problems are studied in which an integral of the form $\int_0^{+\infty} L(k(t), \dot{k}(t))e^{-\rho t}\,dt$ is minimized over a class of arcs $k\colon [0, +\infty) \to R^n$. It is assumed that L is a convex function on $R^n \times R^n$ and that the discount rate ρ is positive. Optimality conditions are expressed in terms of a perturbed Hamiltonian differential system involving a Hamiltonian function $H(k, q)$ which is concave in k and convex in q, but not necessarily differentiable. Conditions are given ensuring that, for ρ sufficiently small, the system has a stationary point, in a neighborhood of which one has classical "saddle point" behavior. The optimal arcs of interest then correspond to the solutions of the system which tend to the stationary point as $t \to +\infty$. These results are motivated by questions in theoretical economics and extend previous work of the author for the case $\rho = 0$. The case $\rho < 0$ is also covered in part.

1. INTRODUCTION

Let $L\colon R^n \times R^n \to (-\infty, +\infty]$ be convex, lower, semicontinuous, and not identically $+\infty$, and let $\rho \geqslant 0$. For each $c \in R^n$, let

$$\phi(c) = \inf \left\{ \int_0^{+\infty} L(k(t), \dot{k}(t))\, e^{-\rho t}\, dt \;\middle|\; k(0) = c \right\}, \tag{1.1}$$

where the infimum is over the class of all *arcs* (taken here to mean *absolutely continuous functions*) $k\colon [0, +\infty) \to R^n$ such that $e^{-\rho t}k(t)$ remains bounded as $t \to +\infty$. The integral in (1.1) has a classical value, possibly infinite, unless neither the positive nor the negative part of the integrand $L(k(t), \dot{k}(t))\, e^{-\rho t}$ (a measurable function of t) is summable over $[0, +\infty)$; in the latter case, we consider the integral to have the value $+\infty$ by convention. The convexity of L implies the convexity of ϕ as an extended-real-valued function on R^n.

Our interest lies in the existence and characterization of the arcs k, if any, for which the infimum in (1.1) is attained. An important aid in this

* Supported in part by Grant AFOSR-72-2269 at the University of Washington.

regard is the study of a generalized Hamiltonian subdifferential "equation"

$$(-e^{\rho t}\dot{w}(t), \dot{k}(t)) \in \partial H(k(t), e^{\rho t}w(t)) \qquad \text{a.e.,} \tag{1.2}$$

which after the change of variables,

$$q(t) = e^{\rho t}w(t), \tag{1.3}$$

can be rewritten in the autonomous form

$$(-\dot{q}(t) + \rho q(t), \dot{k}(t)) \in \partial H(k(t), q(t)) \qquad \text{a.e.} \tag{1.4}$$

The Hamiltonian H is defined here by the conjugacy formula

$$H(k, q) = \sup\{z \cdot q - L(k, z) \mid z \in R^n\}. \tag{1.5}$$

By virtue of the assumptions on L, $H(k, q)$ is concave in k, convex in q, and the inverse formula

$$L(k, z) = \sup\{z \cdot q - H(k, q) \mid q \in R^n\} \tag{1.6}$$

is valid (cf. [3]). The set $\partial H(k, q)$ consists of the *subgradients* of H at (k, q), i.e., the pairs $(r, z) \in R^n \times R^n$ such that

$$H(k, q') \geqslant H(k, q) + (q' - q) \cdot z \qquad \text{for all} \quad q' \in R^n, \tag{1.7}$$

$$H(k', q) \leqslant H(k, q) + (k' - k) \cdot r \qquad \text{for all} \quad k' \in R^n. \tag{1.8}$$

Established theory (cf. [3]) tells us that if $k(t)$ and $q(t)$ satisfy the (perturbed) Hamiltonian system (1.4) over a real interval J, then for every bounded subinterval $[t_0, t_1]$ of J, the integral

$$\int_{t_0}^{t_1} L(k(t), \dot{k}(t))\, e^{-\rho t}\, dt \tag{1.9}$$

is minimized with respect to the class of all arcs over $[t_0, t_1]$ having the same endpoints as k at $t = t_0$ and $t = t_1$. At the same time, there is a dual property: for a certain function M, the integral

$$\int_{t_0}^{t_1} M(q(t), \dot{q}(t) - \rho q(t))\, e^{-\rho t}\, dt \tag{1.10}$$

is minimized with respect to the class of all arcs over $[t_0, t_1]$ having the same endpoints as q at $t = t_0$ and $t = t_1$. The function M is defined by

$$\begin{aligned} M(q, s) &= \sup\{k \cdot s + z \cdot q - L(k, z) \mid (k, z) \in R^n \times R^n\} \\ &= \sup\{k \cdot s + H(k, q) \mid k \in R^n\}, \end{aligned} \tag{1.11}$$

and one has, reciprocally,

$$L(k, z) = \sup\{k \cdot s + z \cdot q - M(q, s) | (q, s) \in R^n \times R^n\}. \quad (1.12)$$

Indeed, in the terminology of [3] the Lagrangians

$$\tilde{L}(t, k, z) = L(k, z)\, e^{-\rho t}, \quad (1.13)$$

$$\tilde{M}(t, w, v) = M(e^{\rho t} w, e^{\rho t} v)\, e^{-\rho t} \quad (1.14)$$

are dual to each other, whence the result [3, p. 213].

Henceforth, let $(\bar{k}, \bar{q})$ denote a pair in $R^n \times R^n$ such that

$$(\rho \bar{q}, 0) \in \partial H(\bar{k}, \bar{q}). \quad (1.15)$$

(We shall comment in Section 6 on the existence of such a pair $(\bar{k}, \bar{q})$.) Relation (1.15) means that $(\bar{k}, \bar{q})$ is a *stationary point* of the system (1.4), in the sense that the constant functions $k(t) \equiv \bar{k}$ and $q(t) \equiv \bar{q}$ satisfy the system over $J = (-\infty, +\infty)$. In a previous paper [2], we investigated for the case $\rho = 0$ the behavior of the system near such a stationary point, particularly the existence of solutions $(k(t), q(t))$ tending to $(\bar{k}, \bar{q})$ as $t \to +\infty$ or as $t \to -\infty$. This was shown to be closely related to question of optimality for the minimization problems in (1.1), as well as for a class of dual problems involving M. The analysis was carried out under the assumption that H was *strictly* concave–convex in a neighborhood of $(\bar{k}, \bar{q})$.

The purpose of the present paper is to extend some of the results to the case of sufficiently small $\rho > 0$, making use of a strengthened strict concavity–convexity assumption on H. Economic motivation may be found in the interesting paper of Case and Shell [1], which contains certain related results based on a somewhat different set of technical assumptions.

It will be convenient to make a translation of variables,

$$x = k - \bar{k}, \qquad p = q - \bar{q}, \quad (1.16)$$

so that the stationary point of the Hamiltonian system appears at the origin and the finiteness of certain integrals is more apparent. Specifically, let

$$H_0(x, p) = H(\bar{k} + x, \bar{q} + p) - H(\bar{k}, \bar{q}) - \rho x \bar{q}. \quad (1.17)$$

Then H_0 is a concave–convex function which, according to (1.15), satisfies

$$(0, 0) \in \partial H_0(0, 0), \quad (1.18)$$

or in other words, the minimax saddle point condition

$$H_0(x, 0) \leqslant H_0(0, 0) \leqslant H_0(0, p) \qquad \text{for all} \quad x \in R^n, p \in R^n. \quad (1.19)$$

Moreover, one has

$$H_0(0, 0) = 0. \quad (1.20)$$

The (perturbed) Hamiltonian system

$$(-\dot{p}(t) + \rho p(t), \dot{x}(t)) \in \partial H_0(x(t), p(t)) \quad (1.21)$$

is clearly equivalent to the previous system (1.4) under (1.16).

Let us also define

$$L_0(x, z) = L(\bar{k} + x, z) - L(\bar{k}, 0) - \bar{q} \cdot (z - \rho x), \quad (1.22)$$

$$M_0(p, u) = M(\bar{q} + p, -\rho\bar{q} + u) - M(\bar{q}, -\rho\bar{q}) - \bar{k} \cdot u. \quad (1.23)$$

These formulas yield (by the theory of conjugate convex functions, cf. [3]) the relations

$$H_0(x, p) = \sup\{z \cdot p - L_0(x, z) \mid z \in R^n\}, \quad (1.24)$$

$$L_0(x, z) = \sup\{z \cdot p - H_0(x, p) \mid p \in R^n\}, \quad (1.25)$$

$$\begin{aligned} M_0(p, u) &= \sup\{x \cdot u + z \cdot p - L_0(x, z) \mid (x, z) \in R^n \times R^n\} \\ &= \sup\{x \cdot u + H_0(x, p) \mid x \in R^n\} \end{aligned} \quad (1.26)$$

$$L_0(x, z) = \sup\{x \cdot u + z \cdot p - M_0(p, u) \mid (p, u) \in R^n \times R^n\}, \quad (1.27)$$

inasmuch as (1.3) implies

$$-L(\bar{k}, 0) = H(\bar{k}, \bar{q}) = M(\bar{q}, -\rho\bar{q}) + \rho\bar{k} \cdot \bar{p}. \quad (1.28)$$

Observe that

$$L_0(x, z) \geqslant L_0(0, 0) = 0 \qquad \text{for all} \quad (x, z) \in R^n \times R^n, \quad (1.29)$$

$$M_0(p, u) \geqslant M_0(0, 0) = 0 \qquad \text{for all} \quad (p, u) \in R^n \times R^n. \quad (1.30)$$

Let us say that a finite function h on a convex set $C \subset R^n$ is *α-convex*, where $\alpha \in R$, if for all $x \in R^n$, $x' \in R^n$ and $\lambda \in [0, 1]$, it is true that

$$h((1 - \lambda) x + \lambda x') \leqslant (1 - \lambda) h(x) + \lambda h(x') - \tfrac{1}{2}\alpha\lambda(1 - \lambda)| x - x' |^2, \quad (1.31)$$

where $|\cdot|$ denotes the Euclidean norm. Obviously, this is ordinary convexity if $\alpha = 0$ and a form of strict convexity (*strong* convexity) if

$\alpha > 0$. One can verify that h is α-convex if and only if the function $h(x) - \frac{1}{2}\alpha \mid x \mid^2$ is convex. Thus, if C is open and h is twice differentiable, α-convexity is equivalent to the condition that

$$v \cdot Q(x)\, v \geqslant \alpha \mid v \mid^2 \qquad \text{for all} \quad x \in C,\ v \in R^n,$$

where $Q(x)$ is the matrix of second partial derivatives of h at x. Another easily derived characterization, for C open, is that h is α-convex if and only if for each $z \in C$, there exists $y \in R^n$ such that

$$h(x') \geqslant h(x) + (x' - x) \cdot y + \tfrac{1}{2}\alpha \mid x' - x \mid^2 \qquad \text{for all} \quad x' \in C. \quad (1.32)$$

At all events, if h is α-convex on C (not necessarily open) and $y \in \partial h(x)$, then (1.32) holds.

We shall say h is α-*concave* if, in place of (1.31), we have

$$h((1 - \lambda)\, x + \lambda x') \geqslant (1 - \lambda)\, h(x) + \lambda h(x') + \tfrac{1}{2}\alpha\lambda(1 - \lambda) \mid x - x' \mid^2. \quad (1.33)$$

CURVATURE ASSUMPTION. *We suppose throughout this paper that, for certain values $\alpha > 0$ and $\beta > 0$, the Hamiltonian H is locally α-concave-β-convex near the stationary point $(\bar{k}, \bar{q})$, or in other words, that there exists a convex neighborhood $U \times V$ of $(0, 0)$ in $R^n \times R^n$ such that $H_0(x, p)$ is (finite and) α-concave in $x \in U$ for each $p \in V$ and β-convex in $p \in V$ for each $x \in U$. Moreover, the discount rate $\rho > 0$ is small enough so that*

$$\rho^2 < 4\alpha\beta. \quad (1.34)$$

Let K_+ denote the set of all pairs $(a, b) \in R^n \times R^n$ such that the Hamiltonian system (1.21) has a solution $(x(t), p(t))$ over $[0, +\infty)$ satisfying

$$(x(0), p(0)) = (a, b), \quad (1.35)$$

$$e^{-\rho t} x(t) \cdot p(t) \to 0 \text{ as } t \to +\infty. \quad (1.36)$$

The first of our main results is the following.

THEOREM 1. *There is an open neighborhood $U_+ \times V_+$ of $(0, 0)$ (arbitrarily small, with $U_+ \subset U$ and $V_+ \subset V$) such that $K_+ \cap (U_+ \times V_+)$ is the graph of a homeomorphism of U_+ onto V_+, and for each $(a, b) \in K_+ \cap (U_+ \times V_+)$ the solution to the system (1.21) over $[0, +\infty)$ satisfying (1.35) and (1.36) is unique, remains in $K_+ \cap (U_+ \times V_+)$ and converges to $(0, 0)$ as $t \to +\infty$.*

To give an interpretation to the set K_+, we introduce the functions

$$f_+(a) = \inf \left\{ \int_0^{+\infty} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \;\middle|\; x(0) = a \right\}, \tag{1.37}$$

$$g_+(b) = \inf \left\{ \int_0^{+\infty} M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \;\middle|\; p(0) = b \right\}, \tag{1.38}$$

where each infimum is taken over the class of *all* arcs (absolutely continuous R^n-valued functions) defined on $[0, +\infty)$ and satisfying the endpoint condition in question. Note that the integrals are well-defined (possibly $+\infty$), due to the lower semicontinuity and nonnegativity of the functions L_0 and M_0. In fact, (1.29) and (1.30) imply that the functions f_+ and g_+ on R^n are nonnegative and vanish at 0. Of course, these functions are all convex; this follows immediately from the convexity of L_0 and M_0.

For $a = c - \bar{k}$, the minimization problem in (1.37) is equivalent to the one defining the value $\phi(c)$ at the beginning of our introduction, as will be demonstrated in the next section (Proposition 2). This equivalence could fail if one were to drop from the definition of $\phi(c)$ the restriction to arcs k such that $e^{-\rho t}k(t)$ remains bounded as $t \to +\infty$; see Example 2 in Section 6.

