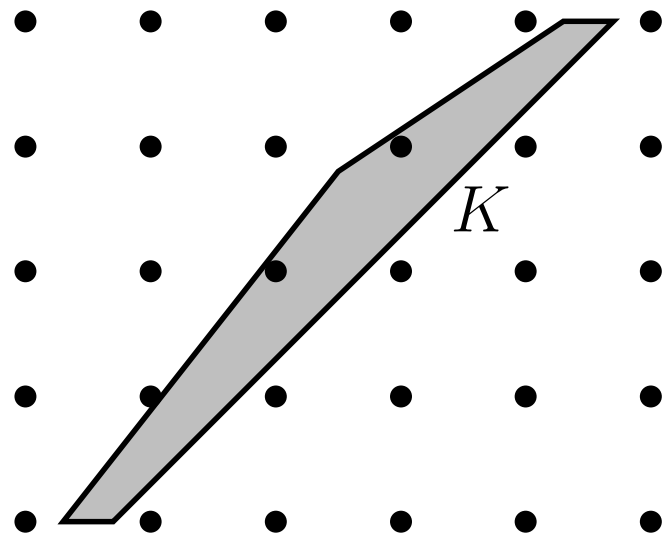
Part 1:
Introduction to Lattices

Thomas Rothvoss

Based the work
[Micciancio, Voulgaris 2010],
[Dadush, Peikert, Vempala 2011],
[Dadush, Vempala 2012], [Dadush 2012]

UNIVERSITY *of*
WASHINGTON

The main goal



Theorem (Reis, R.'23)

For convex body $K \subseteq \mathbb{R}^n$ one can find a point in $K \cap \mathbb{Z}^n$ in time $(\log n)^{O(n)}$.

Previously best known:

- $2^{O(n^2)}$ [Lenstra '83]
- $n^{O(n)}$ [Kannan '87]
- $2^{O(n)}n^n$ [Dadush '12], [Dadush, Eisenbrand, R. '22]

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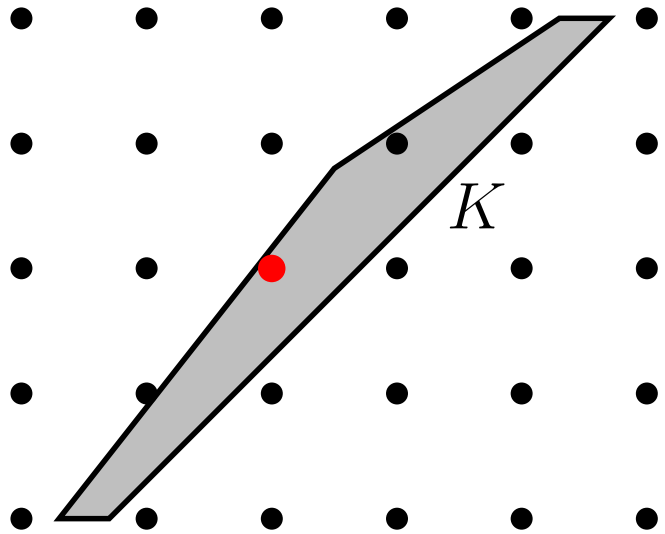

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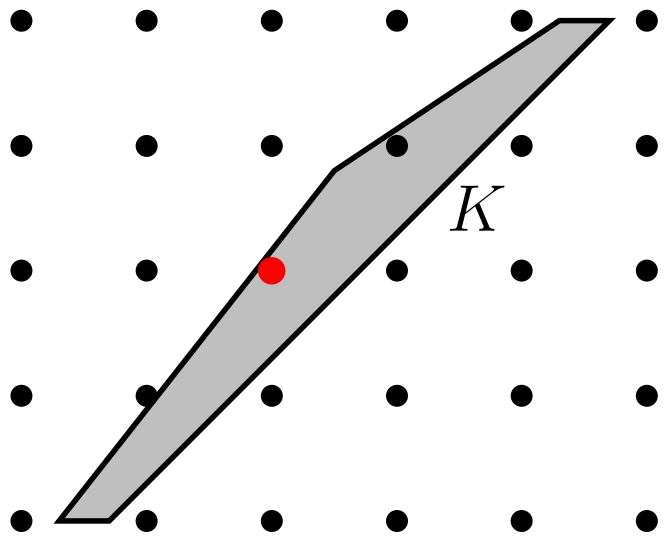

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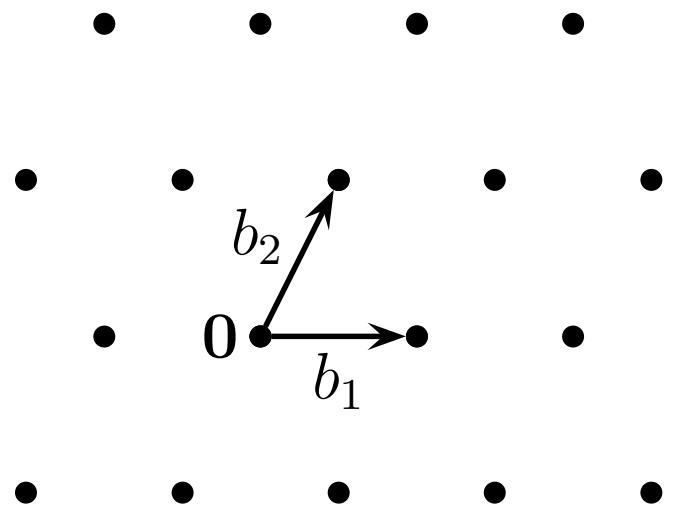
Theorem (Reis, R.'23)

One can solve an integer program $\max\{c^T x \mid Ax \leq b, x \in \mathbb{Z}^n\}$ in time $(\log n)^{O(n)}$.

Lattices

- A **lattice** is a set $\Lambda = \{Bx : x \in \mathbb{Z}^k\}$ where $B \in \mathbb{R}^{n \times k}$ has linearly independent columns.
- $B = (b_1, \ldots, b_k)$ is called **basis** of the lattice.



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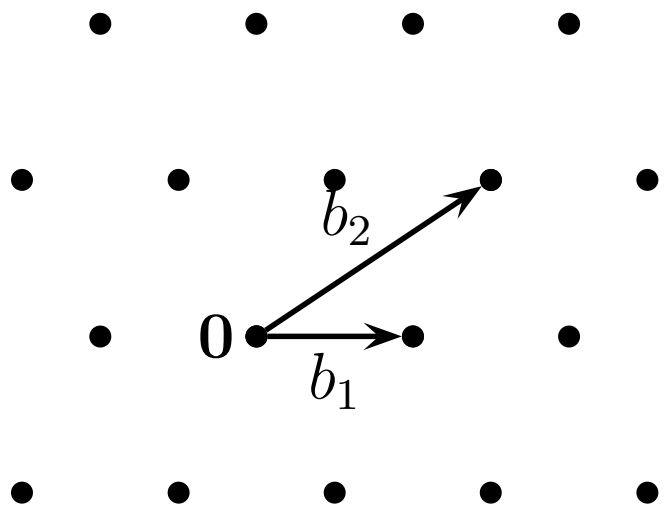

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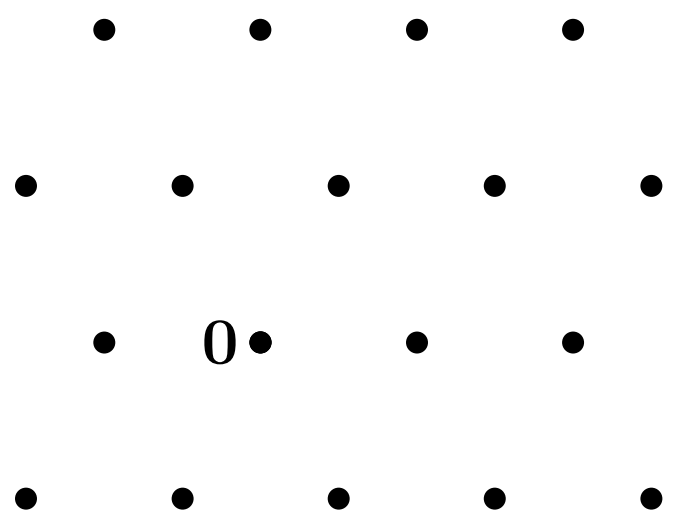

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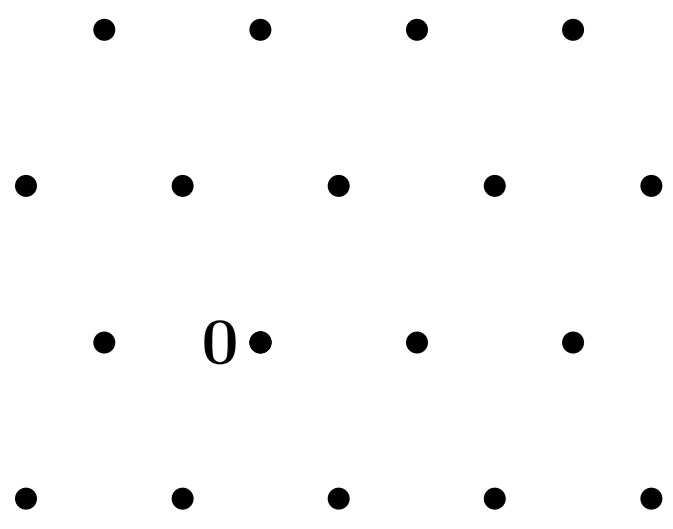
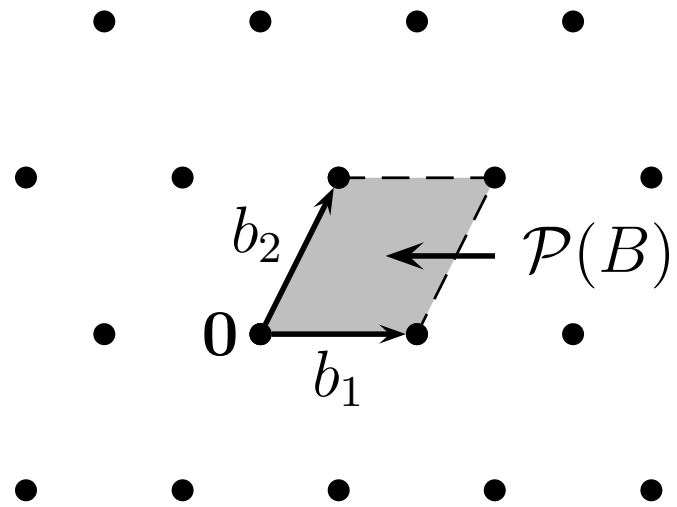


- The **rank** of Λ is $\dim(\text{span}(\Lambda))$
- A lattice has **full rank**, if $n = \text{rank}(\Lambda)$.

