The Subspace Flatness Conjecture and Faster Integer Programming

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Joint work with Victor Reis

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Covering radius

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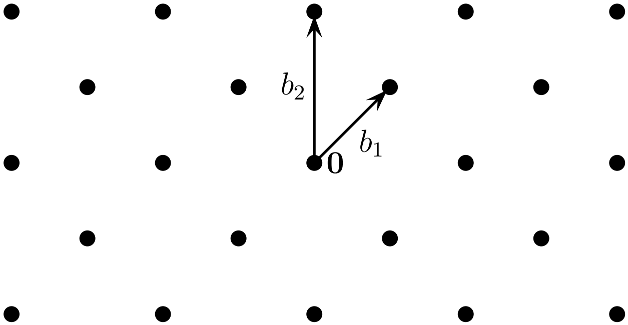

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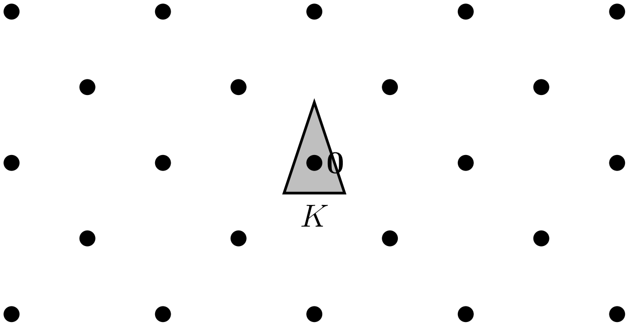

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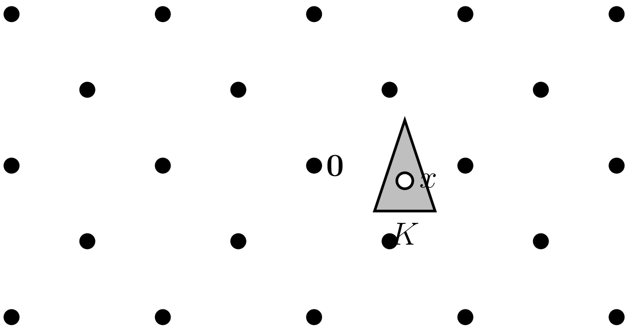

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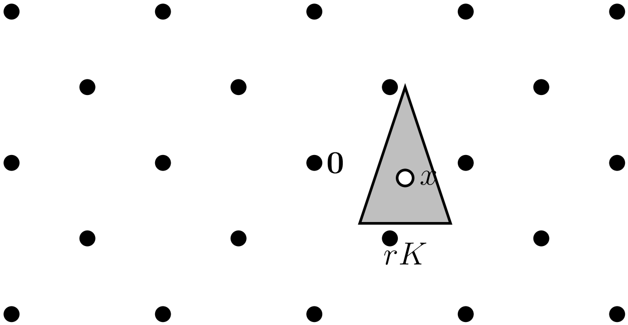

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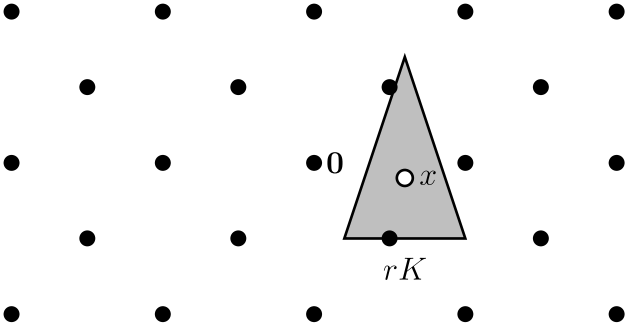
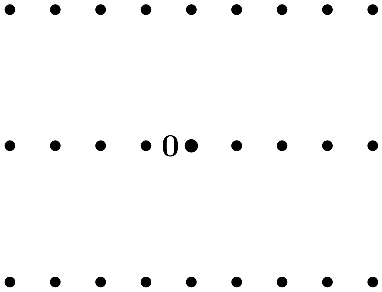


Lower bounds on the covering radius



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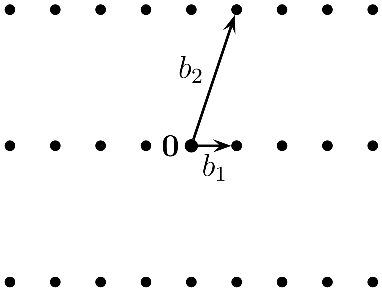