THEOREM 2. *Let U_+ and V_+ be neighborhoods of* 0 *with the properties in Theorem* 1. *Then f_+ is finite and continuously differentiable on U_+, g_+ is finite and continuously differentiable on V_+, and for $(a, b) \in U_+ \times V_+$, one has*

$$f_+(a) + g_+(b) \geqslant -a \cdot b, \tag{1.39}$$

with

$$\begin{aligned} (a, b) \in K_+ &\Leftrightarrow f_+(a) + g_+(b) = -a \cdot b \\ &\Leftrightarrow b = -\nabla f_+(a) \Leftrightarrow a = -\nabla g_+(b). \end{aligned} \tag{1.40}$$

Moreover, if $(x(t), p(t))$ is the solution to the Hamiltonian system (1.21) *over $[0, +\infty)$ corresponding to $(a, b) \in K_+ \cap (U_+ \times V_+)$ as in Theorem* 1, *then the arc x uniquely furnishes the minimum in the definition* (1.37) *of $f_+(a)$, while the arc p uniquely furnishes the minimum in the definition* (1.38) *of $g_+(b)$.*

Complementary results are obtainable for behavior over the interval $(-\infty, 0]$. These are of less interest for economic applications, but they do shed further light on the qualitative nature of the Hamiltonian dynamical system. They can also be interpreted equivalently, under a reversal of time, as results over $[0, +\infty)$ for a *negative discount rate* ρ.

Let K_- denote the set of all pairs $(a, b) \in R^n \times R^n$ such that the system (1.21) has a solution $(x(t), p(t))$ over $(-\infty, 0]$ satisfying (1.35), and

$$e^{-\rho t}x(t) \cdot p(t) \to 0 \text{ as } t \to -\infty. \tag{1.41}$$

Define

$$f_-(a) = \inf \left\{ \int_{-\infty}^{0} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \;\middle|\; x(0) = a \right\}, \tag{1.42}$$

$$g_-(b) = \inf \left\{ \int_{-\infty}^{0} M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \;\middle|\; p(0) = b \right\}. \tag{1.43}$$

The functions f_- and g_- are nonnegative, convex, and they vanish at 0.

THEOREM 1′. *There is an open neighborhood $U_- \times V_-$ of $(0, 0)$ (arbitrarily small, with $U_- \subset U$ and $V_- \subset V$) such that $K_- \cap (U_- \times V_-)$ is the graph of a homeomorphism of U_- onto V_-, and for each $(a, b) \in K_- \cap (U_- \times V_-)$ the solution to the system* (1.21) *over $(-\infty, 0]$ satisfying* (1.35) *and* (1.41) *is unique, remains in $K_- \cap (U_- \times V_-)$ and converges to $(0, 0)$ as $t \to -\infty$. Moreover,*

$$K_+ \cap K_- = \{(0, 0)\}. \tag{1.44}$$

THEOREM 2′. *Let U_- and V_- be neighborhoods of 0 with the properties in Theorem 1′. Then f_- is finite and continuously differentiable on U_-, g_- is finite and continuously differentiable on V_-, and for $(a, b) \in U_- \times V_-$, one has*

$$f_-(a) + g_-(b) \geqslant a \cdot b, \tag{1.45}$$

with

$$\begin{aligned} (a, b) \in K_- &\Leftrightarrow f_-(a) + g_-(b) = a \cdot b \\ &\Leftrightarrow b = \nabla f_-(a) \Leftrightarrow a = \nabla g_-(b). \end{aligned} \tag{1.46}$$

Moreover, if $(x(t), p(t))$ is the solution to the Hamiltonian system (1.21) *over $(-\infty, 0]$ corresponding to $(a, b) \in K_- \cap (U_- \times V_-)$ as in Theorem 1′, then the arc x uniquely furnishes the minimum in the definition* (1.42) *of $f_-(a)$, while the arc p uniquely furnishes the minimum in the definition* (1.43) *of $g_-(b)$.*

Theorems 1 and 1′ say that the behavior of the system (1.21) near the rest point (0, 0) resembles that of a classical saddlepoint in the theory of differential equations. At least locally, K_+ and K_- are n-dimensional manifolds intersecting only at (0, 0), and comprised, respectively, of the trajectories that tend to (0, 0) as $t \to +\infty$ and as $t \to -\infty$.

Theorem 2 will be derived from Theorem 1 in Section 2, while Theorem 1 itself will be established at the end of Section 4. The proofs of Theorem 1′ and 2′ are parallel, but in certain respects simpler; they will be treated in Section 5. Various examples and a result about the existence of points $(\bar{k}, \bar{q})$ satisfying the stationary point condition (1.15) will be treated in Section 6. In Section 3, we develop some facts which enable us, in Section 4, to deduce the local results from more special theorems based on a *global* assumption of α-concavity-β-convexity.

2. Local Behavior and Optimality

We start by establishing some bounds that, as a by-product, make clear the equivalence of the original class of variational problems defining the values $\phi(c)$ and the notationally more convenient class of "translated" problems defining the values $f_+(a)$.

Proposition 1. *The convex function L_0 is finite on a neighborhood of* $(0, 0)$, *and there exist real numbers $\mu_0 > 0$ and μ_1 such that*

$$L_0(x, z) \geqslant \mu_0[|x| + |z - \rho x|] - \mu_1 \qquad \text{for all} \quad (x, z) \in R^n \times R^n. \tag{2.1}$$

Similarly, the convex function M_0 is finite on a neighborhood of $(0, 0)$, *and there exist real numbers $\nu_0 > 0$ and ν_1 such that*

$$M_0(p, s - \rho p) \geqslant \nu_0[|p| + |s - \rho p|] - \nu_1 \qquad \text{for all} \quad (p, x) \in R^n \times R^n. \tag{2.2}$$

Proof. According to our curvature assumption, the convex function $H_0(0, \cdot)$ is finite and strictly convex in a neighborhood of $p = 0$. It follows from this and the minimax property (1.19) that the supremum in formula (1.25) is uniquely attained at $p = 0$. But $H_0(0, \cdot)$ is conjugate to $L_0(0, \cdot)$ by (1.24), so this implies 0 is the unique subgradient of $L_0(0, \cdot)$ at $z = 0$ [5, Theorem 23.5]. Hence, $L_0(0, \cdot)$ is finite on a neighborhood of 0 (since otherwise the subgradient set would have to be unbounded or empty [5, Theorem 23.4]). Thus, the convex set

$$\operatorname{dom} L_0 = \{(x, z) \in R^n \times R^n \mid L_0(x, z) < +\infty\} \tag{2.3}$$

contains $(0, z)$ for all z sufficiently near 0. On the other hand, the image of $\operatorname{dom} L_0$ under the projection $(x, z) \to x$ consists of all x such that the convex function $L_0(x, \cdot)$ is not identically $+\infty$, or what is equivalent in view of the conjugacy relation (1.24), such that the convex function $H_0(x, \cdot)$ nowhere takes the value $-\infty$. This image therefore contains the neighborhood U in our curvature assumption, since a convex function cannot take on the value $-\infty$ anywhere if it is finite on some nonempty open set [5, Theorem 7.2]. Thus 0 lies in the interior of the x-projection of dom L_0, while 0 is also in the interior of the z-cross-section of dom L_0 corresponding to $x = 0$. These properties imply that (0, 0) is an interior point of dom L_0 [5, Theorem 7.8], or in other words that L_0 is finite on a neighborhood of (0, 0). Of course, L_0 is then continuous near (0, 0) by convexity. Thus there exist $\nu_0 > 0$ and ν_1 such that

$$L_0(x, z) \leqslant \nu_1 \qquad \text{if} \quad |x| \leqslant \nu_0 \text{ and } |z| \leqslant \nu_0 .$$

Applying this to formula (1.26) (for M_0 in terms of L_0), we obtain

$$\begin{aligned} M_0(p, u) &\geqslant \sup\{x \cdot p + z \cdot u - \nu_1 \mid |x| \leqslant \nu_0, |z| \leqslant \nu_0\} \\ &= \nu_0 |p| + \nu_0 |u| - \nu_1 , \end{aligned}$$

which is the desired inequality (2.2).

By a parallel argument, the concave function $H_0(\cdot, 0)$ is finite and strictly concave on a neighborhood of $x = 0$, so that by (1.19), the supremum in formula (1.26) (for M_0 in terms of H_0) is attained uniquely for $x = 0$. Since this formula expresses the convex function $M_0(0, \cdot)$ as the conjugate of $-H(\cdot, 0)$, we are able to conclude, just as above, that 0 is the unique subgradient of $M_0(0, \cdot)$ at 0, and, hence, that $M_0(0, \cdot)$ is finite on a neighborhood of 0. The convex set

$$\text{dom } M_0 = \{(p, u) \mid M_0(p, u) < +\infty\} \tag{2.4}$$

therefore contains $(0, u)$ for all u sufficiently near 0. The p-projection of dom M_0 also contains the neighborhood V of 0 in our curvature assumption, because of (1.26). We deduce from this that (0, 0) is an interior point of dom M_0 and consequently a point in a neighborhood of which M_0 is finite and continuous. Let the numbers $\mu > 0$, $\mu' > 0$, and μ_1 be such that

$$M_0(p, u) \leqslant \mu_1 \qquad \text{if} \quad |p| \leqslant \mu \text{ and } |u| \leqslant \mu'.$$

We then have from (1.27) that

$$\begin{aligned} L_0(x, z) &\geqslant \sup\{x \cdot u + p \cdot z - \mu_1 \mid |p| \leqslant \mu, |u| \leqslant \mu'\} \\ &= \mu |x| + \mu' |z| - \mu_1 . \end{aligned}$$

Using the fact that

$$|z| \geqslant |z - \rho x| - \rho |x| ,$$

we get

$$L_0(x, z) \geqslant (\mu - \rho\mu')| x | + \mu' | z - \rho x | - \mu_1 .$$

The desired inequality (2.1) therefore holds for

$$\mu_0 = \min\{\mu - \rho\mu', \mu'\},$$

provided μ' is taken small enough so that this value is positive.

The next proposition is the one specifically establishing the asserted equivalence between the minimization problems defining $\phi(c)$ in (1.1) (where $e^{-\rho t}k(t)$ is bounded) and $f_+(a)$ in (1.37) for $x = k - \bar{k}$, $a = c - \bar{k}$. (In the problem for $f_+(a)$, it is not stipulated in advance that $e^{-\rho t}x(t)$ be bounded, but this turns out to be a consequence of the finiteness of the integral.)

PROPOSITION 2. (a) *If the arc* $x : [0, +\infty) \to R^n$ *is such that* $L_0(x(t), \dot{x}(t))\, e^{-\rho t}$ *is summable in t over* $[0, +\infty)$, *then*

$$\lim_{t\to+\infty} e^{-\rho t}x(t) = 0, \tag{2.5}$$

and for $k(t) = \bar{k} + x(t)$, *we have* $L(k(t), \dot{k}(t))\, e^{-\rho t}$ *summable in t, with*

$$\int_0^{+\infty} L(k(t), \dot{k}(t))\, e^{-\rho t}\, dt = \int_0^{+\infty} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt - [H(\bar{k}, \bar{q})/\rho] - x(0) \cdot \bar{q}. \tag{2.6}$$

On the other hand, if $L(k(t), \dot{k}(t))\, e^{-\rho t}$ *is majorized by a summable function of t over* $[0, +\infty)$, *and also*

$$\limsup_{t\to+\infty} e^{-\rho t}k(t) \cdot \bar{q} > -\infty, \tag{2.7}$$

then $L_0(x(t), \dot{x}(t))\, e^{-\rho t}$ *is indeed summable over* $[0, +\infty)$. (b) *If the arc* $p : [0, +\infty) \to R^n$ *is such that* $M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}$ *is summable in t over* $[0, +\infty)$, *then*

$$\lim_{t\to+\infty} e^{-\rho t}p(t) = 0, \tag{2.8}$$

and for $q(t) = \bar{q} + p(t)$, *we have* $M(q(t), \dot{q}(t) - \rho q(t))\, e^{-\rho t}$ *summable in t, with*

$$\int_0^{+\infty} M(q(t), \dot{q}(t) - \rho q(t))\, e^{-\rho t}\, dt = \int_0^{+\infty} M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt - [H(\bar{k}, \bar{q})/\rho] + k \cdot q(0). \tag{2.9}$$

On the other hand, if $M(q(t), \dot{q}(t) - pq(t))\, e^{-\rho t}$ is majorized by a summable function of t over $[0, +\infty)$, and also

$$\limsup_{t\to+\infty} e^{-\rho t} q(t) \cdot \bar{k} > -\infty, \tag{2.10}$$

then $M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}$ is summable over $[0, +\infty)$.

Proof. If $L_0(x(t), \dot{x}(t))\, e^{-\rho t}$ is summable, then by virtue of inequality (2.1) in Proposition 1, we have

$$\int_0^{+\infty} |x(t)|\, e^{-\rho t}\, dt < +\infty \quad \text{and} \quad \int_0^{+\infty} |\dot{x}(t) - x(t)|\, e^{-\rho t}\, dt < +\infty. \tag{2.11}$$

Setting $v(t) = e^{-\rho t} x(t)$, so that

$$\dot{v}(t) = [\dot{x}(t) - \rho x(t)]\, e^{-\rho t},$$

we see from (2.11) that

$$\int_0^{+\infty} |v(t)|\, dt < +\infty \quad \text{and} \quad \int_0^{+\infty} |\dot{v}(t)|\, dt < +\infty.$$

The finiteness of the second integral shows that $v(t)$ tends to a limit as $t \to +\infty$, while the finiteness of the first integral shows that the limit is 0. Thus (2.5) is true. Since

$$L(k(t), \dot{k}(t))\, e^{-\rho t} = L_0(x(t), \dot{x}(t))\, e^{-\rho t} - H(\bar{k}, \bar{q})\, e^{-\rho t} + (d/dt)\, e^{-\rho t}\, x(t) \cdot \bar{q} \tag{2.12}$$

by (1.22) and (1.28), we then have (2.6) and the summability of $L(k(t), \dot{k}(t))\, e^{-\rho t}$. Conversely, if the latter expression is majorized by a summable function of t, then so is $(d/dt)\, e^{-\rho t} x(t) \cdot \bar{q}$ by (2.12), since $L_0 \geqslant 0$. This implies that $e^{-\rho t} x(t) \cdot \bar{q}$ tends to a certain limit other than $+\infty$ as $t \to +\infty$. The limit cannot be $-\infty$ by assumption (2.7), and, therefore, it is finite. In other words, $(d/dt)\, e^{-\rho t} x(t)$ is actually summable, which leads via (2.12) to the conclusion that $L(k(t), \dot{k}(t))\, e^{-\rho t}$ also majorizes a summable function (the right side of (2.12) with the L_0 term deleted) and, hence, is summable. But then by (2.12), $L_0(x(t), \dot{x}(t))\, e^{-\rho t}$ must likewise be summable.

