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Offprint from
JOURNAL OF MATHEMATICS AND MECHANICS
Vol. 13, No. 6, November, 1964, pp. 1037–1038

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Communicated by G. BIRKHOFF

The purpose of this note is to prove the following theorem:

Theorem. *Let B be a Banach space and X a bounded subset of B. In order that for each bounded real-valued function F on X there exist a continuous linear functional $\phi \in B^*$ such that $\Delta(\phi) = \sup \{|F(x) - \phi(x)| : x \in X\} = \inf \{\Delta(\psi) : \psi \in B^*\}$, it is necessary that 0 be in the interior of the closed, balanced, convex hull K of X relative to its linear span $[K]$.*

In the preceding paper by E. W. Cheney and A. A. Goldstein, it was proved that this condition is also sufficient to insure the existence of best approximations even when X is not bounded. We shall actually show that when the condition fails, there is a *linear* function F on $[K]$ such that

(a) the restriction of F to X is continuous
(b) F can be uniformly approximated on X within any desired degree of accuracy by functionals in B^*, but
(c) the restriction of F to X is not equal to the restriction to X of any functional in B^*.

Let ρ be the norm of B. Suppose K has no interior relative to the ρ-topology on $[K]$. Since K is symmetric, convex, closed, and absorbing *in* $[K]$, there is a norm σ on $[K]$ such that $K = \{x \in [K] : \sigma(x) \leqq 1\}$ (Bourbaki [1, p. 95]). Since K is ρ-bounded, the new topology induced by σ on $[K]$ is finer than the ρ-topology. Because K is ρ-complete, the σ-topology on $[K]$ is complete, so that σ makes $[K]$ into a Banach space (Bourbaki, [1, Chap. I, Cor. of Prop. 8]).

Let us show that $[K]$ cannot be ρ-closed in B. In the contrary case, it would follow from the open mapping theorem that σ and ρ were equivalent norms. (Dunford and Schwartz [2, Theorem II. 2.5]). But this would contradict our hypothesis that K has no interior relative to the ρ-topology on $[K]$.

Thus the ρ-closure L of $[K]$ is actually larger than $[K]$. The inclusion map

* This work was supported by grant AF-AFOSR 62-348.

$T: [K] \to L$ is continuous from the σ-topology to the ρ-topology. Its adjoint T^* maps the ρ-dual L^* of L into the σ-dual $[K]^*$ of $[K]$. Let σ^* be the norm on $[K]^*$ dual to σ. $T^*(L^*)$ cannot be σ^*-closed in $[K]^*$, for if it were, it would follow that $T([K])$ would be ρ-closed in L (Dunford and Schwartz [2, Theorem VI. 6.4]). Thus there is an element of F in the σ^*-closure of $T^*(L^*)$ which does not belong to $T^*(L^*)$.

$T^*(L^*)$ consists of those functionals in $[K]^*$ which are restrictions to $[K]$ of functionals in L^*. The Hahn-Banach theorem shows that in fact $T^*(L^*)$ consists of the restrictions to $[K]$ of functionals in B^* (Dunford and Schwartz [2, Theorem II. 3.11]). Consequently, F is a linear functional on $[K]$ which is not the restriction to $[K]$ of a functional in B^*. Since $[K]$ is the linear span of cl(conv $(X \cup -X)$), the values of a ρ-continuous linear functional on $[K]$ are completely determined by its values on X. This proves (c).

The σ^*-topology is that of uniform convergence on K. Since F lies in the σ^*-closure of $T^*(L^*)$, (b) is proved. Since F is the uniform limit on K of ρ-continuous linear functionals, (a) follows as well.

References

[1] N. Bourbaki, *Espaces Vectoriels Topologiques*, Chaps. I and II, Hermann, Paris, 1953.
[2] Nelson Dunford & Jacob T. Schwartz, *Linear Operators*, Part I, Interscience, New York, 1958.

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