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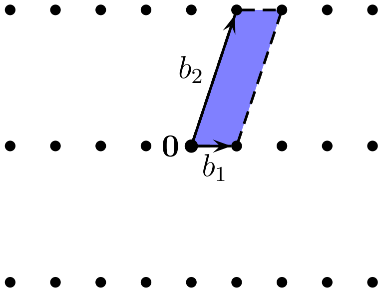

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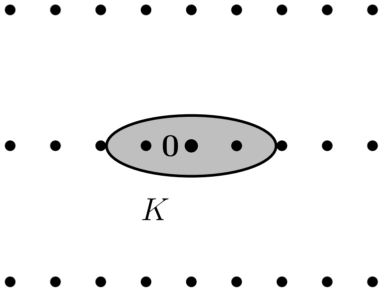

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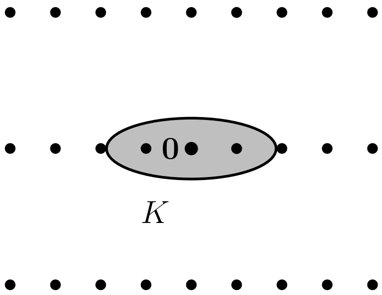

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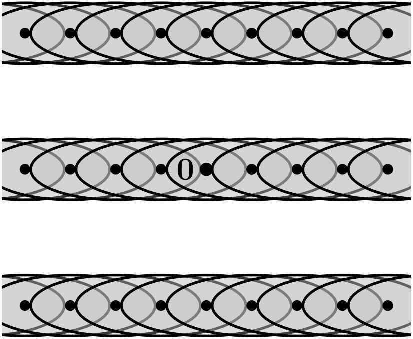

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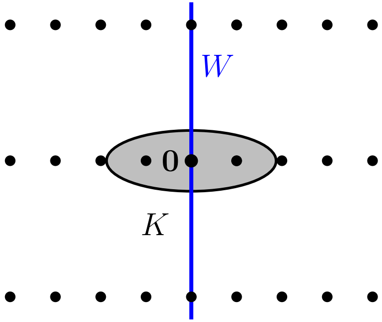

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- Simple lower bound: $\mu(\Lambda, K) \geq (\frac{\det(\Lambda)}{\mathrm{Vol}_n(K)})^{1/n}$
- For any subspace $\mu(\Lambda, K) \geq \mu(\Pi_W(\Lambda), \Pi_W(K))$

Kannan, Lovász (1988)

- Consider the best volume-based lower bound

$$\mu_{KL}(\Lambda, K) = \max_{\substack{W \subseteq \mathrm{span}(\Lambda) \text{ subspace} \\ d := \dim(W)}} \left(\frac{\det(\Pi_W(\Lambda))}{\mathrm{Vol}_d(\Pi_W(K))} \right)^{1/d}$$

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Theorem (Kannan, Lovász (1988))

For any full rank lattice Λ, convex body $K \subseteq \mathbb{R}^n$

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Subspace Flatness Conjecture (Dadush 2012)

For full rank lattice $\Lambda \subseteq \mathbb{R}^n$ and convex body $K \subseteq \mathbb{R}^n$ one has

$$\mu(\Lambda, K) \leq O(\log(n)) \cdot \mu_{KL}(\Lambda, K)$$

- Dadush shows consequences for solving IPs.

Main results

Theorem (Reis, R.'23)

For full rank lattice $\Lambda \subseteq \mathbb{R}^n$ and convex body $K \subseteq \mathbb{R}^n$ one has

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Theorem (Reis, R.'23)

For convex body $K \subseteq \mathbb{R}^n$ one can find a point in $K \cap \mathbb{Z}^n$ in time $(\log n)^{O(n)}$.

Previously best known:

- $2^{O(n^2)}$ [Lenstra '83]
- $n^{O(n)}$ [Kannan '87]
- $2^{O(n)}n^n$ [Dadush '12], [Dadush, Eisenbrand, R. '22]

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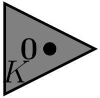
$$\mu(\Lambda, K) \leq O(\log^7(n)) \cdot \mu(\Lambda, K - K)$$



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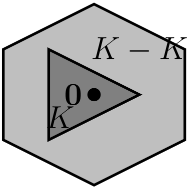

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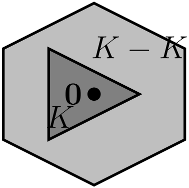


Theorem (Reis, R.'23)

Flatness constant in dimension n is at most $O(n \log^8(n))$.