The proof of part (b) of Proposition 2 is much the same.

COROLLARY. $\phi(\bar{k} + a) = f_+(a) - [H(\bar{k}, \bar{q})/\rho] - a \cdot \bar{q}$.

A fundamental fact about "truncated" variational problems over the finite interval $[0, T]$ will now be stated. Much of our analysis of the problems over $[0, +\infty)$ is dependent on limit arguments concerning what happens to this case as $T \to +\infty$.

PROPOSITION 3. *For arbitrary arcs* $x : [0, T] \to R^n$ *and* $p : [0, T] \to R^n$ *with* $0 < T < +\infty$, *one has*

$$\int_0^T L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt + \int_0^T M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \geqslant e^{-\rho T} x(T) \cdot p(T) - x(0) \cdot p(0). \tag{2.13}$$

Furthermore, equality holds in (2.13) *if and only if* x *and* p *satisfy the Hamiltonian system* (1.21) *over* $[0, T]$.

Proof. Formula (1.26) tells us that

$$L_0(x, z) + M_0(p, u) \geqslant x \cdot u + z \cdot p, \tag{2.14}$$

with equality if and only if

$$(u, p) \in \partial L_0(x, z), \tag{2.15}$$

or equivalently (cf. [5, Theorem 37.5])

$$(-u, z) \in \partial H_0(x, p). \tag{2.16}$$

Therefore,

$$\begin{aligned} L_0(x(t), \dot{x}(t))\, e^{-\rho t} &+ M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t} \\ &\geqslant e^{-\rho t}[x(t) \cdot \dot{p}(t) + \dot{x}(t) \cdot p(t) - \rho x(t) \cdot p(t)] \\ &= (d/dt)\, e^{-\rho t} x(t) \cdot p(t), \end{aligned}$$

with equality if and only if (1.21) holds. The result is then immediate.

COROLLARY. *For arbitrary arcs* $x : [0, +\infty) \to R^n$ *and* $p : [0, +\infty) \to R^n$ *such that* $e^{-\rho t} x(t) \cdot p(t) \to 0$ *as* $t \to +\infty$, *one has*

$$\int_0^{+\infty} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt + \int_0^{+\infty} M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \geqslant -x(0) \cdot p(0), \tag{2.17}$$

with equality if and only if x *and* p *satisfy the Hamiltonian system* (1.21) *over* $[0, +\infty)$. *In particular,*

$$K_+ \subset \{(a, b) \mid f_+(a) + g_+(b) + a \cdot b \leqslant 0\}. \tag{2.18}$$

Proof of Theorem 2 *using Theorem* 1. Fix any $(a, b) \in K_+$. Let x and

p denote the corresponding unique solution to the system (1.21) over $[0, +\infty)$ satisfying (in line with Theorem 1)

$$(x(0), p(0)) = (a, b), \tag{2.19}$$

$$(x(t), p(t)) \to (0, 0) \text{ as } t \to +\infty. \tag{2.20}$$

Then equality holds in (2.17) by the corollary immediately above, so that

$$\begin{aligned} +\infty &> \int_0^{+\infty} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \\ &= -a \cdot b - \int_0^{+\infty} M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt. \end{aligned} \tag{2.21}$$

If $x' : [0, +\infty)$ is any other arc with $x'(0) = a'$, say, and

$$+\infty > \int_0^{\infty} L_0(x\,(t), \dot{x}'(t))\, e^{-\rho t}\, dt,$$

we have

$$\lim_{t\to+\infty} e^{-\rho t} x'(t) \cdot p(t) = 0$$

by (2.20) and property (2.5) of Proposition 2. Therefore, again by the corollary to Proposition 3,

$$\begin{aligned} &\int_0^{+\infty} L_0(x'(t), \dot{x}'(t))\, e^{-\rho t}\, dt \\ &\qquad \geqslant -a' \cdot b - \int_0^{+\infty} M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt. \end{aligned} \tag{2.22}$$

Focusing attention on the case where $a' = a$, we see from (2.21) and (2.22) that

$$\int_0^{+\infty} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt = f_+(a). \tag{2.23}$$

Similar reasoning establishes that

$$\int_0^{+\infty} M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt = g_+(b), \tag{2.24}$$

and in consequence, by way of (2.21),

$$f_+(a) = -a \cdot b - g_+(b) \tag{2.25}$$

We conclude further from (2.22) and the corollary to Proposition 3 that if x' were any arc with $x'(0) = a'$ and

$$\int_0^{+\infty} L_0(x'(t), \dot{x}'(t))\, e^{-\rho t}\, dt = -a' \cdot b - g_+(b),$$

then (x', p) would have to satisfy the Hamiltonian system over $[0, +\infty)$, which by the uniqueness assertion in Theorem 1 would necessitate $x'(t) \equiv x(t)$. In particular, taking $a' = a$, we observe that the arc x uniquely gives the minimum in the definition of $f_+(a)$; in fact, the relation

$$f_+(a') \geqslant -a' \cdot b - g_+(b) \tag{2.26}$$

holds, with equality uniquely when $a' = a$. Analogously, the arc p uniquely gives the minimum in the definition of $g_+(b)$, and one has

$$g_+(b') \geqslant -a \cdot b' - f_+(a)$$

for all $b' \in R^n$. Combining (2.25) and (2.26), we get the subgradient relation

$$f_+(a') \geqslant f_+(a) - (a' - a) \cdot b \qquad \text{for all} \quad a' \in R^n, \tag{2.27}$$

or symbolically, $-b \in \partial f_+(a)$. By the same token, we have

$$g_+(b') \geqslant g_+(b) - (b' - b) \cdot a \qquad \text{for all} \quad b' \in R^n, \tag{2.28}$$

or in other words $-a \in \partial g_+(b)$. This establishes all of Theorem 2 except for the differentiability assertion. For the latter, let θ denote the homeomorphism whose graph is $K_+ \cap (U_+ \times V_+)$; thus, $(a, \theta(a)) \in K_+$ for all $a \in U_+$. Then $-\theta(a) \in \partial f_+(a)$, and since θ is continuous, we must actually have $-\theta(a) = \nabla f_+(a)$ [5, Theorems 25.1, 25.5, and 25.6]. Thus, f_+ is continuously differentiable on U_+; similarly, g_+ is continuously differentiable on V_+. This completes the proof.

In the next two sections, Theorem 1 itself will be proved, but for this purpose a further consequence of Proposition 3 will eventually be required. We state it now for convenience. For $0 < T < +\infty$, let

$$f_T(a, a') = \inf \left\{ \int_0^T L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \;\middle|\; x(0) = a,\, x(T) = a' \right\}, \tag{2.29}$$

$$g_T(b, b') = \inf \left\{ \int_0^T M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \;\middle|\; p(0) = b,\, p(T) = b' \right\}, \tag{2.30}$$

where again the infima are over all arcs (absolutely continuous R^n-valued functions on $[0, T]$) satisfying the given terminal constraints. It is evident from (1.29) and (1.30) that the functions f_T and g_T on $R^n \times R^n$ are convex, nonnegative, and vanish at (0, 0).

PROPOSITION 4. *One has*

$$f_T(a, a') + g_T(b, b') \geqslant e^{-\rho T} a' \cdot b' - a \cdot b \qquad \text{for all} \quad (a, a') \text{ and } (b, b'). \tag{2.31}$$

If $(x(t), p(t))$ satisfies the Hamiltonian system (1.21) over $[0, T]$ with $(x(0), p(0)) = (a, b)$ and $(x(T), p(T)) = (a', b')$, then equality holds in (2.31), x yields the minimum in the definition of $f_T(a, a')$, and p yields the minimum in the definition of $g_T(b, b')$. The converse implication is also true.

Proof. This is obvious from Proposition 3.

3. Reduction from the Local to the Global Case

The next results will be used ultimately to show that, for the purpose of proving Theorem 1, our basic curvature assumption can just as well be cast in a global form. Certain facts about uniqueness of solutions to the Hamiltonian system are also implied by these results.

PROPOSITION 5. *If $(x_1(t), p_1(t))$ and $(y_2(t), p_2(t))$ are solutions to the Hamiltonian system (1.21) over an interval J, then the inequality*

$$(d/dt)\, e^{-\rho t}(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t)) \geqslant 0 \qquad \text{a.e.} \tag{3.1}$$

holds on J, with strict inequality over portions of J where $(x_1(t), p_1(t)) \neq (x_2(t), p_2(t))$ and at least one of the two solutions lies in the neighborhood $U \times V$ in the curvature assumption.

Over portions of J where $(x_1(t), p_1(t)) \neq (x_2(t), p_2(t))$ and both of the solutions lie in $U \times V$, one actually has

$$\begin{aligned}(d/dt)\, e^{-\rho t}&(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t)) \\ &\geqslant \sigma_0 \mid e^{-\rho t}(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t))\mid \qquad \text{a.e.,}\end{aligned} \tag{3.2}$$

where

$$\sigma_0 = 2 \min\{\alpha, \beta\} > 0, \tag{3.3}$$

as well as

$$\begin{aligned}(d/dt)&(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t)) \\ &\geqslant \sigma_1[\mid x_1(t) - x_2(t)\mid + \mid p_1(t) - p_2(t)\mid]^2 \qquad \text{a.e.}\end{aligned} \tag{3.4}$$

where

$$\sigma_1 = (4\alpha\beta - \rho^2)/4(\alpha + \beta + \rho) > 0. \tag{3.5}$$

Proof. We have by definition of ∂H_0 that

$$H_0(x_1(t), p_2(t)) \geqslant H_0(x_1(t), p_1(t)) + \dot{x}_1(t) \cdot (p_2(t) - p_1(t)), \tag{3.6}$$

$$H_0(x_2(t), p_1(t)) \leqslant H_0(x_1(t), p_1(t)) + (-\dot{p}_1(t) + \rho p_1(t)) \cdot (x_1(t) - x_2(t)), \tag{3.7}$$

$$H_0(x_2(t), p_1(t)) \geqslant H_0(x_2(t), p_2(t)) + \dot{x}_2(t) \cdot (p_1(t) - p_2(t)), \tag{3.8}$$

$$H_0(x_1(t), p_2(t)) \leqslant H_0(x_2(t), p_2(t)) + (-\dot{p}_2(t) + \rho p_2(t)) \cdot (x_2(t) - x_1(t)). \tag{3.9}$$

These inequalities yield

$$\begin{aligned}(\dot{x}_1(t) - \dot{x}_2(t)) \cdot (p_1(t) - p_2(t)) + (x_1(t) - x_2(t)) \cdot (\dot{p}_1(t) - \dot{p}_2(t))\\ -\rho(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t)) \geqslant 0 \qquad \text{a.e.}\end{aligned} \tag{3.10}$$

Multiplying the latter by $e^{-\rho t}$, we get (3.1). If $(x_1(t), p_1(t))$, say, lies in $U \times V$, where H_0 is in particular strictly concave–convex, we have strict inequality in (3.6) unless $p_1(t) = p_2(t)$, as well as strict inequality in (3.7) unless $x_1(t) = x_2(t)$. In this context, therefore, strict inequality holds in (3.10), and hence, in (3.1), unless $(x_1(t), p_1(t)) = (x_2(t), p_2(t))$.

Over subintervals where both solutions lie in $U \times V$, we can improve the argument by adding the term $\frac{1}{2}\beta \mid p_1(t) - p_2(t)\mid^2$ to the right sides of (3.6) and (3.8), while subtracting $\frac{1}{2}\alpha \mid x_1(t) - x_2(t)\mid^2$ from the right sides of (3.7) and (3.9). In this way, (3.10) is strengthened to

$$\begin{aligned}(\dot{x}_1(t) - \dot{x}_2(t)) \cdot (p_1(t) - p_2(t)) + (x_1(t) - x_2(t)) \cdot (\dot{p}_1(t) - \dot{p}_2(t))\\ - \rho(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t)) \geqslant \alpha \mid x_1(t) - x_2(t)\mid^2\\ + \beta \mid p_1(t) - p_2(t)\mid^2 \qquad \text{a.e.}\end{aligned} \tag{3.11}$$

Using the fact that

$$\mid u \mid^2 + \mid v \mid^2 \geqslant 2 \mid u \cdot v \mid,$$

we see that

$$\begin{aligned}\alpha \mid x_1(t) - x_2(t)\mid^2 + \beta \mid p_1(t) - p_2(t)\mid^2\\ \geqslant \sigma_0 \mid(x_1(t) - x_2(t)) \cdot (p_1(t) + p_2(t))\mid .\end{aligned} \tag{3.12}$$

When (3.12) is juxtaposed with (3.11) and both sides are multiplied by

$e^{-\rho t}$, we obtain (3.2). In establishing (3.4), the first thing to record is that (3.11) also implies

$$\begin{aligned}(d/dt)&(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t))\\ &\geqslant \alpha \mid x_1(t) - x_2(t)\mid^2 + \rho(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t))\\ &\quad + \beta \mid p_1(t) - p_2(t)\mid^2\\ &\geqslant \alpha \mid x_1(t) - x_2(t)\mid^2 - \rho \mid x_1(t) - x_2(t)\mid \mid p_1(t) - p_2(t)\mid\\ &\quad + \beta \mid p_1(t) - p_2(t)\mid^2 \qquad \text{a.e.}\end{aligned} \tag{3.13}$$

The proof of (3.4) can be completed by showing that for all real numbers $\lambda \geqslant 0$ and $\mu \geqslant 0$, one has

$$\alpha\lambda^2 - \rho\lambda\mu + \beta\mu^2 \geqslant \sigma_1(\lambda + \mu)^2. \tag{3.14}$$

This inequality is trivial of course if $\lambda = 0 = \mu$, so we can suppose that $\lambda + \mu > 0$ and rewrite (3.14) as

$$\alpha\theta^2 - \rho\theta(1 - \theta) + \beta(1 - \theta)^2 \geqslant \sigma_1 , \tag{3.15}$$

where $\theta = \lambda/(\lambda + \mu)$. The validity of (3.15) for all $\theta \in [0, 1]$ is seen by calculating the minimum value of the left side of (3.15) as a quadratic (convex) function of $\theta \in (-\infty, +\infty)$ and showing that it in fact equals σ_1, which is positive by assumption (1.34).