Determinants

- The **fundamental parallelepiped** of Λ is the polytope

$$\mathcal{P}(B) := \left\{ \sum_{i=1}^{k} \lambda_i b_i \mid 0 \leq \lambda_i < 1 \; \forall i \in [k] \right\}$$



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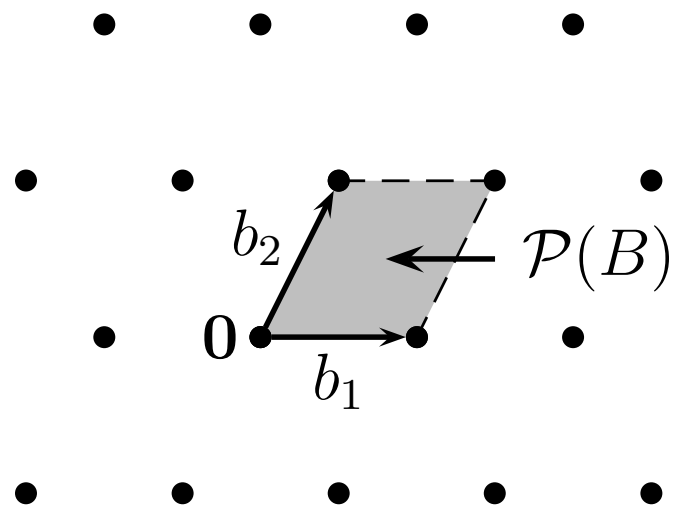
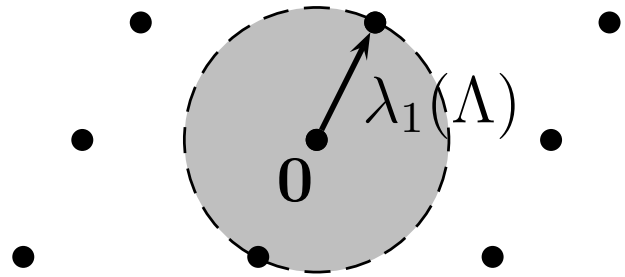
- The **determinant** of lattice is $\det(\Lambda) := \mathrm{Vol}_k(\mathcal{P}(B))$
  In full rank case, $\det(\Lambda) = |\det(B)|$



Shortest vectors

- Length of the **shortest vector** is

$$\lambda_1(\Lambda) := \min\{\|x\|_2 \mid x \in \Lambda \setminus \{\mathbf{0}\}\}$$



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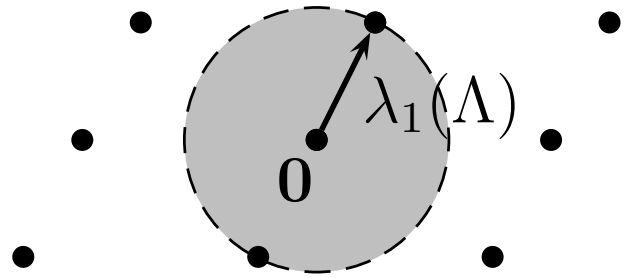

- Finding shortest vector is **NP**-hard (under randomized reductions)
- Can be approximated within 2^n-factor [LLL'82]
- Can be computed in time $2^{O(n)}$ (even w.r.t. arbitrary norms $\|\cdot\|_K$) [Ajtai, Kumar, Sivakumar '01]

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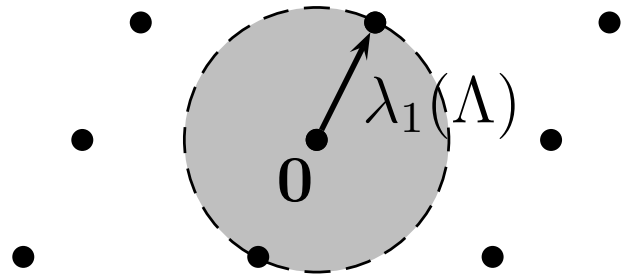

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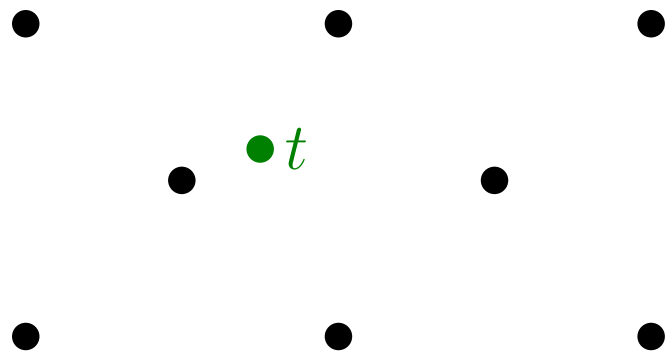
Theorem (Minkowski's Theorem 1889)

Any full rank lattice $\Lambda \subseteq \mathbb{R}^n$ *one has* $\lambda_1(\Lambda) \leq \sqrt{n} \cdot \det(\Lambda)^{1/n}$.

Closest Vector

Closest Vector Problem (CVP): Given lattice Λ and target vector $t \in \mathbb{R}^n$. Find vector attaining $\min\{\|x - t\|_2 : x \in \Lambda\}$.



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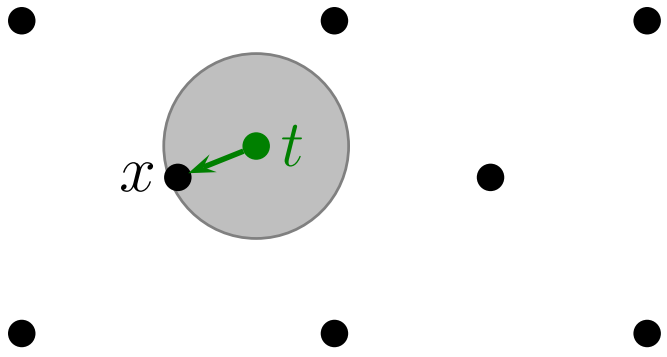

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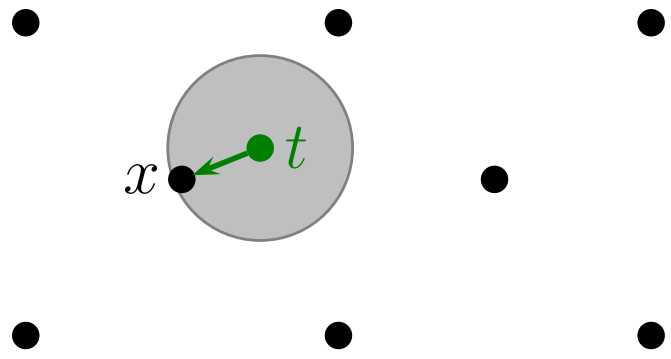


Theorem (Micciancio, Voulgaris '10)

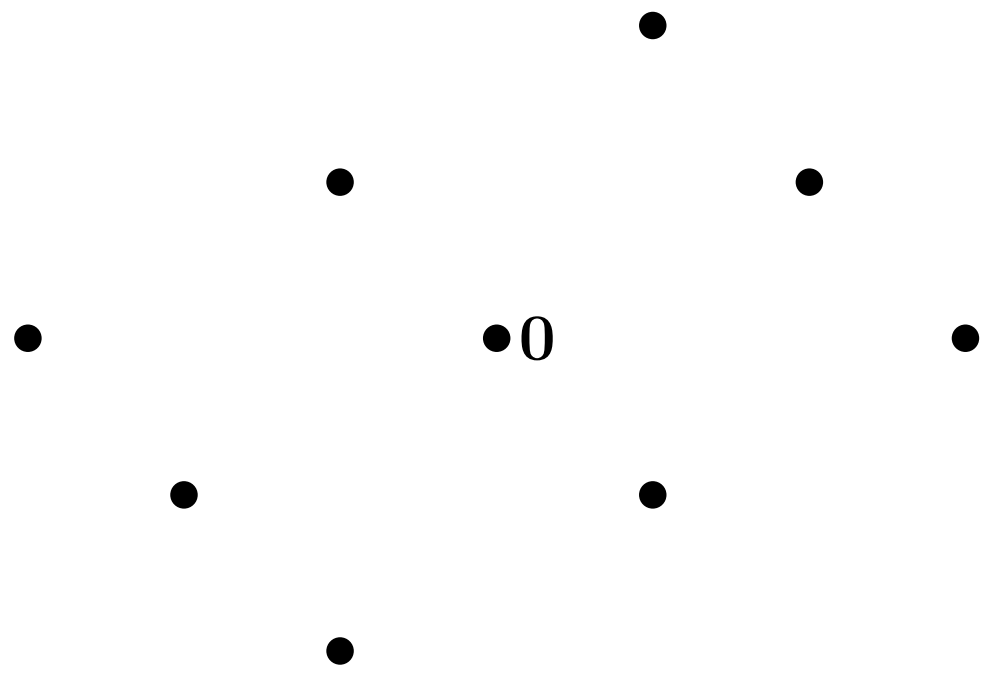
CVP can be solved (deterministically) in time $2^{O(n)}$.

- Only works for $\|\cdot\|_2$

The Voronoi cell

Let $\Lambda \subseteq \mathbb{R}^n$ be full rank lattice. The **Voronoi cell** is

$$\mathcal{V} = \{x \in \mathbb{R}^n : \|x\|_2 \leq \|x - v\|_2 \; \forall v \in \Lambda \setminus \{\mathbf{0}\}\}$$



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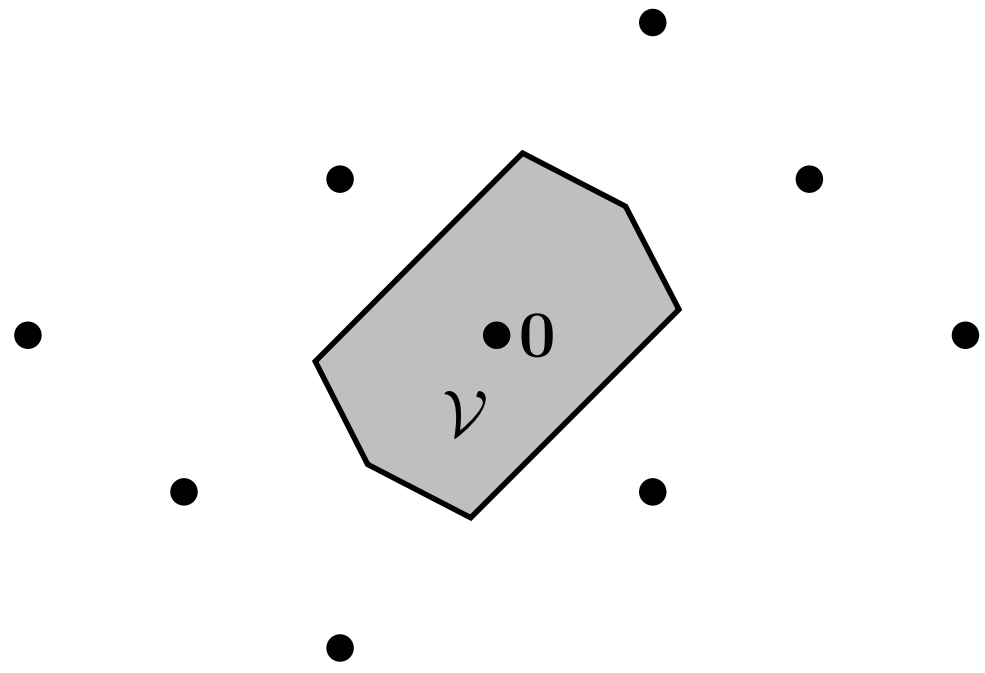