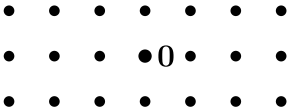
- Previously best known: $O(n^{4/3} \log^{O(1)} n)$ [Rudelson '98+Banaszczyk, Litvak, Pajor, Szarek '99]

The Reverse Minkowski Theorem

Reverse Minkowski Theorem (Regev, Stephens-Da.)

Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice that satisfies $\det(\Lambda') \geq 1$ for all sublattices $\Lambda' \subseteq \Lambda$. Then for $s = \Theta(\log n)$,

$$\rho_{1/s}(\Lambda) = \sum_{x \in \Lambda} \exp(-\pi s^2 \|x\|_2^2) \leq \frac{3}{2}$$



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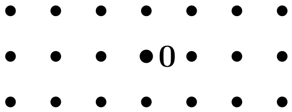

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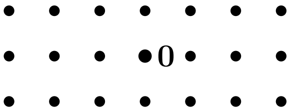

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Definition

A lattice Λ is called **stable** if $\det(\Lambda) = 1$ and $\det(\Lambda') \geq 1$ for all sublattices $\Lambda' \subseteq \Lambda$.

ℓ-position

- For a symmetric convex body $K \subseteq \mathbb{R}^n$,

$$\ell_K = \mathop{\mathbb{E}}_{x \sim N(0, I_n)} [\|x\|_K^2]^{1/2}$$

- **Intuitively:** $\ell_K =$ "average thinness" of K

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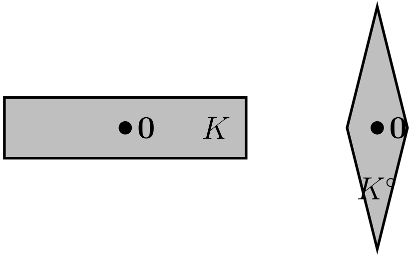
- **Polar** is $K^\circ = \{x \in \mathbb{R}^n \mid \langle x, y \rangle \leq 1 \; \forall y \in K\}$
- Possible that ℓ_K and ℓ_{K° arbitrarily large



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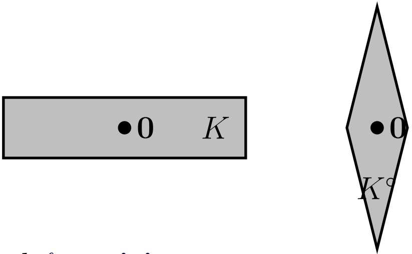
Theorem (Figiel, Tomczak-Jaegerman, Pisier)

For any symmetric convex body $K \subseteq \mathbb{R}^n$, there is an invertible linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ so that $\ell_{T(K)} \cdot \ell_{(T(K))^\circ} \leq O(n \log n)$.



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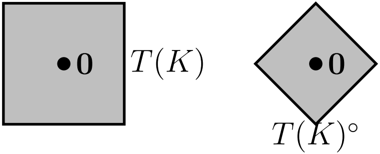

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Covering radius of stable lattice

Lemma

For symmetric convex K and stable lattice Λ one has $\mu(\Lambda, K) \leq O(\log n) \cdot \ell_K$.

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For symmetric convex K and stable lattice Λ one has $\mu(\Lambda, K) \leq O(\log n) \cdot \ell_K$.

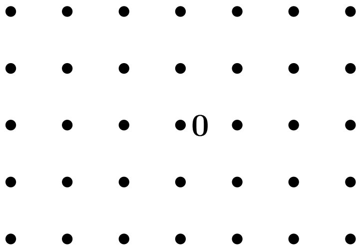
Proof uses:

- Machinery of [Banaszczyk 96]
- $\rho_{1/s}(\Lambda^* \setminus \{\mathbf{0}\}) \leq \frac{1}{2}$ by **Reverse Minkowski Theorem** for $s = \Theta(\log n)$.