COROLLARY 1. *If $(x_1(t), p_1(t))$ and $(x_2(t), p_2(t))$ are solutions to the Hamiltonian system over an interval J, then the expression*

$$\theta(t) = e^{-\rho t}(x_1(t) - x_2(t)) \cdot (p_1(t) - p_2(t)) \tag{3.16}$$

is nondecreasing over J, in fact strictly increasing over those portions of J where $(x_1(t), p_1(t)) \neq (x_2(t), p_2(t))$ and at least one of the solutions $U \times V$.

Over portions of J where $(x_1(t), p_1(t)) = (x_2(t), p_2(t))$ and both of the solutions lie in $U \times V$, one has $e^{-\sigma_0 t}\theta(t)$ nondecreasing where $\theta(t) > 0$, and $e^{\sigma_0 t}\theta(t)$ nondecreasing where $\theta(t) < 0$.

Proof. The function θ is absolutely continuous, so it is nondecreasing over intervals where $\dot\theta(t) \geqslant 0$ almost everywhere, and it is strictly increasing over intervals where $\dot\theta(t) > 0$ almost everywhere. The justification of the final assertion of the corollary is seen by rewriting (3.2) as

$$\dot\theta(t) - \sigma_0 \mid \theta(t)\mid \geqslant 0 \qquad \text{a.e.}$$

Multiplying both sides of this by $e^{-\sigma_0 t}$, one finds that

$$(d/dt)\, e^{-\sigma_0 t}\theta(t) \geqslant 0 \qquad \text{where} \quad \theta(t) > 0,$$

$$(d/dt)\, e^{\sigma_0 t}\theta(t) \geqslant 0 \qquad \text{where} \quad \theta(t) < 0.$$

COROLLARY 2. *If among the solutions to the Hamiltonian system* (1.21) *over* $[0, +\infty)$ *satisfying*

$$(x(0), p(0)) = (a, b) \in U \times V, \tag{3.17}$$

$$\lim_{t\to+\infty} e^{-\rho t}x(t) \cdot p(t) = 0, \tag{3.18}$$

there is one such that $(x(t), p(t))$ *remains bounded and in* $U \times V$ *as* $t \to +\infty$, *then it is the unique solution to the system over* $[0, +\infty)$ *having either* $x(0) = a$ *or* $p(0) = b$.

Proof. Let $(x'(t), p'(t))$ also satisfy (3.18) with either $x'(0) = a$ or $p'(0) = b$. The corollary to Proposition 3 gives us

$$\int_0^{+\infty} L_0(x'(t), \dot{x}'(t))\, e^{-\rho t}\, dt + \int_0^{+\infty} M_0(p'(t), \dot{p}'(t) - \rho p'(t))\, e^{-\rho t}\, dt$$
$$= -x'(0) \cdot p'(0) < +\infty, \tag{3.19}$$

and, hence, by Proposition 2, we have

$$\lim_{t\to+\infty} e^{-\rho t}x'(t) = \lim_{t\to+\infty} e^{-\rho t}p'(t) = 0. \tag{3.20}$$

By virtue of the boundedness of $(x(t), p(t))$, the expression

$$\theta(t) = e^{-\rho t}(x'(t) - x(t)) \cdot (p'(t) - p(t)) \tag{3.21}$$

therefore satisfies

$$\lim_{t\to+\infty} \theta(t) = 0 = \theta(0). \tag{3.22}$$

This implies via Corollary 1 that $x'(t) = x(t)$ and $p'(t) = p(t)$ for all $t \in [0, +\infty)$.

Corollary 2 is the basis for the uniqueness assertion in Theorem 1. The rest of Theorem 1 relates only to the local behavior of H_0 and the corresponding Hamiltonian system near (0, 0). In the proof, therefore, there is no harm in replacing H_0 by any more convenient function that agrees with it on a neighborhood of (0, 0). The proposition below will allow us in this manner to derive Theorem 1 by way of results that are more global in nature.

PROPOSITION 6. *Let $C \times D$ be any compact convex neighborhood of $(0, 0)$ contained in the neighborhood $U \times V$ in the curvature assumption on H_0. Then there exists a function H_1 on $R^n \times R^n$ agreeing with H_0 on $C \times D$, such that $H_1(x, p)$ is everywhere finite, α-concave in x, and β-concave in p.*

Moreover, H_1 can be constructed in such a manner that the corresponding convex Lagrangians

$$L_1(x, z) = \sup\{p \cdot z - H_1(x, p) \mid p \in R^n\}, \tag{3.23}$$

$$M_1(p, u) = \sup\{u \cdot x + H_1(x, p) \mid x \in R^n\}, \tag{3.24}$$

are finite throughout $R^n \times R^n$.

Proof. Let

$$\overline{H}_0(x, p) = H_0(x, p) + \tfrac{1}{2}\alpha \mid x \mid^2 - \tfrac{1}{2}\beta \mid p \mid^2. \tag{3.25}$$

Then H_0 is concave–convex on $U \times V$. The construction given in [2, Proof of Proposition 3.1] furnishes a finite, concave–convex function $\overline{H}_1$ on $R^n \times R^n$ agreeing with H_0 on $C \times D$. Let

$$H_1(x, p) = \overline{H}_1(x, p) - \tfrac{1}{2}\alpha \mid x \mid^2 + \tfrac{1}{2}\beta \mid p \mid^2. \tag{3.26}$$

Then $H_1(x, p)$ is everywhere finite, α-concave in x, β-convex in p, and $H_1(x, p) = H_0(x, p)$ for (x, p) in $C \times D$. Moreover

$$\lim_{\lambda \to +\infty} H_1(x, \lambda p)/\lambda = +\infty, \tag{3.27}$$

$$\lim_{\lambda \to +\infty} H_1(\lambda x, p)/\lambda = -\infty, \tag{3.28}$$

so that the functions L_1 and M_1 in (3.23) and (3.24) must be finite everywhere [5, Corollary 13.3.1].

4. GLOBAL RESULTS

In view of Proposition 6, there is no loss of generality if in the rest of the development of the proof of Theorem 1 we invoke the following.

GLOBAL CURVATURE ASSUMPTION. *The function H_0 is actually finite and α-concave-β-convex throughout $R^n \times R^n$, i.e., the earlier curvature assumption is valid with $U \times V = R^n \times R^n$. Furthermore, the functions L_0 and M_0 are finite throughout $R^n \times R^n$.*

A crucial consequence of this assumption is the following property.

PROPOSITION 7. *Under the global curvature assumption, if $(x(t), p(t))$ satisfies the Hamiltonian system* (1.21) *over* $[0, +\infty)$, *then either*

$$\lim_{t\to+\infty} e^{-\rho t}x(t)\cdot p(t) = +\infty, \tag{4.1}$$

or one has

$$\lim_{t\to+\infty} (x(t), p(t)) = (0, 0). \tag{4.2}$$

Proof. We first apply Corollary 1 of Proposition 5 to $(x_1(t), p_1(t)) \equiv (x(t), p(t))$ and $(x_2(t), p_2(t)) \equiv (0, 0)$ to see that the function

$$\theta(t) = e^{-\rho t}x(t)\cdot p(t)$$

is nondecreasing, and in fact $e^{-\sigma_0 t}\theta(t)$ is nondecreasing on subintervals where $\theta(t) > 0$. Thus (4.1) holds unless $\theta(t) \leqslant 0$ for all $t \geqslant 0$. Suppose now that the latter is true, so that also

$$x(t)\cdot p(t) \leqslant 0 \qquad \text{for all} \quad t \geqslant 0. \tag{4.3}$$

From Proposition 5 we have at the same time

$$(d/dt)\, x(t)\cdot p(t) \geqslant \sigma_1[|x(t)| + |p(t)|]^2 \geqslant \sigma_1 |(x(t), p(t))|^2 \qquad \text{a.e.} \tag{4.4}$$

Hence, for all $T > 0$,

$$\sigma_1 \int_0^T |(x(t), p(t))|^2\, dt \leqslant x(T)\cdot p(T) - x(0)\cdot p(0) \leqslant -x(0)\cdot p(0). \tag{4.5}$$

This yields

$$\int_0^{+\infty} \zeta(t)\, dt < +\infty, \qquad \text{where} \quad \zeta(t) = |(x(t), p(t))|^2. \tag{4.6}$$

Since the concave–convex function H_0 is everywhere finite, its subdifferential multifunction ∂H_0 is bounded on bounded sets [6, Lemma 4], so that in particular there is a number λ such that the elements of the set $\partial H_0(a, b)$ are bounded in norm by λ when $|(a, b)| \leqslant 1$. Then, since $(x(t), p(t))$ satisfies the system (1.21), we have

$$|(\dot{x}(t), \dot{p}(t))| \leqslant \lambda + \rho \qquad \text{whenever} \quad |(x(t), p(t))| \leqslant 1. \tag{4.7}$$

For $\zeta(t)$ as in (4.6), this means that

$$\dot{\zeta}(t) \leqslant 2(\lambda + \rho) \qquad \text{whenever} \quad \zeta(t) \leqslant 1. \tag{4.8}$$

We proceed now to show that (4.6) and (4.8) imply

$$\lim_{t \to +\infty} \zeta(t) = 0, \tag{4.9}$$

a property equivalent, of course, to the desired conclusion (4.2). Fix any $\epsilon \in (0, 1)$ and let

$$S = \{t \in [0, +\infty) \mid \zeta(t) \leqslant \epsilon/2\}.$$

The set S is closed by the continuity of $\zeta(t)$, so its complement in $[0, +\infty)$ is the union of a sequence of intervals. The finiteness of the integral in (4.6) ensures that the intervals among these having length $\epsilon/2(\lambda + \rho)$ or greater are all contained in $[0, T]$ for some T sufficiently large. Then for every $t > T$, there exists $t_0 \in S$ such that $| t - t_0 | < \epsilon/4(\lambda + \rho)$. But (4.8) implies

$$\zeta(t) \leqslant \zeta(t_0) + 2(\lambda + \rho) | t - t_0 | \qquad \text{if} \quad t_0 \in S \text{ and } 2(\lambda + \rho) | t - t_0 | \leqslant \epsilon/2. \tag{4.10}$$

Thus $\zeta(t) \leqslant \epsilon$ if $t > 0$. Since ϵ can be taken arbitrarily small, (4.9) is indeed correct.

We next state a result for the functions f_T and g_T in (2.29) and (2.30) that does not make fullest use of the global curvature assumption, although the latter will enter via Proposition 7 when we argue later by taking the limit as $T \to +\infty$.

THEOREM 3. *Under the global curvature assumption, the function f_T for $0 < T < +\infty$ is everywhere continuously differentiable and strictly convex on $R^n \times R^n$, and the infimum in its definition is always attained by a unique arc. The same properties hold for g_T. Furthermore, one has the conjugacy relations*

$$g_T(b, b') = \max_{(a,a')} \{e^{-\rho T} a' \cdot b' - a \cdot b - f_T(a, a')\} = f_T^*(-b, e^{-\rho T} b'), \tag{4.11}$$

$$f_T(a, a') = \max_{(b,b')} \{e^{-\rho T} a' \cdot b' - a \cdot b - g_T(b, b')\} = g_T^*(-a, e^{-\rho T} a'), \tag{4.12}$$

and the gradient relation

$$(-b, e^{-\rho T} b') = \nabla f_T(a, a') \Leftrightarrow (-a, e^{-\rho T} a') = \nabla g_T(b, b'). \tag{4.13}$$

The conditions in (4.13) *are satisfied if and only if*

$$f_T(a, a') + g_T(b, b') = e^{-\rho T} a' \cdot b' - a \cdot b. \tag{4.14}$$

Proof. Relations (4.11) and (4.12) in "sup" form, and the existence of minimizing arcs, follow from [4, Corollary 2 of Theorem 1] and the global finiteness of the functions L_0 and M_0. This finiteness also implies from the definitions (2.29) and (2.30) that f_T and g_T are finite everywhere, and hence that "sup" can be strengthened to "max" in passing to the conjugate functions f_T^* and g_T^* [5, pp. 217–218]. In view of (4.11) and (4.12), we have the subdifferential relation

$$(-b, e^{-\rho T}b') \in \partial f_T(a, a') \Leftrightarrow (-a, e^{-\rho T}a') \in \partial g_T(b, b'), \qquad (4.15)$$

these conditions being equivalent to (4.14). Suppose now for $i = 1, 2$ that (a_i, a_i') and (b_i, b_i') are such that these conditions hold, and let $x_i(t)$ and $p_i(t)$ be corresponding arcs over $[0, T]$ furnishing the minima in the definition of $f_T(a_i, a_i')$ and $g_T(b_i, b_i')$. Then, according to the converse part of Proposition 4, $(x_i(t), p_i(t))$ satisfies the system (1.21) with

$$(x_i(0), p_i(0)) = (a_i, b_i) \qquad \text{and} \qquad (x_i(T), p_i(T)) = (a_i', b_i'). \qquad (4.16)$$

Invoking Corollary 1 of Proposition 5, we see that the function θ in (3.16) satisfies $\theta(T) > \theta(0)$, unless $(x_1(t), p_1(t)) = (x_2(t), p_2(t))$ for all $t \in [0, T]$. But

$$\theta(0) = (a_1 - a_2) \cdot (b_1 - b_2), \qquad \theta(T) = e^{-\rho T}(a_1' - a_2') \cdot (b_1' - b_2'). \qquad (4.17)$$

Therefore, the equation $(a_1, a_1') = (a_2, a_2')$ implies $(b_1, b_1') = (b_2, b_2')$, and conversely. This shows that the subdifferential multifunctions ∂f_T and ∂g_T are actually one-to-one functions, so that f_T and g_T must be differentiable and strictly convex [5, Corollary 26.3.1]. The argument also shows the uniqueness of the minimizing arcs over $[0, T]$, and the proof of Theorem 3 is, therefore, complete.