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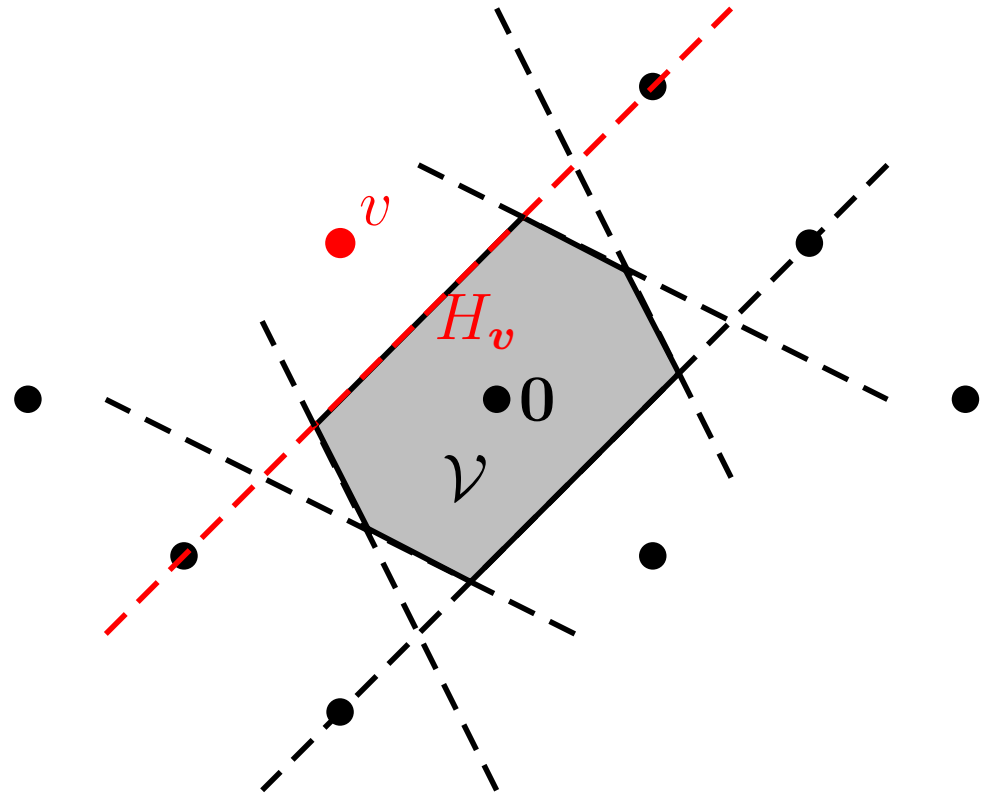

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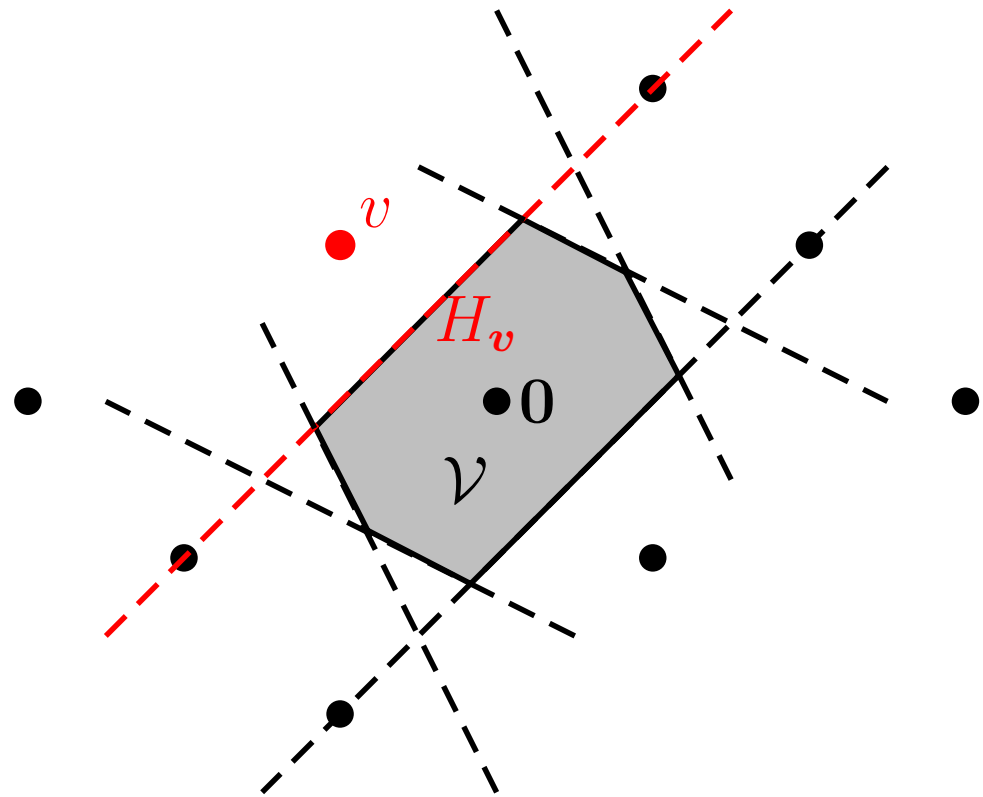
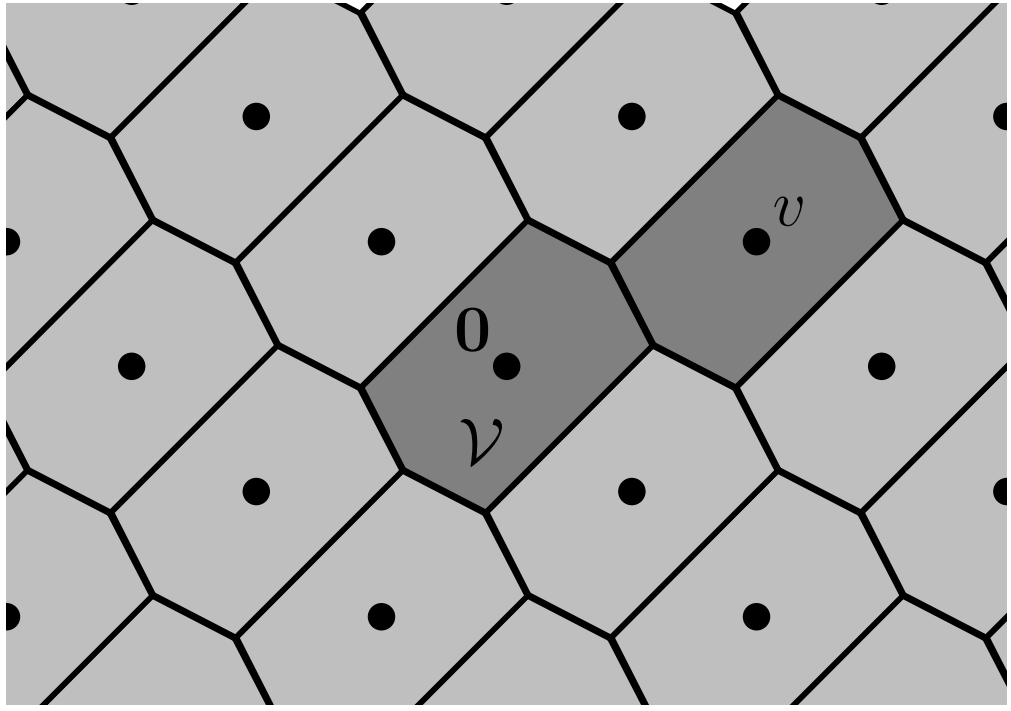


- $\mathcal{V}$ is a symmetric, convex, compact set
- $t \in \mathcal{V} \Leftrightarrow \min\{\|x - t\|_2 : x \in \Lambda\}$ attained by $\mathbf{0}$

The Voronoi relevant vectors

- $v \in \Lambda$ is called **Voronoi relevant** if H_v is a facet of $\mathcal{V}$.



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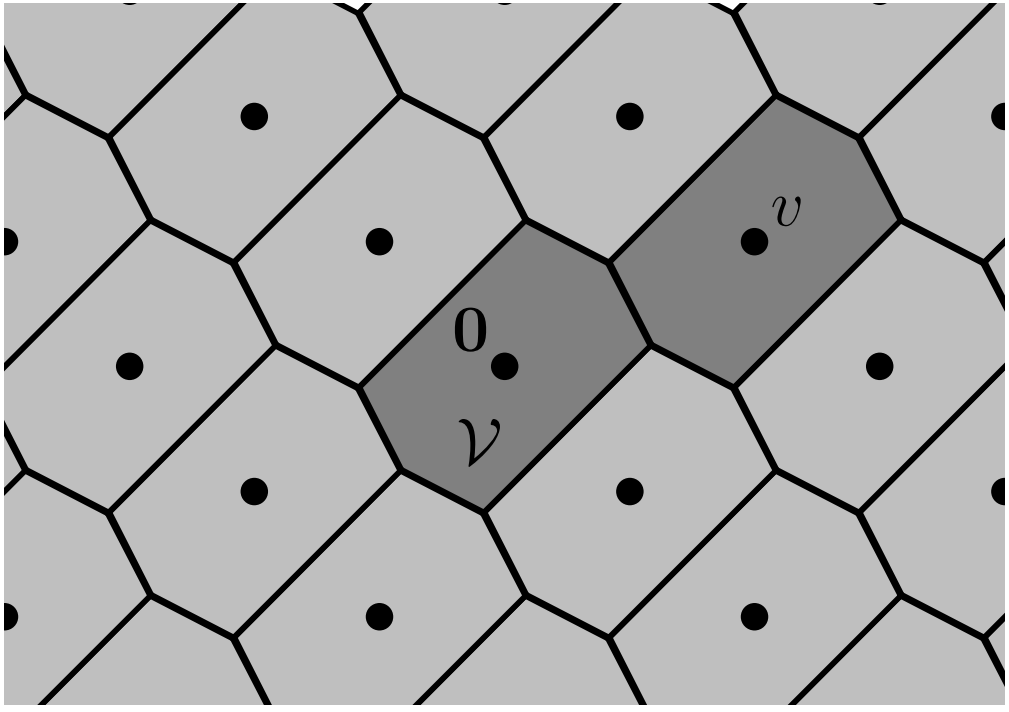

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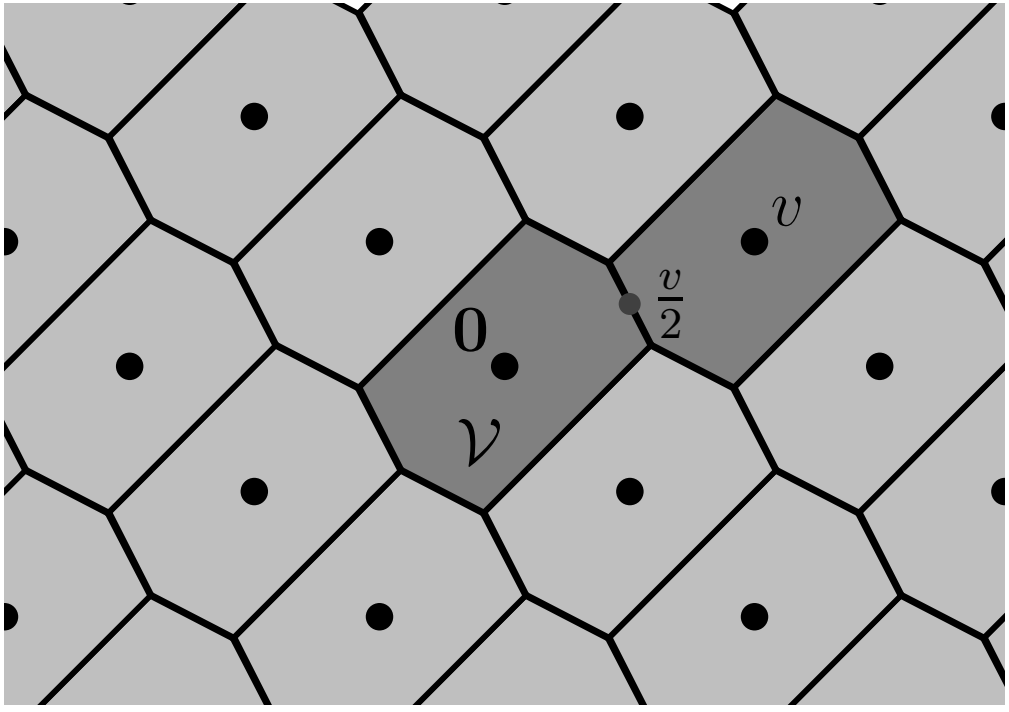
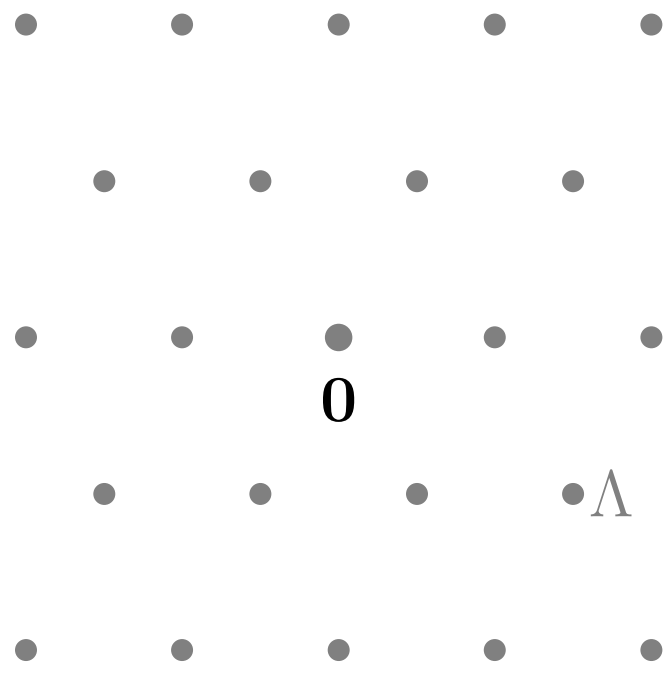
- **Obs 1:** $\mathcal{V} = \bigcap_{v \text{ Voronoi rel}} H_v$
- **Obs 2:** If v Voronoi relevant, then unique closest lattice points to $\frac{v}{2}$ are $\mathbf{0}$ and v.