Quotient lattices

Definition

Consider a lattice $\Lambda \subseteq \mathbb{R}^n$ with a primitive sublattice $\Lambda' \subseteq \Lambda$. The **quotient lattice** is $\Lambda/\Lambda' = \Pi_{\text{span}(\Lambda')^\perp}(\Lambda)$.



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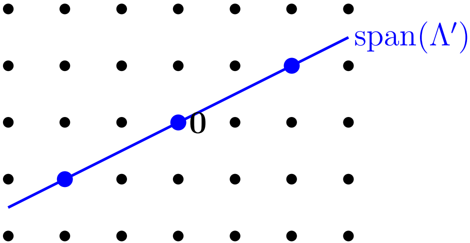

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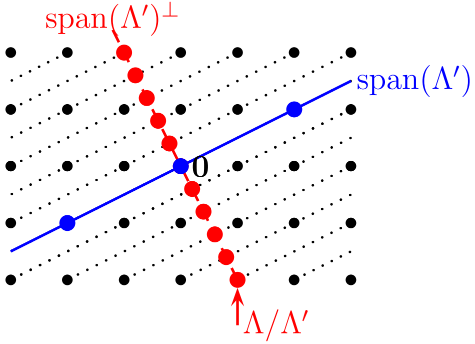

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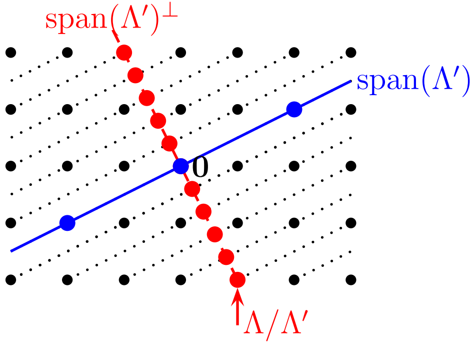

- **Intuition:** We can factor Λ into Λ' and Λ/Λ'
- For example $\det(\Lambda) = \det(\Lambda') \cdot \det(\Lambda/\Lambda')$.

The canonical filtration

- For lattice $\Lambda \subseteq \mathbb{R}^n$, consider **canonical plot**

$$Q := \left\{ \big(\mathrm{rank}(\Lambda'), \ln(\det(\Lambda'))\big) \mid \text{sublattice } \Lambda' \subseteq \Lambda \right\}$$

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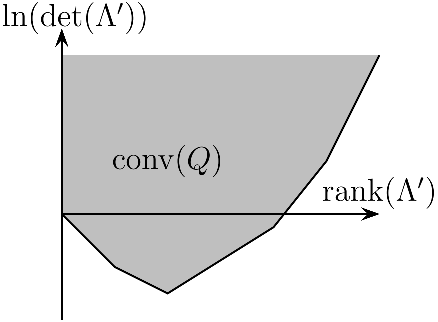

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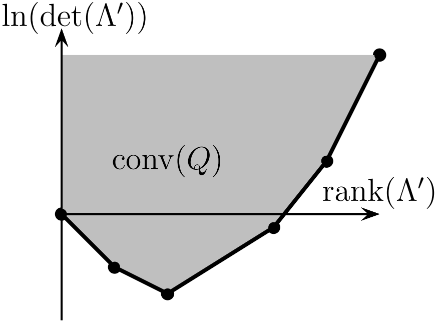
- Lower envelope of $\operatorname{conv}(Q)$ is called **canonical polygon**



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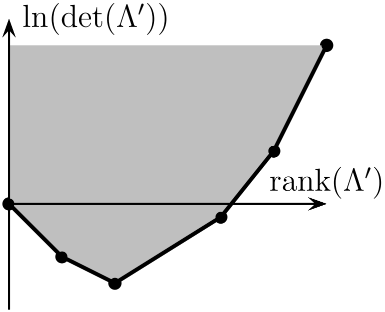

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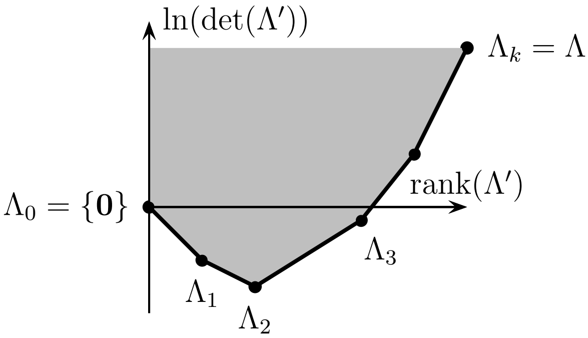