The first consequence of Theorem 3 which we derive concerns the existence of minimizing arcs in the definitions of the functions f_+ and g_+.

PROPOSITION 8. *Under the global curvature assumption, the convex function f_+ is everywhere finite on R^n, and for each $a \in R^n$, there is a unique arc over $[0, +\infty)$ furnishing the minimum in the definition of $f_+(a)$.*

Proof. Since L_0 is finite and has the properties (1.29), it is evident that

$$0 \leqslant f_+(a) \leqslant f_T(a, 0) < +\infty \qquad \text{for all} \quad T > 0. \qquad (4.18)$$

Hence, f_+ is finite. Now fix any $a \in R^n$. The definitions of f_+ and f_T yield the identity

$$f_+(a) = \inf_{a' \in R^n} \{f_T(a, a') + e^{-\rho T} f_+(a')\} \qquad \text{for all} \quad T > 0. \qquad (4.19)$$

We know from Theorem 3 that f_T is finite, strictly convex, and cofinite (i.e., has an everywhere-finite conjugate function $f_T{}^*$). In particular, this implies that, as a function of a', $f_T(a, a')$ is strictly convex and satisfies the growth condition

$$\lim_{\lambda \to +\infty} [f_T(a, a' + \lambda a'') - f_T(a, a')]/\lambda = +\infty \quad \text{whenever} \quad a'' \neq 0 \tag{4.20}$$

[5, Corollary 13.3.1]. The convexity of f_+ ensures, of course, that the difference quotient $[f_+(a' + \lambda a'') - f_+(a')]/\lambda$ is always nondecreasing in λ, so it follows that for each $T > 0$, the function

$$h_T(a') = f_T(a, a') + e^{-\rho T} f_+(a') \tag{4.21}$$

is everywhere finite on R^n, strictly convex and satisfies

$$\lim_{\lambda \to +\infty} [h_T(a' + \lambda a'') - h_T(a')]/\lambda = +\infty \quad \text{whenever} \quad a'' \neq 0. \tag{4.22}$$

Therefore, h_T attains its minimum over R^n at a unique point [5, Theorem 27.2]. Let us denote this point by $x(T)$, defining also $x(0) = a$. We then have the identiy

$$f_+(a) = f_T(a, x(T)) + e^{-\rho T} f_+(x(T)) \qquad \text{for all} \quad T > 0. \tag{4.23}$$

Note that this identity would also have to be satisfied by any arc giving the minimum in the definition of $f_+(a)$, so the function $x : [0, +\infty) \to R^n$ that we have constructed is the unique candidate for such an arc. To verify that x is absolutely continuous, we temporarily fix T and let y denote the unique arc over $[0, T]$ giving the minimum in the definition of $f_T(a, x(T))$ (cf. Theorem 3). For all $S \in (0, T)$, it is true that

$$f_T(a, x(T)) = f_S(a, y(S)) + e^{-\rho S} f_{T-S}(y(S), x(T)), \tag{4.24}$$

$$f_+(y(S)) \leqslant f_{T-S}(y(S), x(T)) + e^{-\rho(T-S)} f_+(x(T)). \tag{4.25}$$

combining (4.24) and (4.25) with (4.23), we obtain

$$f_+(a) \geqslant f_S(a, y(S)) + e^{-\rho S} f_+(y(S)). \tag{4.26}$$

On the other hand, the formula

$$f_+(a) = \min_{a' \in R^n} \{f_S(a, a') + e^{-\rho S} f_+(a')\} \tag{4.27}$$

holds, with the minimum attained uniquely at the point $x(S)$. Therefore (4.26) implies $y(S) = x(S)$. This is true for all $S \in (0, T)$, so the arcs x and

y coincide over $[0, T]$. Thus x is absolutely continuous, and for every $T > 0$ we have

$$f_T(a, x(T)) = \int_0^T L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt. \tag{4.28}$$

Plugging the latter into (4.23), we get

$$f_+(a) \geqslant \int_0^T L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \qquad \text{for all} \quad T > 0,$$

and, therefore,

$$f_+(a) \geqslant \int_0^{+\infty} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt. \tag{4.29}$$

Thus x must give the minimum in the definition of $f_+(a)$ and is the unique arc to do so.

The argument establishing the assertions of Proposition 8 about g_+ is entirely parallel.

PROPOSITION 9. *Under the global curvature assumption, we have*

$$f_+(a) = \lim_{T \to +\infty} [\min_{a' \in R^n} f_T(a, a')], \tag{4.30}$$

$$g_+(b) = \lim_{T \to +\infty} [\min_{b' \in R^n} g_T(b, b')]. \tag{4.31}$$

Proof. Let

$$\psi_T(a) = \min_{a' \in R^n} f_T(a, a'). \tag{4.32}$$

(The "min", in place of "inf", is appropriate because of the growth property of f_T displayed in (4.20).) The identity

$$f_T(a, a') = \inf_{c \in R^n} \{f_S(a, c) + e^{-\rho S} f_{T-S}(c, a')\} \qquad \text{for} \quad 0 < S < T \tag{4.33}$$

shows that

$$f_T(a, a') \geqslant \inf_{c \in R^n} f_S(a, c) = \psi_S(a), \tag{4.34}$$

and it is true, therefore, that

$$\psi_T(a) \geqslant \psi_S(a) \qquad \text{for} \quad 0 < S < T < +\infty. \tag{4.35}$$

At the same time we have

$$f_+(a) \leqslant \psi_T(a) \qquad \text{for all } T > 0 \tag{4.36}$$

by (4.21) and the nonnegativity of f_+. Hence, the limit

$$\psi_+(a) \triangleq \lim_{t\to+\infty} \psi_T(a) \tag{4.37}$$

exists and satisfies

$$0 \leqslant \psi_+(a) \leqslant f_+(a) < +\infty. \tag{4.38}$$

We must show that also $\psi_+(a) \geqslant f_+(a)$. The convexity of f_T implies that of the functions ψ_T and, hence, that of ψ_+. The next step consists of demonstrating that ψ_+, like f_+, satisfies

$$\psi_+(a) = \min_{a' \in R^n} \{f_T(a, a') + e^{-\rho T}\psi_+(a')\} \qquad \text{for all} \quad T > 0. \tag{4.39}$$

Certainly the definitions imply

$$\psi_{T+S}(a) = \inf_{a' \in R^n} \{f_T(a, a') + e^{-\rho T}\psi_S(a')\} \tag{4.40}$$

for all $T > 0$, $S > 0$. In particular, then, we have

$$\psi_{T+S}(a) \leqslant f_T(a, a') + e^{-\rho T}\psi_S(a') \qquad \text{for all} \quad a' \in R^n, \tag{4.41}$$

and passing to the limit as $S \to +\infty$, we get

$$\psi_+(a) \leqslant f_T(a, a') + e^{-\rho T}\psi_+(a') \qquad \text{for all} \quad a' \in R^n. \tag{4.42}$$

On the other hand, let us fix any $a \in R^n$ and $T > 0$, and consider the function

$$k_S(a') = f_T(a, a') + e^{-\rho T}\psi_S(a'). \tag{4.43}$$

Since ψ_S is finite and convex, while f_T is strictly convex with the growth property (4.20), we have k_S strictly convex with

$$\lim_{\lambda\to+\infty} [k_S(a' + \lambda a'') - k_S(a')]/\lambda = +\infty \qquad \text{if} \quad a'' \neq 0. \tag{4.44}$$

It follows that k_S attains its minimum over R^n at a unique point a_S'. Then

$$\psi_{T+S}(a) = f_T(a, a_S') + e^{-\rho T}\psi_S(a_S'). \tag{4.45}$$

Observe that this relation entails

$$f_T(a, a_S') \leqslant \psi_{T+S}(a) \leqslant \psi_+(a),$$

and, hence,

$$a_S' \in B = \{a' \in R^n \mid f_T(a, a') \leqslant \psi_+(a)\}. \tag{4.46}$$

The set B is bounded, by virtue of the growth property (4.20) [5, Theorems 8.4 and 27.1(f)], so that (4.46) implies the existence of a cluster point of a_S' as $S \to +\infty$. Since the functions ψ_S are convex and converge pointwise to ψ_+ on R^n, they actually converge uniformly on all bounded sets [5, Theorem 10.8]. Therefore, in passing to the limit as $S \to +\infty$ in (4.45), we have

$$\psi_+(a) = f_T(a, a') + e^{-\rho T}\psi_+(a'), \tag{4.47}$$

where a' is any cluster point of a_s' as $s \to +\infty$. Thus, equality does hold in (4.42) for some a', and (4.39) is correct.

We next apply to (4.39) the same argument we applied in the proof of Proposition 8 to the parallel formula for f_+. This yields the existence for each $a \in R^n$ of an arc x over $[0, +\infty)$ satisfying $x(0) = a$,

$$\psi_+(a) = f_T(a, x(T)) + e^{-\rho T}\psi_+(x(T)), \tag{4.48}$$

$$f_T(a, x(T)) = \int_0^T L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt, \tag{4.49}$$

for all $T > 0$. But then

$$\psi_+(a) \geqslant \int_0^{+\infty} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \geqslant f_+(a). \tag{4.50}$$

This completes the proof of Proposition 9, the argument for g_+ being parallel.

PROPOSITION 10. *Let F_+ and G_+ be the functions defined like f_+ and g_+ in* (1.37) *and* (1.38), *but with the infima taken only over arcs which are bounded over* $[0, +\infty)$. *Then*

$$\lim_{T\to+\infty} f_T(a, a') = F_+(a) \qquad \textit{for all} \quad a' \in R^n, \tag{4.51}$$

$$\lim_{T\to+\infty} g_T(b, b') = G_+(b) \qquad \textit{for all} \quad b' \in R^n. \tag{4.52}$$

Proof. The properties (1.29) of L_0 imply that $f_T(a, 0)$ is nonincreasing as a function of $T > 0$, and $f_T(a, 0) \geqslant F_+(a)$. (Any arc x over $[0, T]$ with $x(0) = a$ and $x(T) = 0$ can be continued over $[0, +\infty)$ by defining $x(t) = 0$ for all $t > T$.) Thus, the function

$$\tilde{F}_+(a) \triangleq \lim_{T\to+\infty} f_T(a, 0) \geqslant F_+(a) \tag{4.53}$$

is well defined. For $T > S > 0$, we have

$$f_{T+S}(a, 0) \leqslant f_T(a, a') + e^{-\rho T} f_S(a', 0), \tag{4.54}$$

$$f_T(a, a') \leqslant f_{T-S}(a, 0) + e^{-\rho(T-S)} f_S(0, a'). \tag{4.55}$$

Taking the limit in these inequalities as $T \to +\infty$ for fixed S and a', we obtain, respectively,

$$\tilde{F}_+(a) \leqslant \liminf_{T \to +\infty} f_T(a, a'), \tag{4.56}$$

$$\limsup_{T \to +\infty} f_T(a, a') \leqslant \tilde{F}_+(a), \tag{4.57}$$

and, hence,

$$\lim_{T \to +\infty} f_T(a, a') = \tilde{F}_+(a). \tag{4.58}$$

We finish the proof of (4.51) by showing that $\tilde{F}_+(a) \leqslant F_+(a)$. Fix $a \in R^n$ and $\epsilon > 0$, and let x be a bounded arc over $[0, +\infty)$ such that $x(0) = a$ and

$$\int_0^\infty L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt < F_+(a) + \epsilon. \tag{4.59}$$

Then for all $T > 0$, it is true that

$$F_+(a) + \epsilon > \int_0^T L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \geqslant f_T(a, x(T)). \tag{4.60}$$

According to (4.58), the functions $a' \to f_T(a, a')$ converge as $T \to +\infty$ to the constant function $a' \to \tilde{F}_+(a)$, and since the functions are convex, the convergence must be uniform on all bounded sets [5, Theorem 10.8], in particular on the set $\{x(T) \mid 0 \leqslant T < +\infty\}$ Therefore,

$$\lim_{T \to +\infty} f_T(a, x(T)) = \tilde{F}_+(a), \tag{4.61}$$

and (4.60) thus implies $F_+(a) + \epsilon \geqslant \tilde{F}_+(a)$. Since $\epsilon > 0$ was arbitrary, we are able to conclude $F_+(a) \geqslant \tilde{F}_+(a)$ as aimed.

The argument for g_+ and G_+ is parallel.

Our main "global" result can now be treated.