The Voronoi relevant vectors (2)

Lemma

The number of Voronoi relevant vectors is $|R| \leq 2^{n+1}$.



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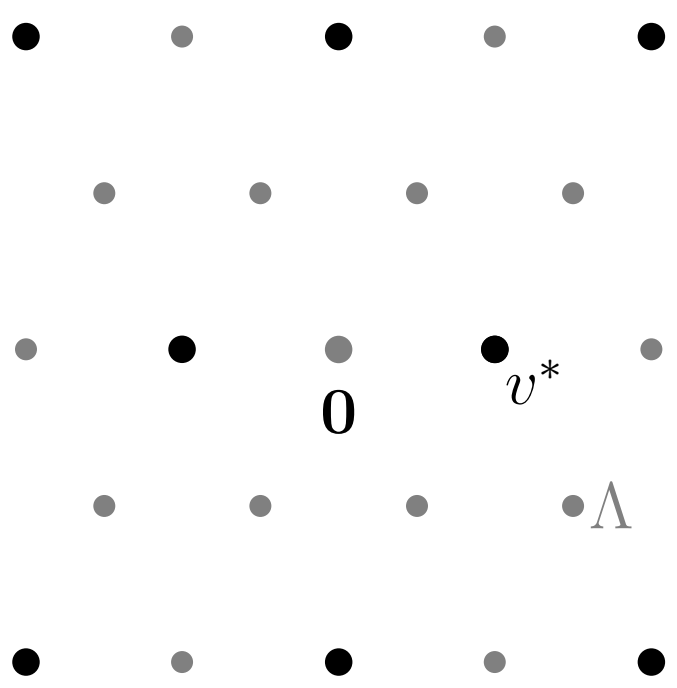
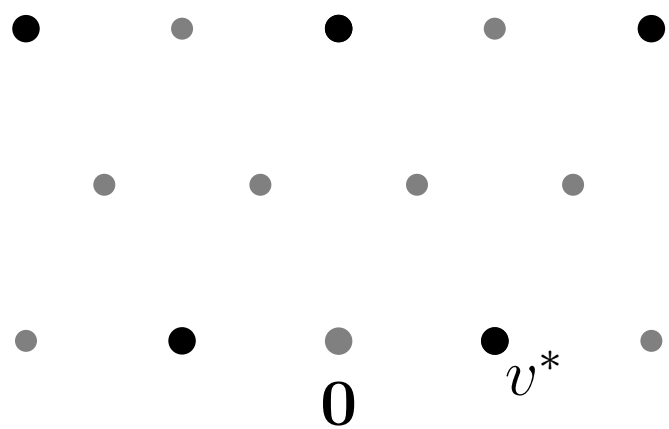
Claim. For $v \in \Lambda$, consider coset $\Lambda' := v + 2\Lambda$. Apart from $v^* := \operatorname{argmin}\{\|x\|_2 : x \in \Lambda'\}$ and $-v^*$, there is no other Voronoi relevant vector in Λ'.



The Voronoi relevant vectors (3)

Proof.



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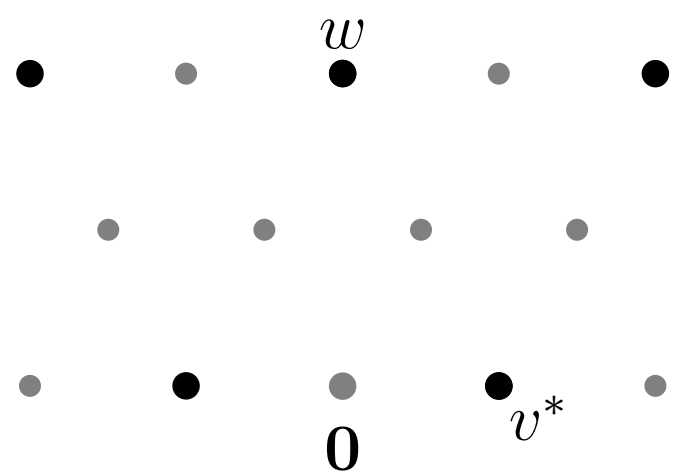
- Suppose that $w \in \Lambda'$ is Voronoi-relevant where $w \notin \{v^*, -v^*\}$



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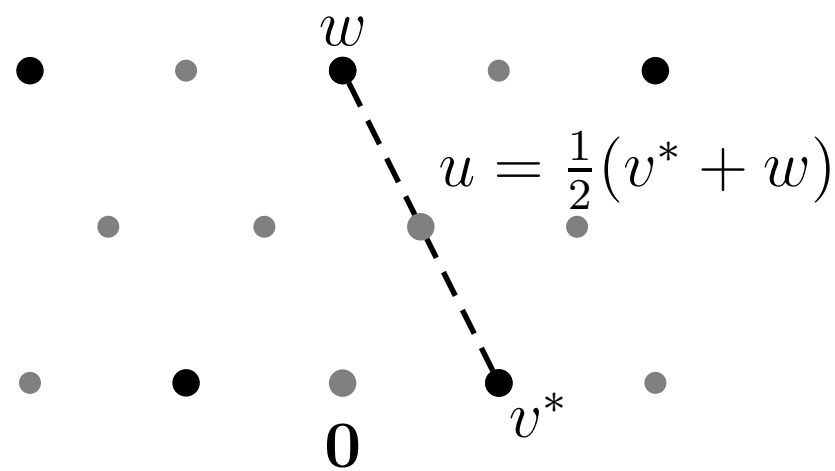

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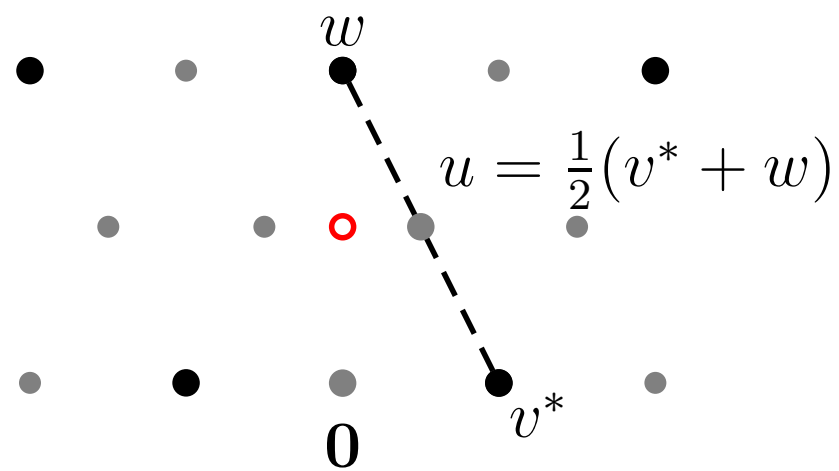
- Consider $u := \frac{1}{2}(v^* + w) \in \Lambda$
- Then

$$\left\| \frac{w}{2} - u \right\|_2 = \left\| \frac{w}{2} - \frac{1}{2}(v^* + w) \right\|_2 = \frac{1}{2}\|v^*\|_2 \leq \frac{1}{2}\|w\|_2$$

which is a contradiction. □



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Observation: We can find all Voronoi-relevant vectors by solving 2^n many CVPs in the same lattice, because

$$\min\{\|x\|_2 : x \in v + 2\Lambda\} = \min\{\|x - v\|_2 : x \in 2\Lambda\}$$

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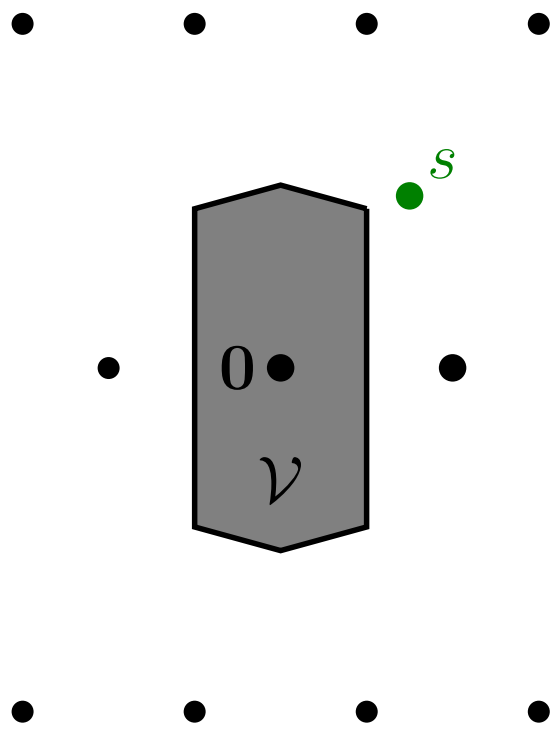
Now assume:

- We know the Voronoi-relevant vectors R
- We have a target vector $t \in 2\mathcal{V}$ (scaling handles the general case)

The Voronoi cell algorithm

Algorithm:

(1) Set $s := t$
(2) WHILE $s \notin \mathcal{V}$ DO
  (3) Set $\delta := \|s\|_{\mathcal{V}}$
  (4) Find $v \in R$ so that s lies on the boundary of δH_v
  (5) Update $s := s - v$
(6) Return $t - s$



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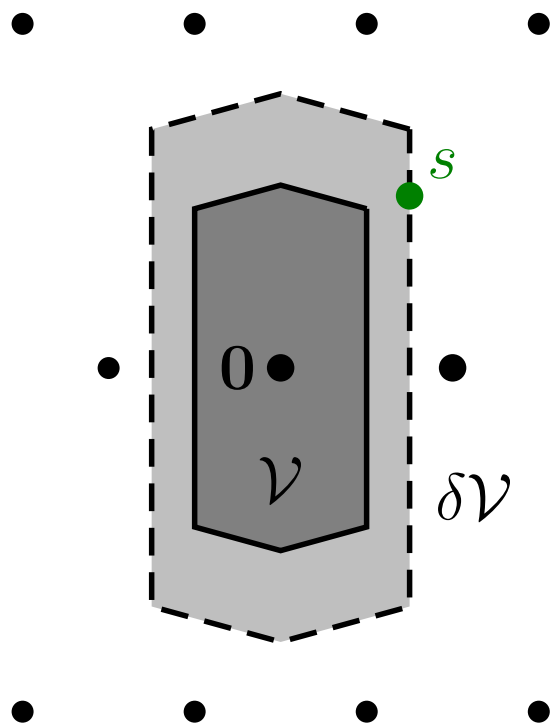