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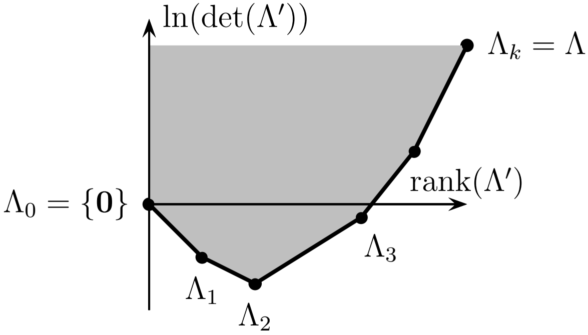

Theorem (Canonical filtration)

(a) *The vertices of the canonical plot form a chain*

$$\{\mathbf{0}\} = \Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_k = \Lambda.$$

(b) $r_i := \det(\Lambda_i/\Lambda_{i-1})^{1/rank(\Lambda_i/\Lambda_{i-1})}$ *satisfy* $r_1 < \ldots < r_k$

(c) *Each* $\frac{1}{r_i}(\Lambda_i/\Lambda_{i-1})$ *is stable.*

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- Define

$$d_i := \operatorname{rank}(\Lambda_i / \Lambda_{i-1}) \quad \text{and} \quad r_i := \det(\Lambda_i / \Lambda_{i-1})^{1/d_i}$$

- **Goal:**
  - Bound $\mu(\Lambda, K)$ in terms of r_i, d_i.
  - Bound r_i, d_i in terms of $\mu_{KL}(\Lambda, K)$.

Main proof (2)

We bound

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$$\mu(\Lambda, K) \overset{\text{triangle ineq.}}{\leq} \sum_{i=1}^{k} \mu\big(\Lambda_i/\Lambda_{i-1}, \Pi_{\mathrm{span}(\Lambda_{i-1})^\perp}(K \cap \mathrm{span}(\Lambda_i))\big)$$

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- Recall: $\frac{1}{r_i}(\Lambda_i/\Lambda_{i-1})$ stable

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$$\lesssim \sum_{i=1}^{k} \log(n) \cdot r_i \cdot \ell_K$$

- Recall: $\frac{1}{r_i}(\Lambda_i/\Lambda_{i-1})$ stable
- Use monotonicity of ℓ-value: $\ell_{K\cap W} \leq \ell_K$ for subspace W (potentially huge loss!!)

Main proof (2)

We bound

$$\begin{aligned}
\mu(\Lambda, K) &\overset{\text{triangle ineq.}}{\leq} \sum_{i=1}^{k} \mu\Big(\underbrace{\Lambda_i/\Lambda_{i-1}}_{r_i\cdot\text{stable lattice}}, \Pi_{\text{span}(\Lambda_{i-1})^\perp}(K \cap \text{span}(\Lambda_i))\Big) \\
&\lesssim \sum_{i=1}^{k} \log(n) \cdot r_i \cdot \ell_K \\
&\lesssim \log(n) \cdot r_k \cdot \ell_K
\end{aligned}$$

- Recall: $\frac{1}{r_i}(\Lambda_i/\Lambda_{i-1})$ stable
- Use monotonicity of ℓ-value: $\ell_{K\cap W} \leq \ell_K$ for subspace W (potentially huge loss!!)
- Group indices of similar density together: $r_i \leq \frac{1}{2} r_{i+2}$

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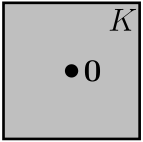

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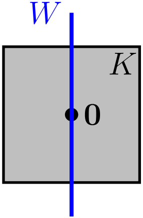

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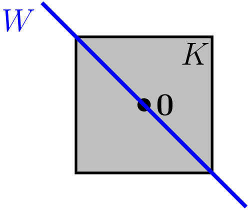