THEOREM 4. *Under the global curvature assumption, the functions f_+ and g_+ are everywhere continuously differentiable and strictly convex on R^n, and they satisfy the conjugacy relations*

$$g_+(b) = \max_{a \in R^n} \{-a \cdot b - f_+(a)\} = f_+^*(-b), \tag{4.62}$$

$$f_+(a) = \max_{b \in R^n} \{-a \cdot b - g_+(b)\} = g_+^*(-a). \tag{4.63}$$

Furthermore, one has

$$\begin{aligned}(a, b) \in K_+ &\Leftrightarrow f_+(a) + g_+(b) + a \cdot b = 0 \\ &\Leftrightarrow b = -\nabla f_+(a) \Leftrightarrow a = -\nabla g_+(b).\end{aligned} \tag{4.64}$$

For each $(a, b) \in K_+$, there is a unique solution $(x(t), p(t))$ to the Hamiltonian system (1.21) *over* $[0, +\infty)$ *satisfying* (1.35) *and* (1.36), *and it tends to* $(0, 0)$ *as* $t \to +\infty$. *In fact, x is the unique arc furnishing the minimum in the definition of $f_+(a)$, while p is the unique arc furnishing the minimum in the definition of $g_+(b)$.*

Proof. Defining ψ_T as in (4.32), we have from Theorem 3 the relation

$$\begin{aligned} g_T(b, 0) &= \max_{(a,a')} \{-a \cdot b - f_T(a, a')\} \\ &= \max_a \{-a \cdot b - \psi_T(a)\} = \psi_T{}^*(-b). \end{aligned} \tag{4.65}$$

As $T \to +\infty$, the convex functions ψ_T converge pointwise to f_+ by Proposition 9, while the convex functions $b \to g_T(b, 0)$ converge pointwise to G_+ by Proposition 10. Thus ψ_T and the conjugate $\psi_T{}^*$ converge to finite limit functions as $T \to +\infty$, implying that these limit functions must be conjugate to each other (cf. [6; 7; 5, Theorem 10.8]). Therefore, $G_+(b) = f_+{}^*(-b)$, so that (again by virtue of the finiteness of the two functions)

$$G_+(b) = \max_{a \in R^n} \{-a \cdot b - f_+(a)\} \tag{4.66}$$

and, reciprocally,

$$f_+(a) = \max_{b \in R^n} \{-a \cdot b - G_+(b)\}. \tag{4.67}$$

The next thing to note is that G_+ satisfies the identity

$$G_+(b) = \inf_{b' \in R^n} \{g_T(b, b') + e^{-\rho T} G_+(b')\} \qquad \text{for all} \quad T > 0. \tag{4.68}$$

This is evident from the definition of G_+ in Proposition 10. The same argument used in connection with Formula (4.19) in the proof of Proposition 8 shows for each $b \in R^n$ the existence of a unique arc p over $[0, +\infty)$ satisfying $p(0) = b$ and

$$G_+(b) = g_T(b, p(T)) + e^{-\rho T} G_+(p(T)) \qquad \text{for all} \quad T > 0, \tag{4.69}$$

$$g_T(b, p(T)) = \int_0^T M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \qquad \text{for all} \quad T > 0. \tag{4.70}$$

Now fix any $a \in R^n$ and let x be the unique arc over $[0, +\infty)$ giving the minimum in the definition of $f_+(a)$, as exists by Proposition 8. Then

$$f_+(a) = f_T(a, x(T)) + e^{-\rho T} f_+(x(T)) \qquad \text{for all} \quad T > 0, \tag{4.71}$$

$$f_T(a, x(T)) = \int_0^T L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \qquad \text{for all} \quad T > 0. \tag{4.72}$$

Let b be such that the maximum in (4.67) is attained, and let p be a corresponding arc over $[0, +\infty)$ with $p(0) = b$ satisfying (4.69) and (4.70). We then have from (4.69) and (4.71) that

$$\begin{aligned}0 &= f_+(a) + G_+(b) + a \cdot b \\ &= f_T(a, x(T)) + g_T(b, p(T)) + e^{-\rho T}[f_+(x(T)) + G_+(p(T))] + a \cdot b \\ &= f_T(a, x(T)) + g_T(b, p(T)) - e^{-\rho T} x(T) \cdot p(T) + a \cdot b \\ &\quad + e^{-\rho T}[f_+(x(T)) + G_+(p(T)) + x(T) \cdot p(T)],\end{aligned} \tag{4.73}$$

and, consequently, by virtue of the conjugacy relations (4.66) and (4.11),

$$f_T(a, x(T)) + g_T(b, p(T)) - e^{-\rho T} x(T) \cdot p(T) + a \cdot b = 0 \quad \text{for all} \quad T > 0, \tag{4.74}$$

$$f_+(x(T)) + G_+(p(T)) + x(T) \cdot p(T) = 0 \quad \text{for all} \quad T > 0. \tag{4.75}$$

Substituting (4.70) and (4.72) into (4.74) and applying Proposition 3, one sees that $(x(t), p(t))$ is a solution to the Hamiltonian system (1.21). Moreover, (4.74) implies in conjunction with (4.69) and (4.71) that

$$\begin{aligned}\limsup_{T \to +\infty} e^{-\rho T} x(T) \cdot p(T) &= \limsup_{T \to +\infty} [f_T(a, x(T)) + g_T(b, p(T)) + a \cdot b] \\ &\leqslant f_+(a) + G_+(b) + a \cdot b < +\infty,\end{aligned} \tag{4.76}$$

and, hence, via Proposition 7, that

$$\lim_{t \to +\infty} (x(t), p(t)) = (0, 0). \tag{4.77}$$

The minimizing arc x is thus bounded over $[0, +\infty)$, and therefore $f_+(a) = F_+(a)$. This has been verified for an arbitrary $a \in R^n$, so actually $f_+ \equiv F_+$.

A parallel argument shows that likewise $g_+ \equiv G_+$. Thus, the formulas derived above for f_+ and G_+ are actually valid for f_+ and g_+; both (4.62) and (4.63) are valid, and for each (a, b) in the set

$$\begin{aligned}K_+' &= \{(a, b) \mid f_+(a) + g_+(b) + a \cdot b = 0\} \\ &= \{(a, b) \mid f_+(a) + g_+(b) + a \cdot b \leqslant 0\} \\ &= \{(a, b) \mid -b \in \partial f(a)\} = \{(a, b) \mid -a \in \partial g_+(b)\},\end{aligned} \tag{4.78}$$

there exists a solution to the system (1.21) over $[0, +\infty)$ with $(x(0), p(0)) = (a, b)$, satisfying (4.77) and (by (4.75))

$$(x(T), p(T)) \in K_+' \quad \text{for all} \quad T > 0. \tag{4.79}$$

But this implies $K_+' \subset K_+$, whereas on the other hand, the inclusion $K_+ \subset K_+'$ follows from the corollary to Proposition 3. Therefore, $K_+' = K_+$, and for each $(a, b) \in K_+$, there is a solution to the system (1.21) over $[0, +\infty)$ with $(x(0), p(0)) = (a, b)$ which remains in K_+ and tends to $(0, 0)$ as $t \to +\infty$. Corollary 2 of Proposition 5 tells us this must be the unique solution to (1.21) satisfying

$$\lim_{t\to\infty} e^{-\rho t} x(t) \cdot p(t) = 0,$$

and having either $x(0) = a$ or $p(0) = b$. Hence, for each $a \in R^n$, there is no more than one $b \in R^n$ with $(a, b) \in K_+$, and for each $b \in R^n$ there is no more than one $a \in R^n$ with $(a, b) \in K_+$. Since $K_+ = K_+'$, this says that the subgradient sets $\partial f_+(a)$ and $\partial g_+(b)$ never contain more than one element. It follows that f_+ and g_+ are continuously differentiable [5, Theorems 23.4, 25.1, and 25.5] and in view of their conjugacy relationship, also strictly convex [5, Theorem 26.3].

COROLLARY. *Under the global curvature assumption, one has* $F_+ = f_+$ *and* $G_+ = g_+$.

Proof of Theorem 1. As already has been noted, there is no loss of generality if the global curvature assumption is invoked in the proof of Theorem 1. (The uniqueness assertion is covered by Corollary 2 of Proposition 3.) Thus, we can place ourselves in the context of Theorem 4, according to which K_+ is the graph of a homeomorphism from R^n onto R^n (namely the mapping $-\nabla f_+$, whose inverse is $-\nabla g_+$). The task is to show that, given any neighborhood $U \times V$ of $(0, 0)$ in $R^n \times R^n$, there exists an open neighborhood $U_+ \times V_+$ of $(0, 0)$ such that $K_+ \cap (U_+ \times V_+)$ is the graph of a homeomorphism from U_+ onto V_+, and for each $(a, b) \in K_+ \cap (U_+ \times V_+)$, the solution to the system (1.21) starting at (a, b) and tending to $(0, 0)$ stays entirely within $K_+ \cap (U_+ \times V_+)$. Since K_+ is already the graph of a global homeomorphism, the local homeomorphism property will certainly be satisfied if U_+ and V_+ are taken to be of the form

$$\begin{aligned} U_+ &= \{a \mid \exists b \quad \text{with} \quad (a, b) \in K_+ \cap W\}, \\ V_+ &= \{b \mid \exists a \quad \text{with} \quad (a, b) \in K_+ \cap W\}, \end{aligned} \tag{4.80}$$

where W is some open neighborhood of $(0, 0)$; one then has

$$K_+ \cap (U_+ \times V_+) = K_+ \cap W. \tag{4.81}$$

Thus the proof is reduced to showing that every neighborhood $U \times V$ of

$(0, 0)$ contains an open neighborhood W of $(0, 0)$ such that, for each $(a, b) \in K_+ \cap W$, the solution to the system (1.21) starting at (a, b) and tending to $(0, 0)$ stays entirely within $K_+ \cap W$. Actually, such a solution $(x(t), p(t))$ remains in K_+ by definition and, hence, by Theorem 4, it satisfies

$$f_+(x(t)) + g_+(p(t)) + x(t) \cdot p(t) = 0 \qquad \text{for all} \quad t > 0. \tag{4.82}$$

Moreover, $x(t) \cdot p(t)$ is nondecreasing in t; this follows from Proposition 5 in the case of $(x_1(t), p_1(t)) = (x(t), p(t))$ and $(x_2(t), p_2(t)) \equiv (0, 0)$. Hence, the expression $f_+(x(t)) + g_+(p(t))$ is nonincreasing. This indicates that the desired properties can be obtained by taking W to be of the form

$$W = \{(a, b) \mid f_+(a) + g_+(b) < \epsilon\}, \tag{4.83}$$

provided this set is indeed, for $\epsilon > 0$ sufficiently small, an open neighborhood of $(0, 0)$ contained in whatever neighborhood $U \times V$ has been specified. But the latter properties follow from the fact that f_+ and g_+ are finite, strictly convex functions (hence, continuous) satisfying by definition

$$f_+(a) \geqslant f_+(0) = 0 \qquad \text{and} \qquad g_+(b) \geqslant g_+(0) = 0 \tag{4.84}$$

(cf. [5, Theorem 27.2]).

5. Results for the Interval $(-\infty, 0]$

Most of the results for $[0, +\infty)$ can easily be derived in a parallel form for $(-\infty, 0]$ but with some important simplifications. In building up to the proofs of Theorems 1′ and 2′, we begin with facts corresponding to those in Proposition 2.

False

Proposition 2′. *The inequalities*

$$\int_{-\infty}^{0} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt < +\infty$$

and (5.1)

$$\int_{-\infty}^{0} M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt < +\infty$$

imply, respectively, that

$$\lim_{t \to -\infty} e^{-\rho t} x(t) = 0 \qquad \textit{and} \qquad \lim_{t \to -\infty} e^{-\rho t} p(t) = 0. \tag{5.2}$$

$L_0(x,v) = \frac{\alpha}{2}x^2 + \frac{\beta}{2}v^2$ on $R \times R$ satisfies the curvature condition globally. But for $x(t) = e^{\frac{3}{4}\rho t}$ with arb. $\rho > 0$ one has $\int_{-\infty}^{0} L_0(x, \dot{x}) e^{-\rho t} dt =$ (const) $\int_{-\infty}^{0} e^{\frac{\rho}{2}t} dt +$ (const) $\int_{-\infty}^{0} e^{\frac{\rho}{2}t} dt < \infty$ yet $x(t) e^{-\rho t} = e^{-\frac{\rho}{4}t} \to \infty$ as $t \to -\infty$

Proof. The argument is based on Proposition 1 and follows exactly the same lines as the first part of the proof of Proposition 2.

COROLLARY. *If* (5.1) *holds, then*

$$\lim_{t\to-\infty} e^{-\rho t}x(t)\cdot p(t) = 0. \tag{5.3}$$

PROPOSITION 3′. *For* $-\infty < T < 0$, *one always has*

$$\int_T^0 L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt + \int_T^0 M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt$$
$$\geqslant x(0)\cdot p(0) - e^{-\rho T}x(T)\cdot p(T). \tag{5.4}$$

Furthermore, equality holds here if and only if $(x(t), p(t))$ *satisfies the Hamiltonian system* (1.21) *over* $[T, 0]$.

Proof. Same argument as for Proposition 3.

COROLLARY 1. *For arbitrary arcs* x *and* p *over* $(-\infty, 0]$, *one has*

$$\int_{-\infty}^0 L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt + \int_{-\infty}^0 M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt$$
$$\geqslant x(0)\cdot p(0), \tag{5.5}$$

with equality if and only if $(x(t), p(t))$ *is a solution to system* (1.21) *over* $(-\infty, 0]$ *which satisfies* (5.3).

COROLLARY 2. *One has*

$$f_-(a) + g_-(b) \geqslant a\cdot b \qquad \textit{for all} \quad (a, b) \in R^n \times R^n, \tag{5.6}$$

$$K_- \subset \{(a, b) \mid f_-(a) + g_-(b) = a\cdot b\}. \tag{5.7}$$

In fact, K_- *consists precisely of the pairs* (a, b) *belonging to the set on the right in* (5.7) *such that the infima in the definitions of* $f_-(a)$ *and* $g_-(b)$ *are both attained by arcs* x *and* p, *respectively; such pairs of arcs give the solutions* $(x(t), p(t))$ *to the system* (1.21) *described in the definition of* K_-, *and they always satisfy* (5.2) *and consequently*

$$\lim_{t\to-\infty} (x(t), p(t)) = (0, 0). \tag{5.8}$$

Note the absence in Corollary 1 of any assumption on $e^{-\rho t}x(t)\cdot p(t)$, which is justified by the corollary to Proposition 2′. This has produced the general inequality (5.6). We were able to derive the similar inequality for f_+ and g_+ only under the *global* curvature assumption.

Proof of Theorem 2′ Using Theorem 1′. This is obvious in view of Corollary 2 above, except for the assertions about ∇f_- and ∇g_-. The relations (5.6) and (5.7) do imply that

$$K_- \subset \{(a, b) \mid b \in \partial f_-(a)\}, \tag{5.9}$$

and, hence, the homeomorphism Θ in Theorem 1′ from U_- onto V_- whose graph is $K_- \cap (U_- \times V_-)$ satisfies

$$\Theta(a) \in \partial f_-(a) \qquad \text{for all} \quad a \in U_-. \tag{5.10}$$

But f_- is a convex function which is finite on U_-. Therefore, $\partial f_-(a)$ reduces almost everywhere on U_- to the gradient $\nabla f_-(a)$ [5, Theorem 25.5]. Moreover, at the remaining points $a \in U_-$, $\partial f_-(a)$ can be constructed as the convex hull of the limiting values of gradients $\nabla f_-(a')$ existing at points a' near a [5, Theorem 25.6]. The existence of a homeomorphism Θ satisfying (5.10) therefore implies that $\partial f_-(a)$ reduces everywhere on U_- to a single element, and this must then be the gradient $\nabla f_-(a)$. Thus (5.10) is equivalent to

$$\Theta(a) = \nabla f_-(a) \qquad \text{for all} \quad a \in U_-, \tag{5.11}$$

and in particular f_- is continuously differentiable on U_-. This, with a similar argument for g_-, justifies the second and third equivalences in (1.46).