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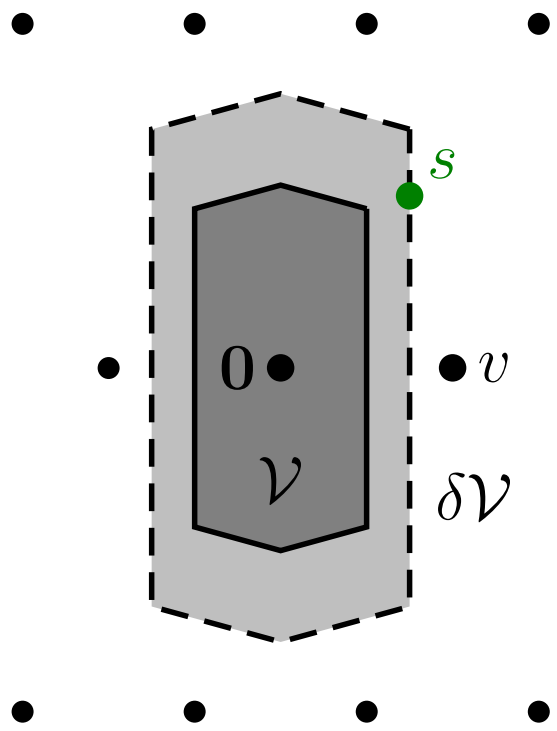

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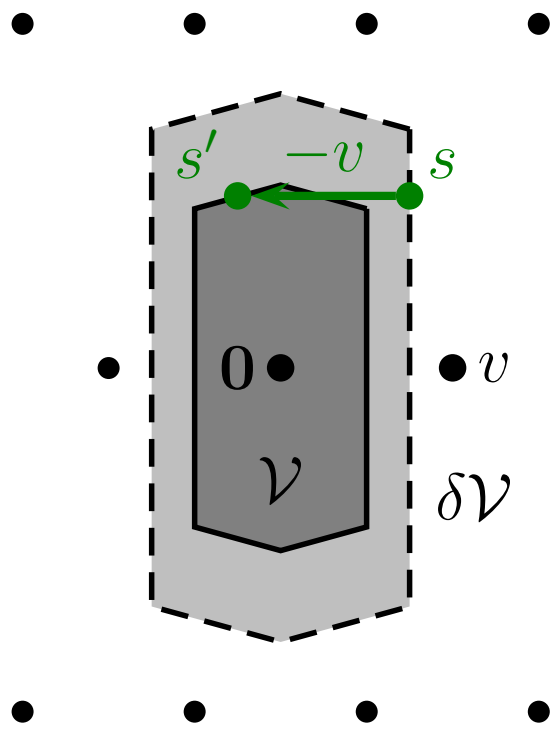
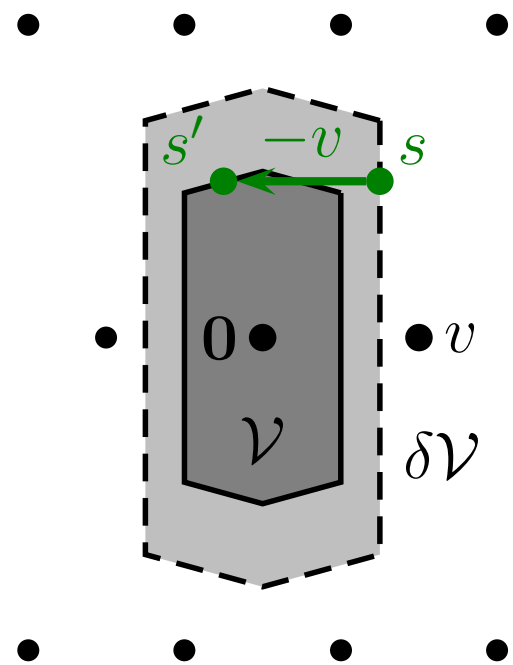


The Voronoi cell algorithm (2)

Claim. In each iteration $\|s - v\|_{\mathcal{V}} \leq \|s\|_{\mathcal{V}}$ and $\|s - v\|_2 < \|s\|_2$.



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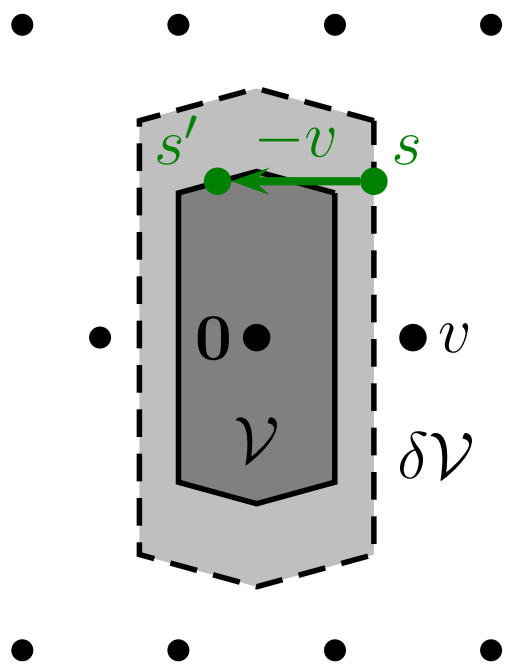
Claim. In each iteration $\|s-v\|_{\mathcal{V}} \leq \|s\|_{\mathcal{V}}$ and $\|s-v\|_2 < \|s\|_2$.
Proof.

- By assumption $\frac{s}{\delta} \in v + \mathcal{V}$ and so by triangle inequality

$$\|s-v\|_{\mathcal{V}} \leq \underbrace{\left\|\frac{s}{\delta} - v\right\|_{\mathcal{V}}}_{\leq 1} + \left(1-\frac{1}{\delta}\right) \underbrace{\|s\|_{\mathcal{V}}}_{\leq \delta} \leq \delta$$

- Bound on $\|\cdot\|_2$ clear.



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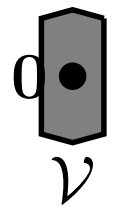
Lemma

For any $t \in \mathbb{R}^n$, $|(t - \Lambda) \cap 2\mathcal{V}| \leq 2^{O(n)}$.



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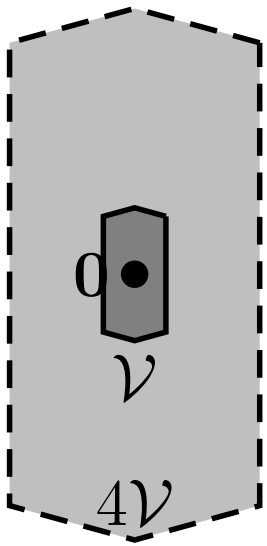

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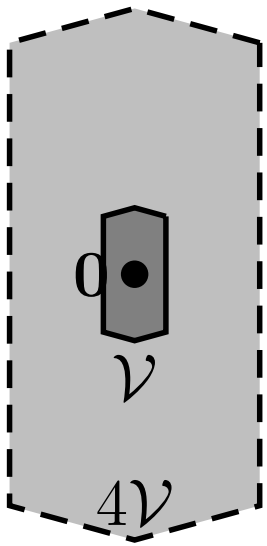

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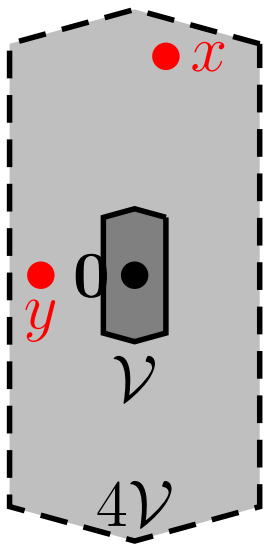

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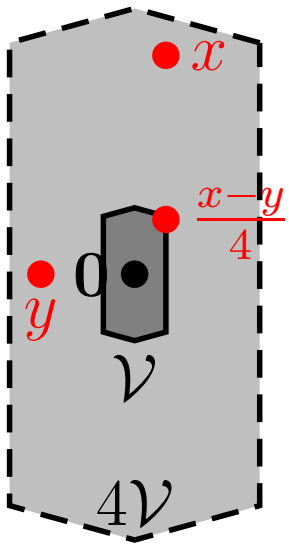

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More generally: given $t \in 2^k\mathcal{V}$, after $2^{O(n)}$ iteration we are in $2^{k-1}\mathcal{V}$ using update steps $2^{k-1}v$ with $v \in \Lambda$.

Putting things together

- Define

$$\begin{aligned} T_{\text{Voronoi}}(n) &= \text{time to compute Voronoi cell for } \Lambda \subseteq \mathbb{R}^n \\ T_{\text{CVP}}(n,k) &= \text{time to solve } k \text{ many CVPs in the} \\ &\quad \text{same } n\text{-dim. lattice} \end{aligned}$$

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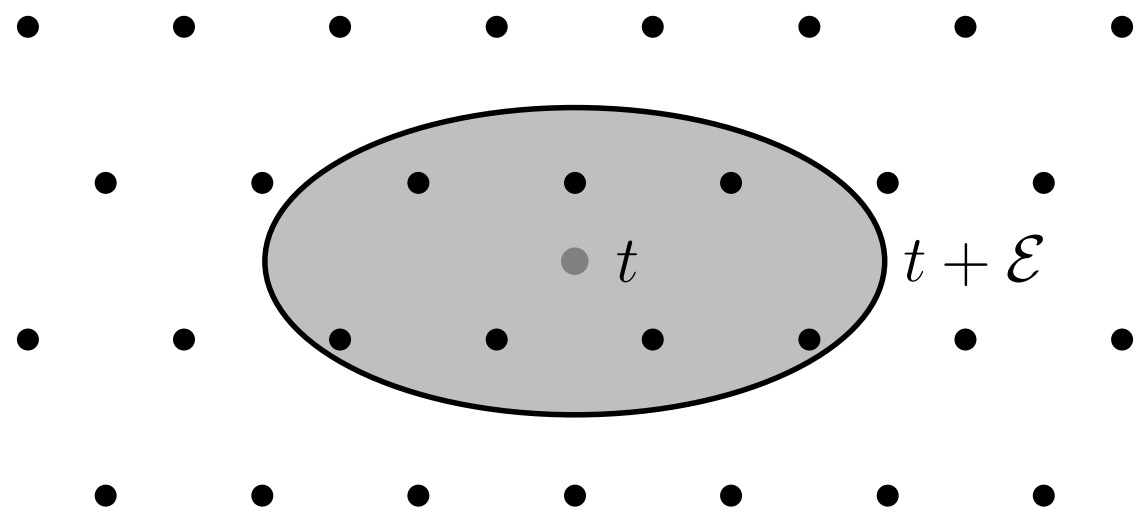
Enumeration of points

Theorem

For any ellipsoid $\mathcal{E}$, one can enumerate points $S := \Lambda \cap (\mathcal{E} + t)$ in time $2^{O(n)} \cdot (|S| + 1)$.