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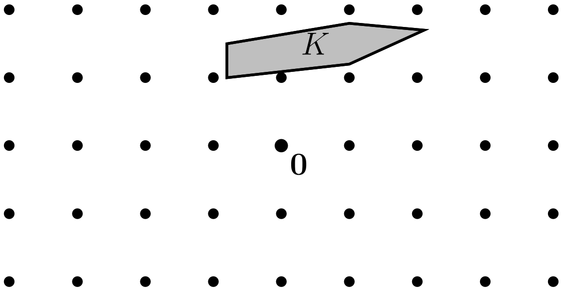
- **Case 1.** For some $i \geq i^*$, $d_i \geq \Omega(\frac{n}{\log(n)})$. **Done!**
- **Case 2.** All $i \geq i^*$ have $d_i \leq o(\frac{n}{\log(n)})$. Then
  $k - i^* \geq \omega(\log(n))$ and $r_k \geq n^{100} r_{i^*}$. **Done!** □

Dadush's algorithm



Input: $K \subseteq \mathbb{R}^n$, lattice Λ
Output: Point in $K \cap \Lambda$

(1) Shrink K so that $\mu(\Lambda, K) \geq 1$
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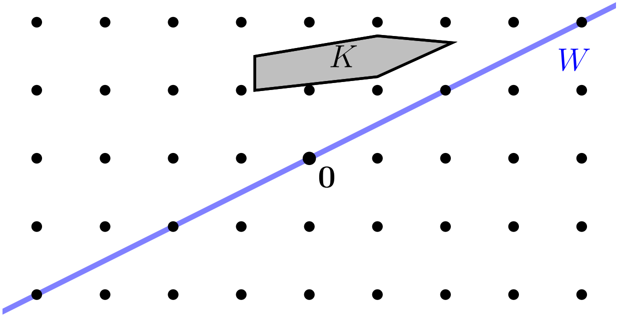

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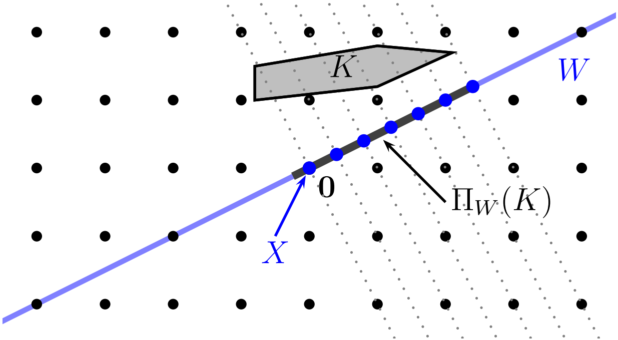

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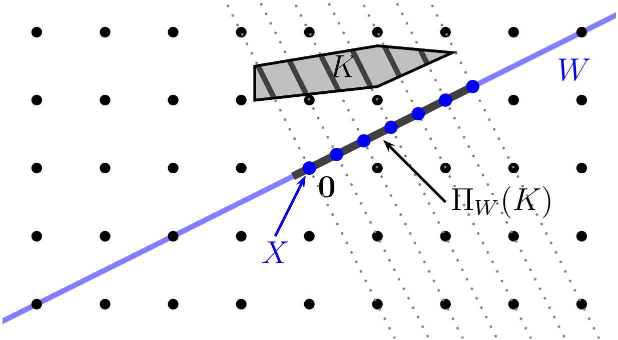

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Theorem (Dadush)

For full rank lattice $\Lambda \subseteq \mathbb{R}^n$ and convex body $P \subseteq \mathbb{R}^n$ one has

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Can compute points in same time.

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Can one solve every n variable IP in time $2^{O(n)}$?

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Thanks for your attention!

Generalization to non-symmetric K

- Translate K so that **Santaló point** is at $\mathbf{0}$ (i.e. barycenter of K° is at $\mathbf{0}$.
- Run proof with $K_{\mathrm{sym}} := K \cap (-K)$ (inner symmetrizer)
- We need:

$$\mathrm{Vol}_d(\Pi_W(K))^{1/d} \lesssim \left(\frac{n}{d}\right)^3 \cdot \mathrm{Vol}_d(\Pi_W(K_{\mathrm{sym}}))^{1/d}$$

More involved! This is a polar version of Rudelson's result on sections of the difference body:

$$\mathrm{Vol}_d((K-K)\cap W)^{1/d} \lesssim \frac{n}{d} \cdot \max_{x\in\mathbb{R}^n} \left\{ \mathrm{Vol}_d(K \cap (x+W))^{1/d} \right\}$$