The next result corresponds to Proposition 7 and Corollary 2 of Proposition 5.

PROPOSITION 11. *Let $(x(t), p(t))$ be a solution to the system* (1.21) *over* $(-\infty, 0]$ *with* $(x(t), p(t)) = (a, b)$. *Then either*

$$\lim_{t \to -\infty} e^{-\rho t} x(t) \cdot p(t) = -\infty \tag{5.12}$$

or one has (5.3), *and hence* $(a, b) \in K_+$. *If* (5.3) *holds and* $(x(t), p(t))$ *remains in* $U \times V$ *(the neighborhood of* $(0, 0)$ *in the basic curvature assumption) for all* $t \in (-\infty, 0]$, *then any solution* $(x'(t), p'(t))$ *to* (1.21) *likewise with property* (5.3) *and having either* $x'(0) = a$ *or* $p'(0) = b$, *must satisfy*

$$(x'(t), p'(t)) = (x(t), p(t)) \qquad \textit{for all} \quad t \in (-\infty, 0]. \tag{5.13}$$

Proof. From Proposition 3′, we have

$$e^{-\rho T}x(T)\cdot p(T) = x(0)\cdot p(0) - \int_T^0 L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt - \int_T^0 M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \qquad (5.14)$$

for all $T < 0$. If the limit of the right side of (5.11) is not $-\infty$, then (5.3) holds by the corollary to Proposition 2′, and indeed the stronger relations (5.2) are valid. If $(x'(t), p'(t))$ also has these properties and has either $x'(0) = a$ or $p'(0) = b$, the function

$$\theta(t) = e^{-\rho t}(x'(t) - x(t))\cdot(p'(t) - p(t)) \qquad (5.15)$$

satisfies

$$\lim_{t\to-\infty} \theta(t) = 0 = \theta(0). \qquad (5.16)$$

Assuming that $(x(t), p(t)) \in U \times V$ for all $t\,(-\infty, 0]$, we must have (5.13) because of Corollary 1 of Proposition 5.

PROPOSITION 8′. *Under the global curvature assumption, the convex function f_- is everywhere finite on R^n, and for each $a \in R^n$, there is a unique arc x over $(-\infty, 0]$ furnishing the minimum in the definition of $f_-(a)$. Similarly, g_- is finite everywhere on R^n, and for each $b \in R^n$, there is a unique arc p over $(-\infty, 0]$ furnishing the minimum in the definition of $g_-(b)$.*

Proof. Closely parallel to Proposition 8. No new version of Theorem 3 need be stated for the intervals $[T, 0]$, since for $T < 0$, one has

$$\inf\left\{\int_T^0 L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \,\middle|\, x(T) = a', x(0) = a\right\} = e^{-\rho T} f_{-T}(a', a), \qquad (5.17)$$

$$\inf\left\{\int_T^0 M_0(p(t), \dot{p}(t) - \rho p(t))\, e^{-\rho t}\, dt \,\middle|\, p(T) = b', p(0) = b\right\} = e^{-\rho T} g_{-T}(b', b). \qquad (5.18)$$

We next derive facts corresponding to Propositions 9 and 10.

PROPOSITION 12. *Under the global curvature assumption, one has*

$$\lim_{T\to-\infty} e^{-\rho T} f_{-T}(a', a) = f_-(a) \quad \text{if } a' = 0, \qquad = +\infty \quad \text{if } a' \neq 0, \qquad (5.19)$$

$$\lim_{T\to-\infty} e^{-\rho T} g_{-T}(b', b) = g_-(b) \quad \text{if } b' = 0, \qquad = +\infty \quad \text{if } b' \neq 0, \qquad (5.20)$$

and consequently also

$$\lim_{T\to-\infty} [\min_{a' \in R^n} e^{-\rho T} f_{-T}(a', a)] = f_-(a), \tag{5.21}$$

$$\lim_{T\to-\infty} [\min_{b' \in R^n} e^{-\rho T} g_{-T}(b', b)] = g_-(b). \tag{5.22}$$

Proof. Note first from (5.14) that

$$e^{-\rho T} f_{-T}(0, a) \geqslant f_-(a) \qquad \text{for all} \quad T < 0, \tag{5.23}$$

where the left side is nonincreasing as $T \to -\infty$ and, hence, approaches a limit. To show that the limit is $f_-(a)$, select an arbitrary $\epsilon > 0$ and any arc x over $(-\infty, 0]$ such that $x(0) = a$ and

$$\int_{-\infty}^{0} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt < f_-(a) + \epsilon. \tag{5.24}$$

For all T sufficiently low, one will have

$$\int_{T+1}^{0} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt < f_-(a) + \epsilon. \tag{5.25}$$

Let $\bar{x}$ denote the arc corresponding to the minimum in the definition of $f_1(0, x(T+1))$ (as exists by Theorem 3), and define the arc x' over $[T, 0]$ by

$$\begin{aligned} x'(t) &= \bar{x}(t - T) &\quad \text{for} \quad T \leqslant t \leqslant T+1, \\ &= x(t) &\quad \text{for} \quad T+1 \leqslant t \leqslant 0. \end{aligned} \tag{5.26}$$

Then, by (5.17),

$$\begin{aligned} e^{-\rho T} f_{-T}(0, a) &\leqslant \int_{T}^{0} L_0(x'(t), \dot{x}'(t))\, e^{-\rho t}\, dt \\ &= \int_{T}^{T+1} L_0(\bar{x}(t), \dot{\bar{x}}(t))\, e^{-\rho t}\, dt + \int_{T+1}^{0} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt \\ &= e^{-\rho T} f_1(0, x(T+1)) + \int_{T+1}^{0} L_0(x(t), \dot{x}(t))\, e^{-\rho t}\, dt. \end{aligned} \tag{5.27}$$

On the other hand, since f_1 is convex with $0 = f_1(0, 0) = \min f_1$, it is true (for $T < 0$) that

$$0 \leqslant e^{-\rho T} f_1(0, x(T+1)) \leqslant f_1(0, e^{-\rho T} x(T+1)). \tag{5.28}$$

But (5.24) implies via Proposition 2′ that

$$\lim_{T\to-\infty} e^{-\rho T} x(T+1) = 0. \tag{5.29}$$

Hence,

$$\lim_{T\to-\infty} e^{-\rho T} f_1(0, x(T+1)) = 0, \tag{5.30}$$

because f_1 is continuous (Theorem 3). From this and inequalities (5.25) and (5.27), it is clear that one will have

$$e^{-\rho T} f_{-T}(0, a) < f_-(a) + \epsilon \tag{5.31}$$

for all T sufficiently low. The first limit assertion in (5.19) is thereby established. The second part of (5.19) follows from the fact that

$$\lim_{T\to-\infty} f_{-T}(a', a) = f_+(a') > 0 \qquad \text{if} \quad a' \neq 0, \tag{5.32}$$

which is valid because of Proposition 10 and the corollary to Theorem 4; since f_+ is nonnegative and strictly convex by Theorem 4, it cannot vanish except at the origin. The implication from (5.19) to (5.21) can be verified using the convexity of the expression $e^{-\rho T} f_{-T}(a', a)$ as a function of a' for fixed a.

The proofs of (5.20) and (5.22) are analogous.

THEOREM 4′. *Under the global curvature assumption, the functions f_- and g_- are everywhere continuously differentiable and strictly convex on R^n, and they satisfy the conjugacy relations*

$$g_-(b) = \max_{a\in R^n} \{a \cdot b - f_-(a)\} = f_-^*(b), \tag{5.33}$$

$$f_-(a) = \max_{b\in R^n} \{a \cdot b - g_-(b)\} = g_-^*(a). \tag{5.34}$$

Furthermore, one has

$$\begin{aligned}(a, b) \in K_- &\Leftrightarrow f_-(a) + g_-(b) = a \cdot b \\ &\Leftrightarrow b = \nabla f_-(a) \Leftrightarrow a = \nabla g_-(b).\end{aligned} \tag{5.35}$$

For each $(a, b) \in K_-$, there is a unique solution $(x(t), p(t))$ to the Hamiltonian system (1.21) *over $(-\infty, 0]$ satisfying $(x(0), p(0)) = (a, b)$ and* (5.3), *and it tends to $(0, 0)$ as $t \to -\infty$. In fact, x is the unique arc furnishing the minimum in the definition of $f_-(a)$, while p is the unique arc furnishing the minimum in the definition of $g_-(b)$.*

Proof. Let

$$\phi_T(a) = \min_{a'\in R^n} e^{-\rho T} f_{-T}(a', a) \qquad \text{for} \quad T < 0. \tag{5.36}$$

Then ϕ_T is a finite convex function. Using the conjugacy relations in Theorem 3, we calculate that

$$\begin{aligned}\phi_T^*(b) &= \max_{a\in R^n} \{a \cdot b - \phi_T(a)\} \\ &= e^{-\rho T} \max_{(a,a')} \{e^{\rho T} a \cdot b - a \cdot 0 - f_{-T}(a', a)\} \\ &= e^{-\rho T} g_{-T}(0, b).\end{aligned} \tag{5.37}$$

By Proposition 12, therefore, the functions ϕ_T converge pointwise to f_- as $T \to -\infty$, while their conjugates ϕ_T^* converge pointwise to g_-. Since f_- and g_- are finite everywhere (Proposition 8′), this implies that f_- and g_- are conjugate to each other (cf. [6, 7]), i.e., Formulas (5.33) and (5.34) are correct. (The finiteness justifies writing "max" in place of "inf" [5, Theorems 23.4 and 23.5].) Proposition 8′ and Corollary 2 of Proposition 3′ give us now the facts stated after (5.35), as well as the first equivalence in (5.35). Since f_- and g_- are conjugate to each other, we certainly have

$$f_-(a) + g_-(b) = a \cdot b \Leftrightarrow b \in \partial f_-(a) \Leftrightarrow a \in \partial g_-(b) \tag{5.38}$$

[5, Theorem 23.5], and, hence,

$$K_- = \{(a, b) \mid b \in \partial f_-(a)\} = \{(a, b) \mid a \in \partial g_-(b)\}. \tag{5.39}$$

The uniqueness properties in Proposition 11 assert that K_- is the graph of a one-to-one mapping, and it therefore follows that f_- and g_- are continuously differentiable and strictly convex [5, Corollary 26.3.1]. The proof of Theorem 4′ is thereby finished.

Proof of Theorem 1′. First we prove (1.44). It is clear that

$$(0, 0) \in K_+ \cap K_-,$$

since the Hamiltonian system (1.21) is satisfied by $(x(t), p(t)) \equiv (0, 0)$. Suppose that also $(a, b) \in K_+ \cap K_-$. Then there is a solution $(x(t), p(t))$ to the Hamiltonian system over $(-\infty, +\infty)$ satisfying $(x(0), p(0)) = (a, b)$ and

$$0 = \lim_{t\to+\infty} \theta(t) = \lim_{t\to-\infty} \theta(t), \qquad \text{where} \quad \theta(t) = e^{-\rho t} x(t) \cdot p(t). \tag{5.40}$$

Applying Corollary 1 of Proposition 5 to $(x_1(t), p_1(t)) \equiv (x(t), p(t))$ and $(x_2(t), p_2(t)) \equiv (0, 0)$, we find that this implies $(x(t), p(t)) = (0, 0)$ for all $t \in (-\infty, +\infty)$. In particular, $(a, b) = (0, 0)$. Therefore (1.44) is correct.

The assertion in Theorem 1′ about uniqueness of solutions is covered by Proposition 11. The rest of the proof is just like that of Theorem 1. We invoke the global curvature assumption and apply Theorem 4. The sets U_- and V_- are defined by

$$U_- = \{a \mid \exists b \quad \text{with} \quad (a, b) \in K_- \cap W\}, \tag{5.41}$$

$$V_- = \{b \mid \exists a \quad \text{with} \quad (a, b) \in K_- \cap W\}, \tag{5.42}$$

where

$$W = \{(a, b) \mid f_-(a) + g_-(b) \leqslant \epsilon\} \tag{5.43}$$

for ϵ sufficiently small. For each $(a, b) \in K_- \cap W$, the corresponding solution $(x(t), p(t))$ to the system (1.21) over $(-\infty, 0]$, as in the definition of K_-, remains in K_- and, hence, satisfies (by Theorem 4)

$$f_-(x(t)) + g_-(p(t)) = x(t) \cdot p(t) \qquad \text{for} \quad -\infty < t \leqslant 0. \tag{5.44}$$

But the right side of (5.44) is nondecreasing as a function of t by Proposition 5, and, hence, the left side is also nondecreasing. This shows that $(x(t), p(t))$ remains in W for all $t \in (-\infty, 0]$. The details of the proof can thus be effected as in the case of Theorem 1.

6. Some Complementary Results

We present now two counterexamples, as well as a theorem on the existence of a stationary point $(\bar{k}, \bar{q})$ in certain cases where the discount rate ρ is sufficiently small. For other existence results of somewhat different import, see [1].

Example 1. This will demonstrate that strict concavity–convexity of H_0 near (0, 0) is not enough to guarantee "saddle point behavior" of the dynamical system for a given $\rho > 0$, and that something like the inequality in the basic curvature assumption is necessary. Let $n = 1$ and

$$H(x, p) = -\tfrac{1}{2}x^2 + \tfrac{1}{2}p^2 + xp \qquad \text{for all} \quad (x, p) \in R \times R. \tag{6.1}$$

Then H is strictly concave in x and strictly convex in p, with $H(0, 0) = 0$ and $\nabla H(0, 0) = (0, 0)$. Thus $(\bar{k}, \bar{q}) = (0, 0)$ satisfies the rest-point condition (1.15), and we have $H_0 = H$. The perturbed Hamiltonian system (1.21) reduces to

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & \rho - 1 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}, \tag{6.2}$$

a differential equation whose behavior around $(0, 0)$ is easily analyzed. One finds that there is "saddle point behavior" if $0 \leqslant \rho < 2$, but if $\rho > 2$, then *all* solutions to (6.2) (other than $(x(t), p(t)) \equiv (0, 0)$) diverge from $(0, 0)$ as $t \to +\infty$.