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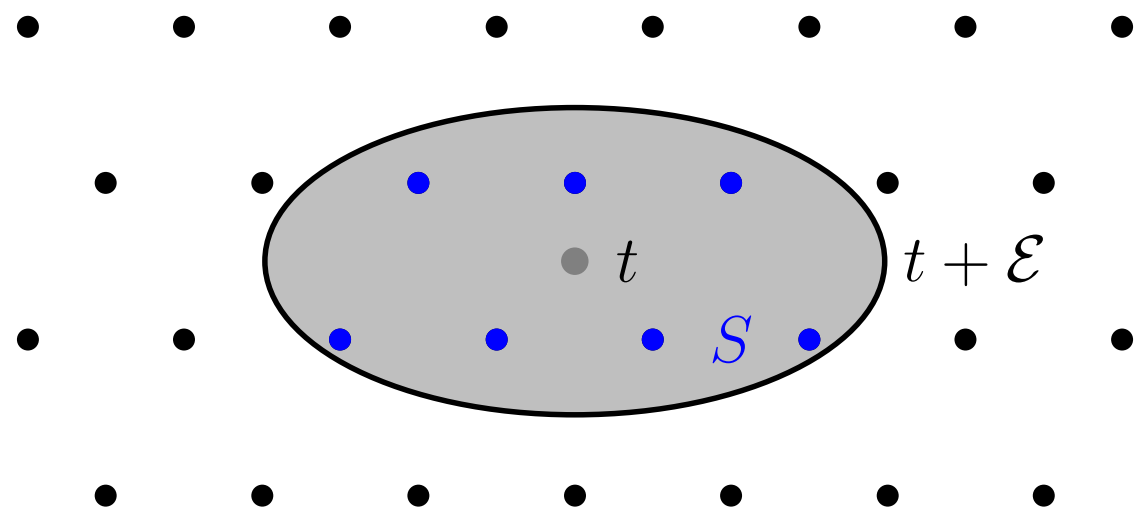

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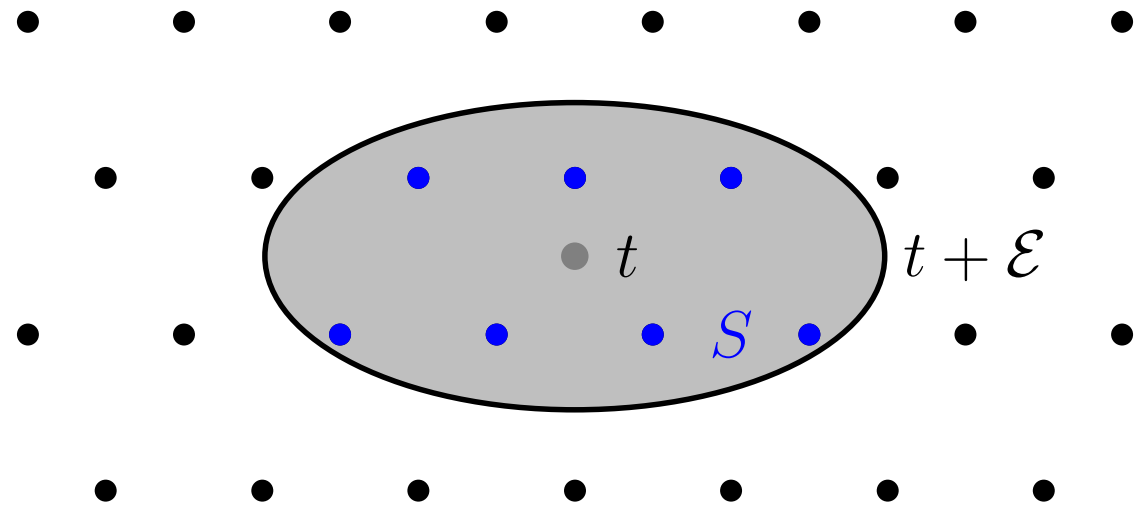

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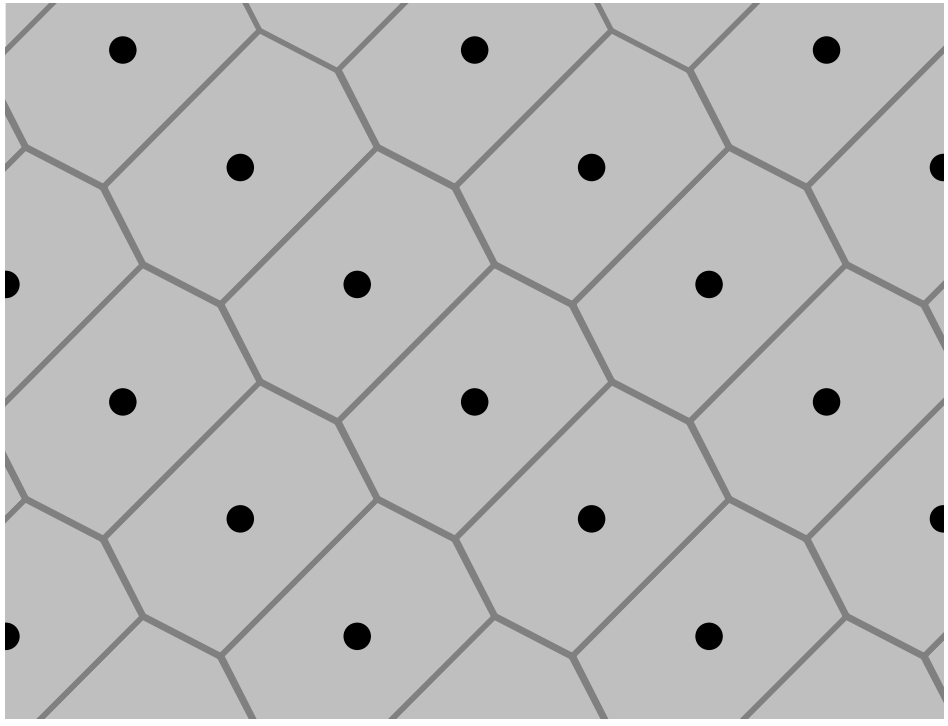

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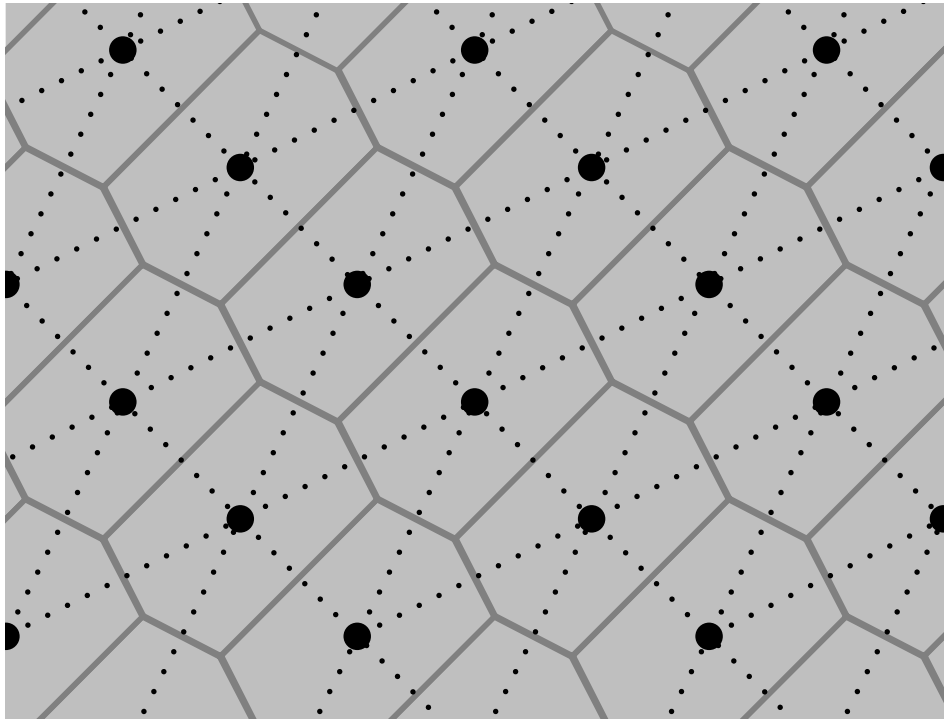

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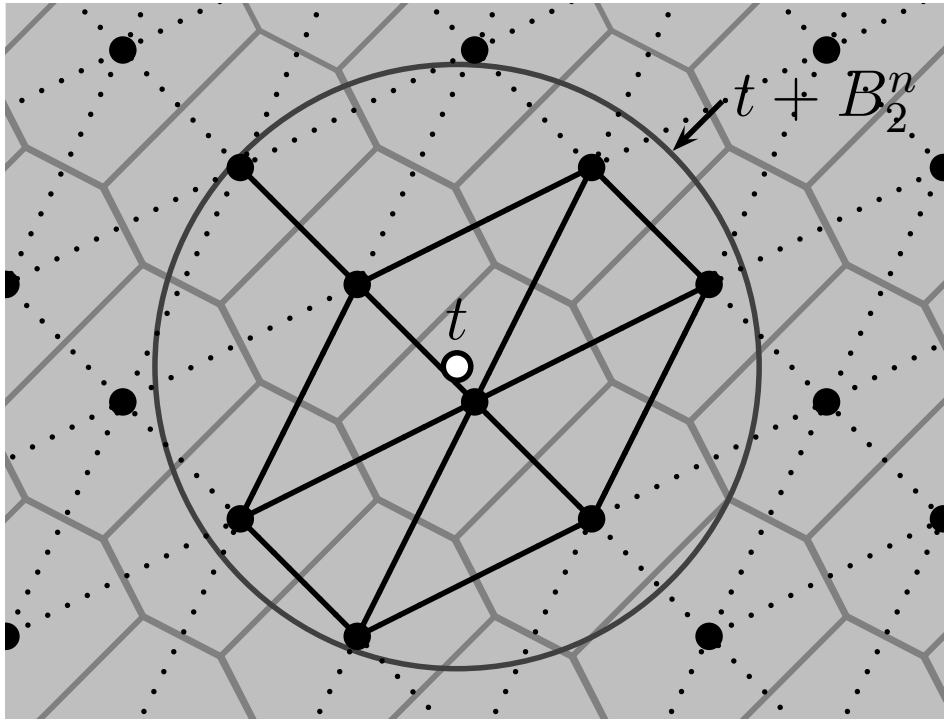

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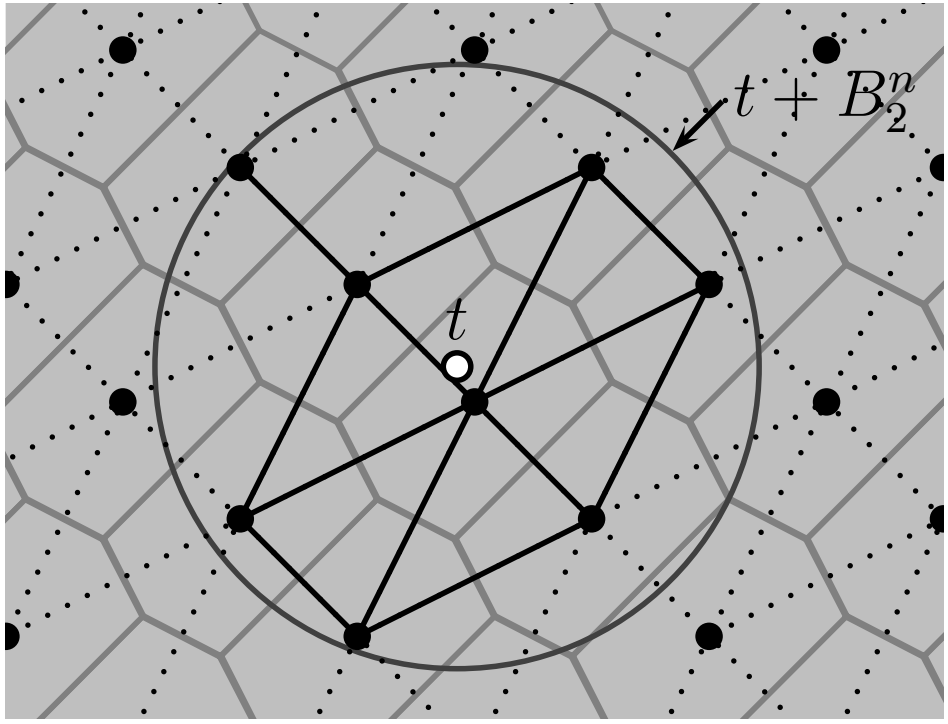

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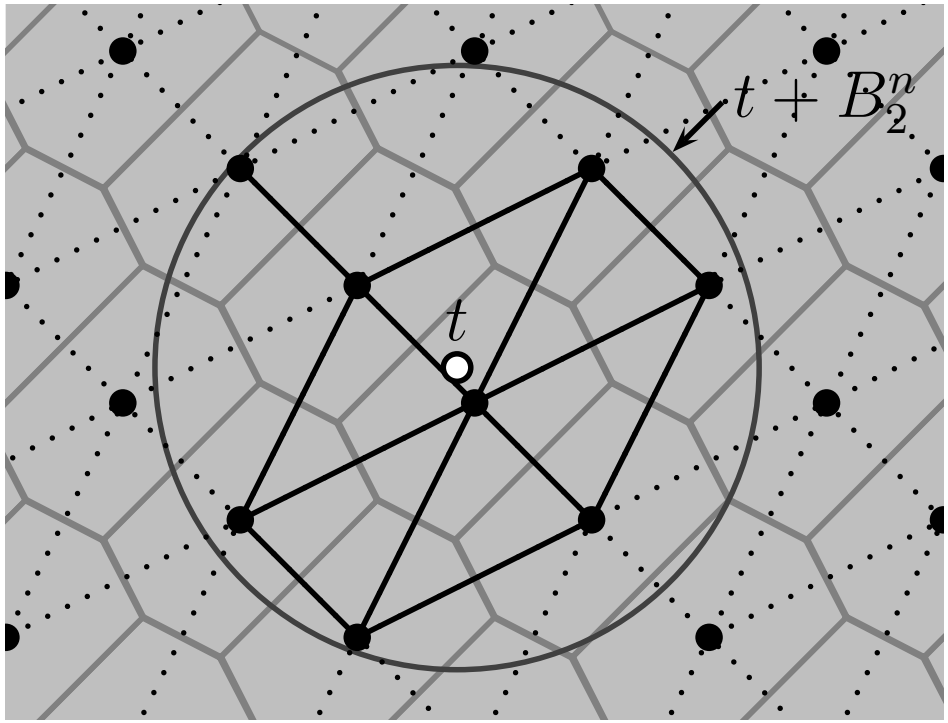

- Then explore $G[S]$ from point attaining $\mathrm{CVP}(\Lambda, t)$

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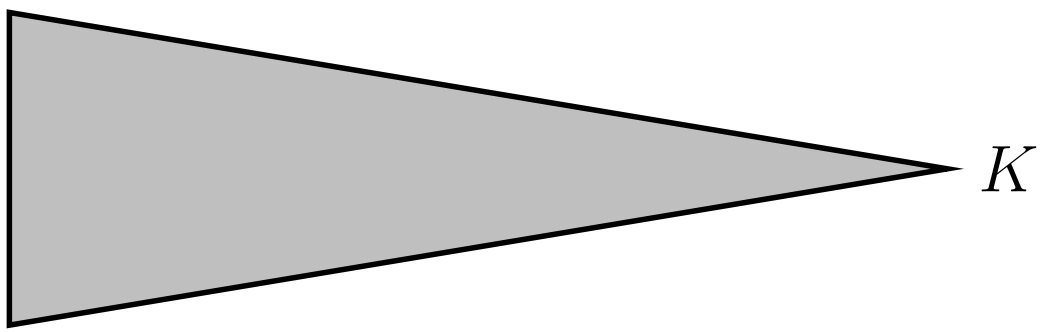

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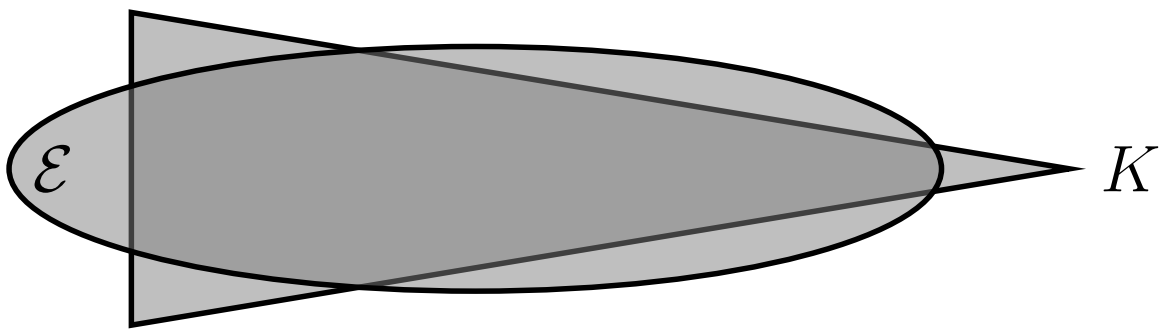

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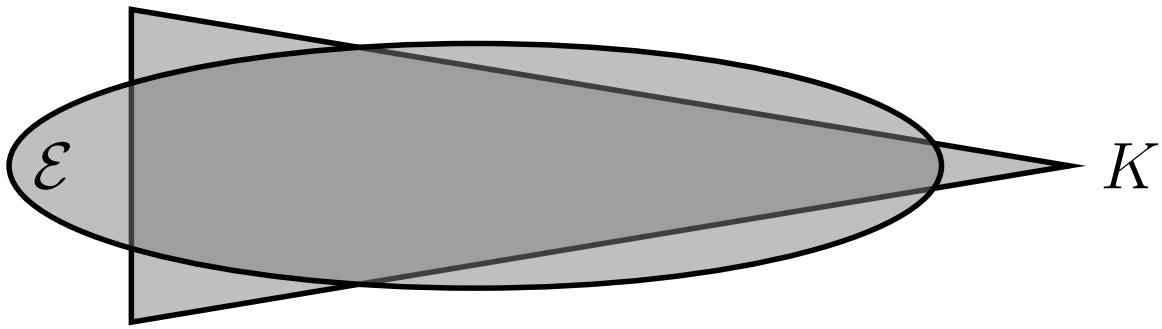

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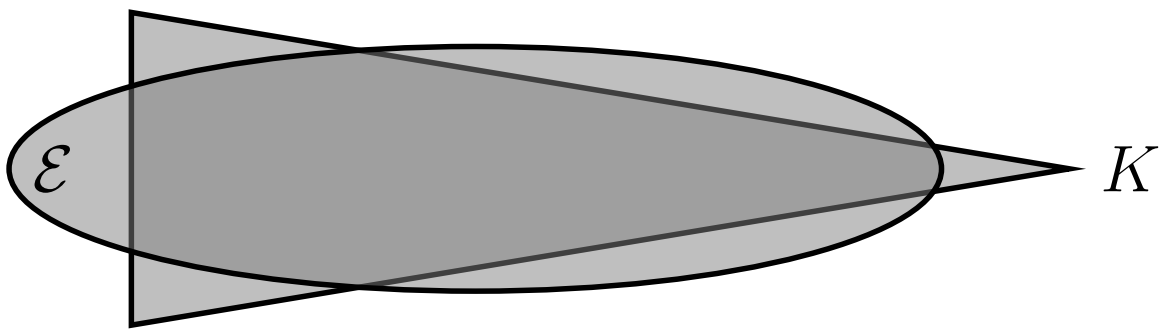

- In convex geometry these are called **M-ellipsoids**

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Idea: Cover K by $2^{O(n)}$ many M-ellipsoids, then find/enumerate points in ellipsoids.

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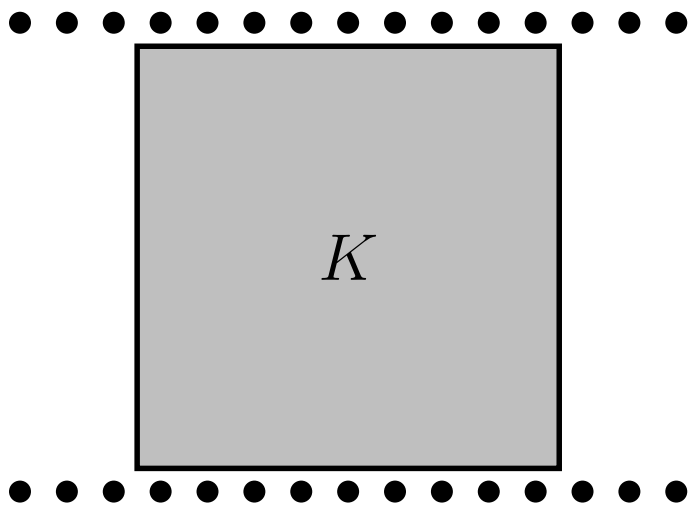

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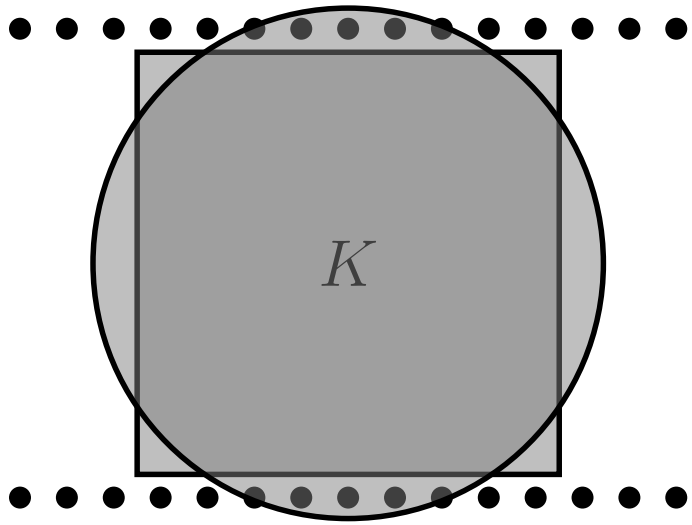

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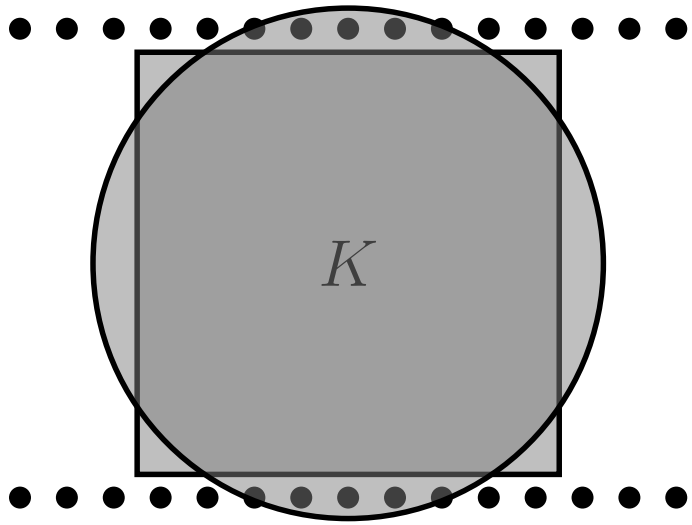

- But the method does actually work if the **covering radius** is lower bounded..