EXAMPLE 2. The purpose of this example is to show that the problem in the definition (1.1) of $\phi(c)$, if deprived of the boundedness restriction on $e^{-\rho t}k(t)$, may fail to be reducible to the problem in the definition (1.37) of $f_+(a)$ as in Proposition 2 and its corollary. Taking $n = 1$, we define

$$L(k, z) = r(k) + r(z) + 4(z - \rho k) \qquad \text{for} \quad (k, z) \in R \times R, \tag{6.3}$$

where

$$\begin{aligned} r(z) &= \tfrac{1}{2}z^2 && \text{if } \; |z| \leqslant 1, \\ &= |z| && \text{if } \; |z| \geqslant 1. \end{aligned} \tag{6.4}$$

Note that r is a differentiable convex function on the real line whose conjugate is

$$\begin{aligned} r^*(p) &= \tfrac{1}{2}\,|p|^2 && \text{if } \; |p| \leqslant 1, \\ &= +\infty && \text{if } \; |p| > 1. \end{aligned} \tag{6.5}$$

From the definition (1.5), we have

$$H(k, q) = -r(k) + r^*(q - 4) + 4\rho k. \tag{6.6}$$

Thus, the rest point condition (1.15) is satisfied by $(\bar{k}, \bar{q}) = (0, 4)$, and H is α-concave–β-convex near $(\bar{k}, \bar{q})$ for $\alpha = \beta = 1$. The basic curvature condition is therefore fulfilled, assuming $0 < \rho < 2^{1/2}$.

We claim that $\phi(c)$ is finite for all $c \in R$, while the function $\tilde{\phi}$ obtained by dropping the boundedness restriction in the infimum defining $\phi(c)$ has in fact $\tilde{\phi}(c) = -\infty$ for all $c \in R$; thus, the boundedness restriction plays an essential role. To see this, consider first an arbitrary arc k over $[0, +\infty)$ with $k(0) = c$ and $e^{-\rho t}k(t)$ bounded. If $L(k(t), \dot{k}(t))\, e^{-\rho t}$ is majorized by a summable function over $[0, +\infty)$, Proposition 2(a) implies that $L_0(k(t), \dot{k}(t))\, e^{-\rho t}$ is summable over $[0, +\infty)$ (here $k(t) = x(t)$ because $\bar{k} = 0$), and furthermore $e^{-\rho t}k(t) \to 0$ as $t \to +\infty$. In consequence,

$$\begin{aligned} &\int_0^{+\infty} L(k(t), \dot{k}(t))\, e^{-\rho t}\, dt \\ &\quad = \int_0^{+\infty} [r(k(t)) + r(\dot{k}(t))]\, e^{-\rho t}\, dt + 4 \int_0^{\infty} (d/dt)\, e^{-\rho t}k(t)\, dt \\ &\quad \geqslant -4k(0) = -4c. \end{aligned} \tag{6.7}$$

This demonstrates that

$$\phi(c) \geqslant -4c > -\infty \qquad \text{for all} \quad c \in R. \tag{6.8}$$

On the other hand, the finiteness of L ensures that $\phi(c) < +\infty$. Therefore ϕ is indeed finite. Observe at the same time that $r(z) \leqslant |z|$ and, hence,

$$\begin{aligned} L(k, z) &\leqslant |k| + |z - \rho k + \rho k| + 4(z - \rho k) \\ &\leqslant (1 + \rho)|k| + |z - \rho k| + 4(z - \rho k). \end{aligned} \tag{6.9}$$

For the arc $k(t) = -e^{2\rho t}$ (which does *not* have $e^{-\rho t}k(t)$ bounded over $[0, +\infty)$), we obtain from (6.9) that

$$\begin{aligned} \int_0^{+\infty} L(k(t), \dot{k}(t))\, e^{-\rho t}\, dt &\leqslant \int_0^{+\infty} [(1 + \rho)\, e^{2\rho t} + \rho e^{2\rho t} - 4\rho e^{2\rho t}]\, e^{-\rho t}\, dt \\ &= [1 - 2\rho] \int_0^{+\infty} e^{\rho t}\, dt = -\infty \end{aligned} \tag{6.10}$$

if $\rho > \frac{1}{2}$. In this case,

$$-\infty = \tilde{\phi}(k(0)) = \tilde{\phi}(-1),$$

and since $\tilde{\phi}$ is a convex function with $\tilde{\phi}(c) \leqslant \phi(c) < +\infty$, it follows that $\tilde{\phi}(c) = -\infty$ for all $c \in R$ [5, Theorem 7.2].

Thus if $\frac{1}{2} < \rho < 2^{1/2}$, we have $\phi(c) \neq \tilde{\phi}(c)$ for all c, even though the basic curvature assumption is fulfilled.

We conclude now with our theorem on the existence of stationary points for sufficiently small values of the discount rate ρ.

THEOREM 5. *Suppose that* $(\bar{k}_0, \bar{q}_0)$ *satisfies the stationary point condition* (1.15) *for* $\rho = 0$ *and that* H *is* α*-concave–*β*-convex in some neighborhood of* $(\bar{k}_0, \bar{q}_0)$. *Then for each* $\rho > 0$ *sufficiently small, there exists a unique pair* $(\bar{k}_\rho, \bar{q}_\rho)$ *satisfying the stationary point condition for that value of* ρ, *and one has*

$$\lim_{\rho \to 0} (\bar{k}_\rho, \bar{q}_\rho) = (0, 0). \tag{6.11}$$

Proof. We know that H is globally concave–convex (since it is related to the convex function L by (1.5)), and our hypothesis entails the finiteness of H near (0, 0). Therefore, H is continuous near (0, 0) [5, Theorem 35.1]. Let $C \times D$ be a compact convex neighborhood of $(\bar{k}_0, \bar{q}_0)$ such that, on $C \times D$, $H(k, q)$ is finite and continuous, α-concave in k and β-convex in q. In particular, for each $q' \in R^n$ and $\rho \geqslant 0$ the function

$$(k, q) \to H(k, q) - \rho k q' \tag{6.12}$$

is strictly concave–convex on $C \times D$ and has a unique saddle point relative to $C \times D$ (in the minimax sense), which, as is easy to see from the continuity of H and definition of "saddle point," must depend continuously on q' and ρ. Let

$$\Phi_\rho : C \times D \to C \times D \tag{6.13}$$

be the continuous mapping which assigns to each $(k', q') \in C \times D$ the saddlepoint (k'', q'') of the function (6.12) relative to $C \times D$. Then Φ_ρ has at least one fixed point. Such fixed points are the pairs $(\bar{k}, \bar{q}) \in C \times D$ such that

$$H(\bar{k}, \bar{q}) - \rho k \cdot \bar{q} - \rho k \cdot q \leqslant H(\bar{k}, \bar{q}) - \rho \bar{k} \cdot \bar{q} \qquad \text{for all} \quad k \in C, \tag{6.14}$$

$$H(\bar{k}, q) - \rho \bar{k} \cdot \bar{q} \geqslant H(\bar{k}, \bar{q}) - \rho \bar{k} \cdot \bar{q} \qquad \text{for all} \quad q \in D. \tag{6.15}$$

Of course, the term $\rho \bar{k} \cdot \bar{q}$ can be dropped from both sides of (6.15). The theory of the minimum of a convex function informs us that (6.15) holds if and only if there is a vector $\bar{m}$ such that

$$\bar{m} \cdot q \leqslant \bar{m} \cdot \bar{q} \qquad \text{for all} \quad q \in D, \tag{6.16}$$

$$H(\bar{k}, q) + \bar{m} \cdot q \geqslant H(\bar{k}, \bar{q}) + \bar{m} \cdot \bar{q} \qquad \text{for all} \quad q \in R^n \tag{6.17}$$

(cf. [5, Theorem 27.4]). Likewise, (6.14) holds if and only if there exists a vector $\bar{n}$ such that

$$k \cdot \bar{n} \leqslant \bar{k} \cdot \bar{n} \qquad \text{for all} \quad k \in C, \tag{6.18}$$

$$H(\bar{k}, \bar{q}) - \rho k \cdot \bar{q} - k \cdot \bar{n} \leqslant H(\bar{k}, \bar{q}) - \rho \bar{k} \cdot \bar{q} - \bar{k} \cdot \bar{n} \qquad \text{for all} \quad k \in R^n. \tag{6.19}$$

Observe that (6.17) and (6.19) can be combined into

$$(\rho \bar{q} + \bar{n}_0 - \bar{m}) \in \partial H(\bar{k}, \bar{q}). \tag{6.20}$$

If $\bar{n} = 0$ and $\bar{m} = 0$, as must be true by (6.16) and (6.17) if $(\bar{k}, \bar{q})$ is an interior point of $C \times D$, then (6.20) reduces to the stationary point condition (1.15).

We have seen that for each ρ there is at least one set of vectors $\bar{k}, \bar{q}, \bar{m}, \bar{n}$ with $(\bar{k}, \bar{q}) \in C \times D$ such that (6.16)–(6.19) are satisfied. To investigate this further, consider also another set of vectors $\bar{k}', \bar{q}', \bar{m}', \bar{n}'$ satisfying these conditions for a possibly different value ρ'. Since H is α-concave–β—convex on $C \times D$, (6.17) implies

$$H(\bar{k}, q) + \bar{m} \cdot q \geqslant H(\bar{k}, \bar{q}) + \bar{m} \cdot \bar{q} + \tfrac{1}{2}\beta \mid q - \bar{q} \mid^2 \qquad \text{for all} \quad q \in C, \tag{6.21}$$

while (6.19) implies

$$H(k, \bar{q}) - k \cdot (\rho\bar{q} + \bar{n}) \\ \leqslant H(\bar{k}, \bar{q}) - k \cdot (\rho\bar{q} + \bar{n}) - \tfrac{1}{2}\alpha \mid k - \bar{k} \mid^2 \quad \text{for all} \quad k \in D. \tag{6.22}$$

Therefore,

$$H(\bar{k}, \bar{q}') - H(\bar{k}, \bar{q}) \geqslant \bar{m} \cdot (\bar{q} - \bar{q}') + \tfrac{1}{2}\beta \mid \bar{q}' - \bar{q} \mid^2, \tag{6.23}$$

$$H(\bar{k}, \bar{q}) - H(\bar{k}', \bar{q}) \geqslant (\bar{k} - \bar{k}') \cdot (\rho\bar{q} + \bar{n}) + \tfrac{1}{2}\alpha \mid \bar{k}' - \bar{k} \mid^2, \tag{6.24}$$

and dually,

$$H(\bar{k}', \bar{q}) - H(\bar{k}', \bar{q}') \geqslant \bar{m}' \cdot (\bar{q}' - \bar{q}) + \tfrac{1}{2}\beta \mid \bar{q} - \bar{q}' \mid^2, \tag{6.25}$$

$$H(\bar{k}', \bar{q}') - H(\bar{k}, \bar{q}') \geqslant (\bar{k}' - \bar{k}) \cdot (\rho'\bar{q}' + \bar{n}') + \tfrac{1}{2}\alpha \mid \bar{k} - \bar{k}' \mid^2. \tag{6.26}$$

Adding the last four inequalities and using the fact that by (6.16) and (6.18),

$$\bar{m} \cdot (\bar{q} - \bar{q}') \geqslant 0, \qquad (\bar{k} - \bar{k}') \cdot \bar{n} \geqslant 0,$$
$$\bar{m}' \cdot (\bar{q}' - \bar{q}) \geqslant 0, \qquad (\bar{k}' - \bar{k}) \cdot \bar{n}' \geqslant 0,$$

we obtain

$$0 \geqslant (\bar{k} - \bar{k}')(\rho\bar{q} - \rho'\bar{q}') + \alpha \mid \bar{k}' - \bar{k} \mid^2 + \beta \mid \bar{q}' - \bar{q} \mid^2, \tag{6.27}$$

or equivalently,

$$(\rho' - \rho)(\bar{k} - \bar{k}') \cdot \bar{q}' \geqslant \alpha \mid \bar{k}' - \bar{k} \mid^2 + \rho(\bar{k}' - \bar{k}) \cdot (\bar{q}' - \bar{q}) + \beta \mid \bar{q}' - \bar{q} \mid^2. \tag{6.28}$$

Defining

$$\gamma = \max \{ \mid (k - k') \cdot q' \mid \,\big|\, k \in C, k' \in C, q' \in D \}, \tag{6.29}$$

we can convert (6.28) into the bound

$$\gamma \mid \rho' - \rho \mid \geqslant \alpha \mid \bar{k}' - k \mid^2 - \rho \mid \bar{k}' - \bar{k} \mid \cdot \mid \bar{q}' - \bar{q} \mid + \beta \mid \bar{q}' - \bar{q} \mid^2 \\ \geqslant \sigma_1[\mid \bar{k}' - \bar{k} \mid^2 + \mid \bar{q}' - \bar{q} \mid]^2, \tag{6.30}$$

where

$$\sigma_1 = (4\alpha\beta - \rho^2)/4(\alpha + \beta + \rho); \tag{6.31}$$

the proof of the second inequality in (6.30) is given at the end of the proof of Proposition 5. Let us suppose that $\rho^2 < 4\alpha\beta$, so that $\sigma_1 > 0$. Then two conclusions are apparent from (6.30). First, if $\rho' = \rho$, then $\bar{k}' = \bar{k}$ and $\bar{q}' = \bar{q}$. In other words, $(\bar{k}, \bar{q})$ is *uniquely* determined by conditions (6.16)–(6.19). Second, taking $\rho' = 0$ and $(\bar{k}', \bar{q}') = (\bar{k}_0, \bar{q}_0)$, we see that as $\rho \downarrow 0$, the point $(\bar{k}, \bar{q})$ must tend to $(\bar{k}_0, \bar{q}_0)$. Hence, for ρ sufficiently small, $(\bar{k}, \bar{q})$ must be an interior point of $C \times D$, so that $\bar{m} = 0 = \bar{n}$ as already explained, and the stationary point condition (1.15) is actually satisfied. These two conclusions immediately yield the assertions in the theorem.

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Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.