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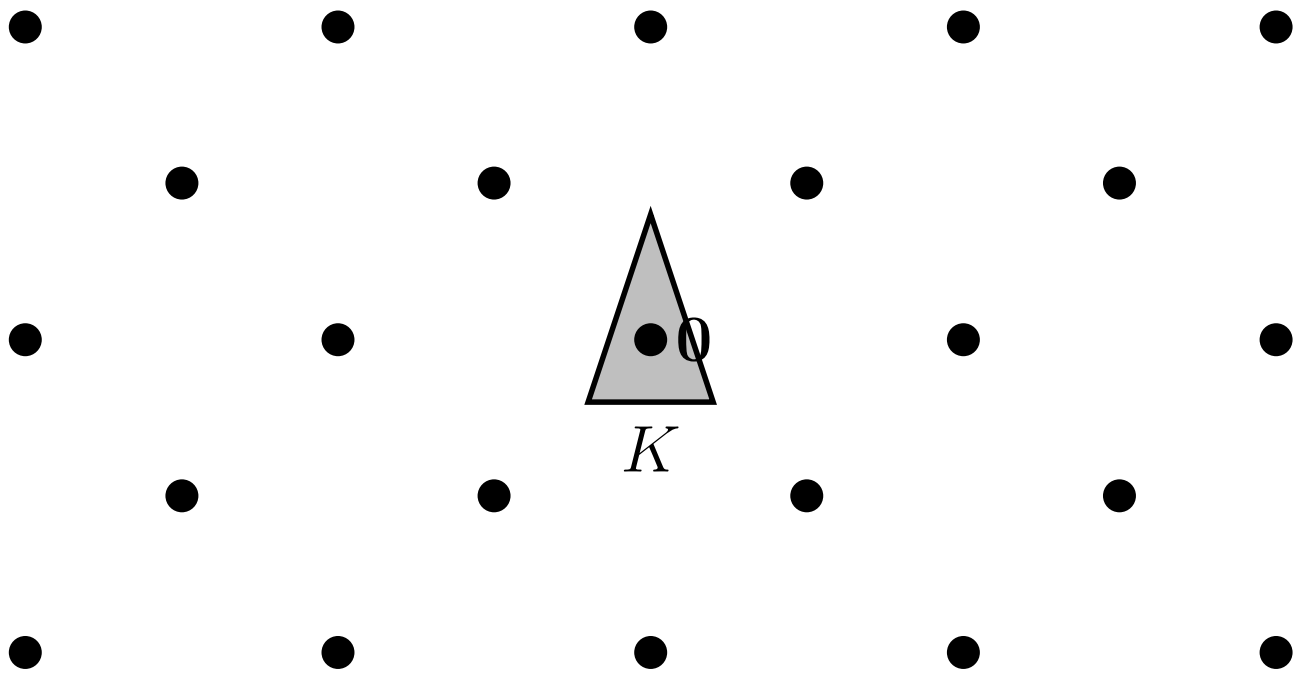

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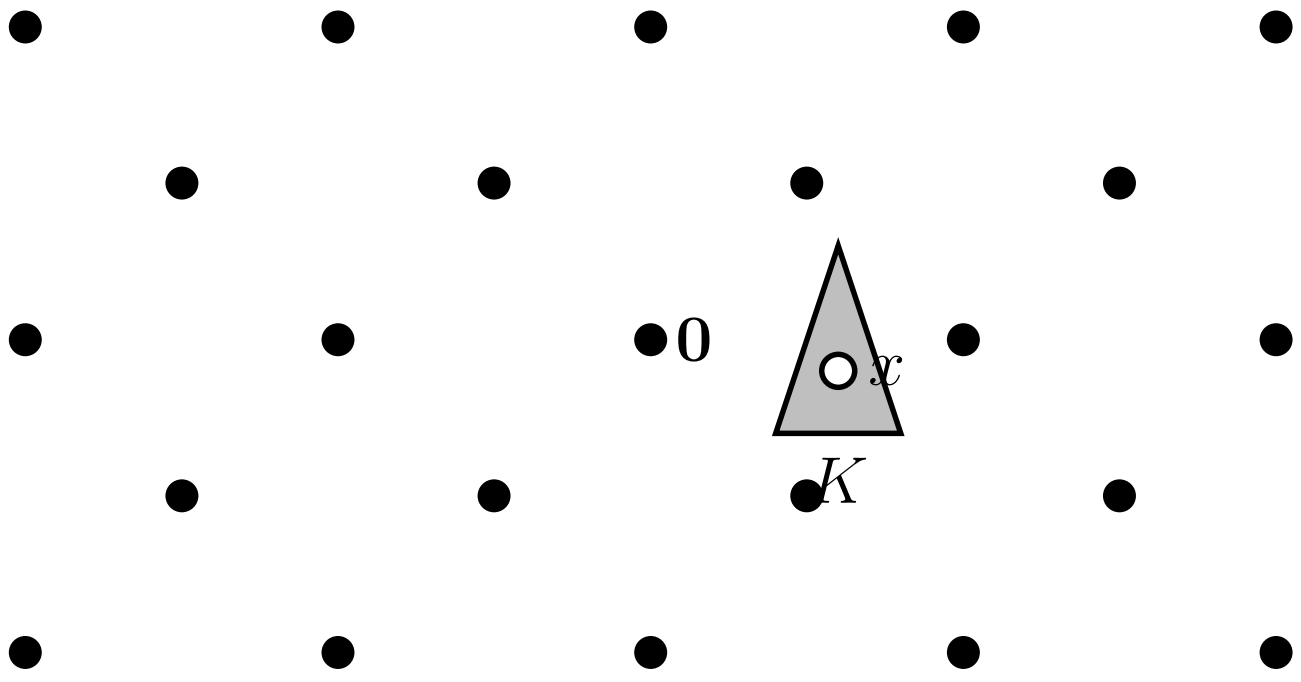

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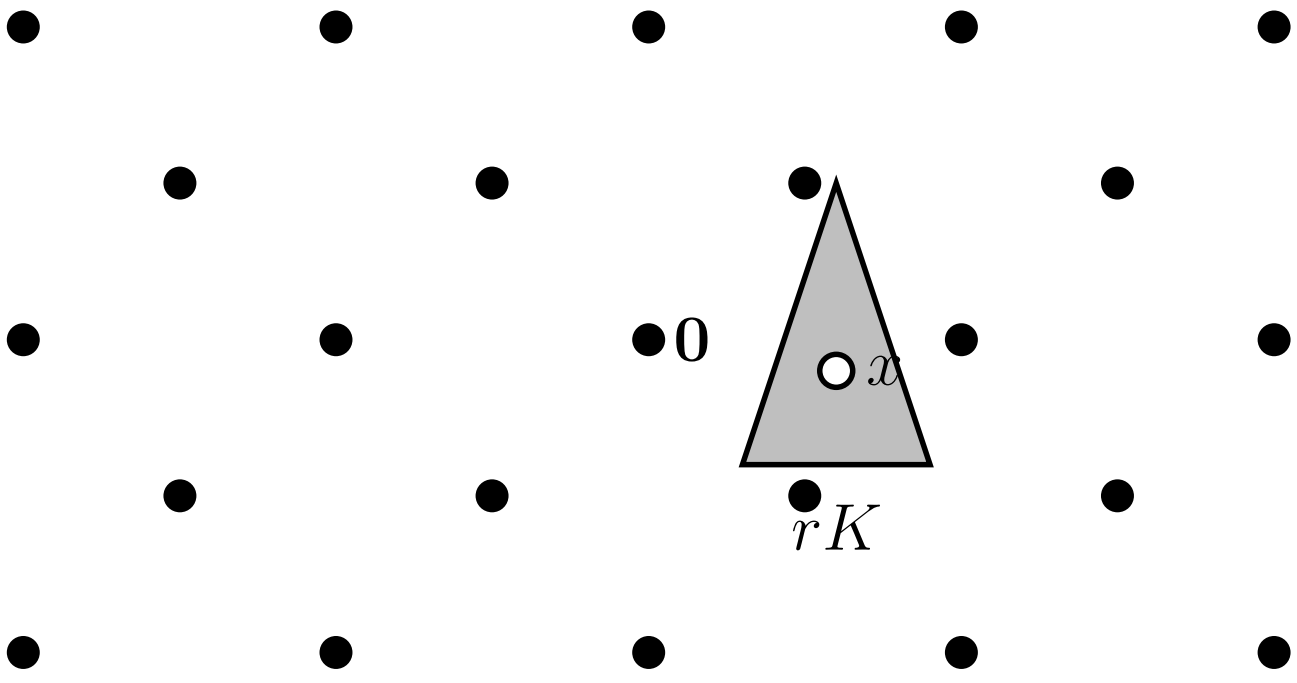

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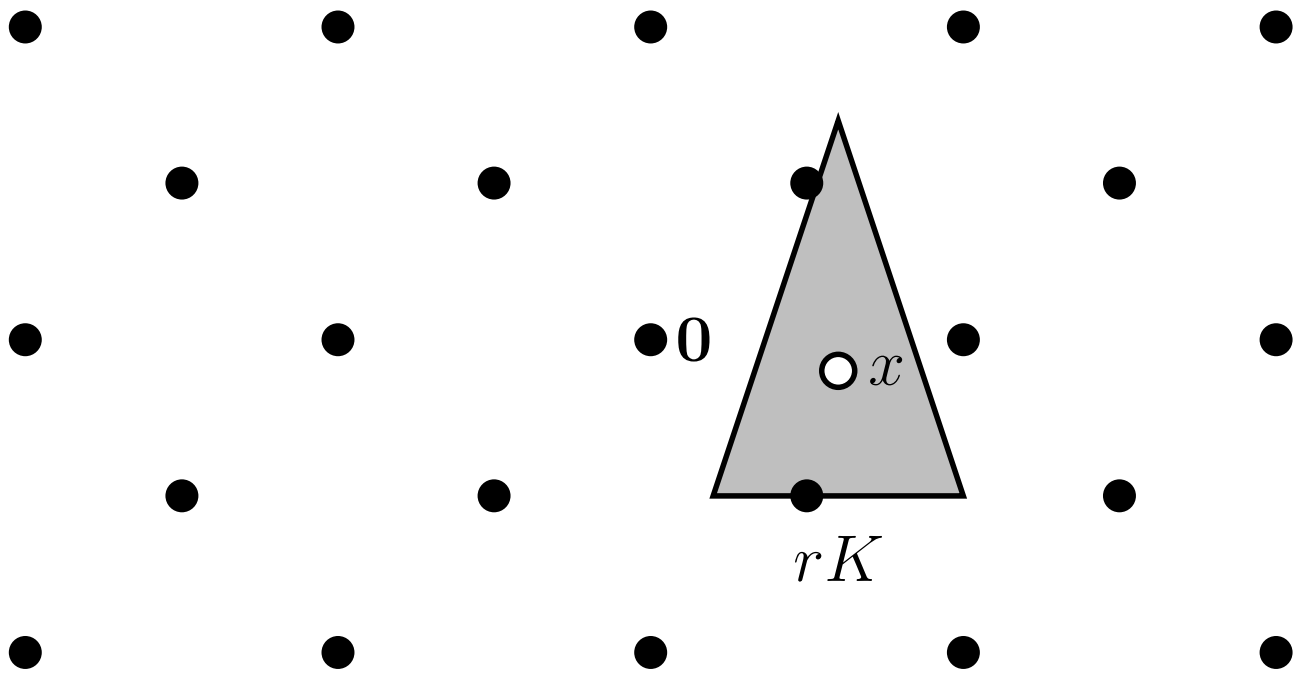


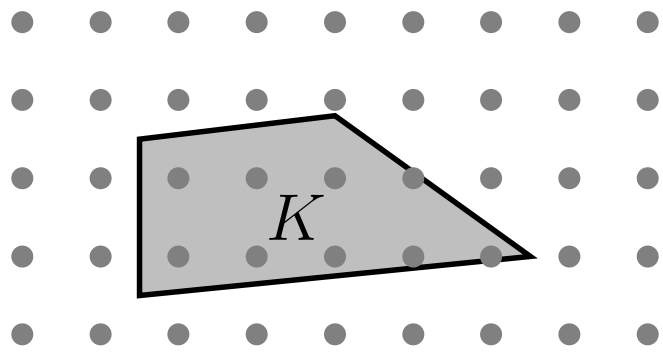
Dadush's lattice point upper bound

Theorem (Dadush 2012)

For full rank lattice $\Lambda \subseteq \mathbb{R}^n$ and convex body $K \subseteq \mathbb{R}^n$ one has

$$|K \cap \Lambda| \leq N := 2^n \max\{\mu(\Lambda, K)^n, 1\} \cdot \frac{Vol_n(K)}{\det(\Lambda)}.$$

Moreover, can compute points in same time.



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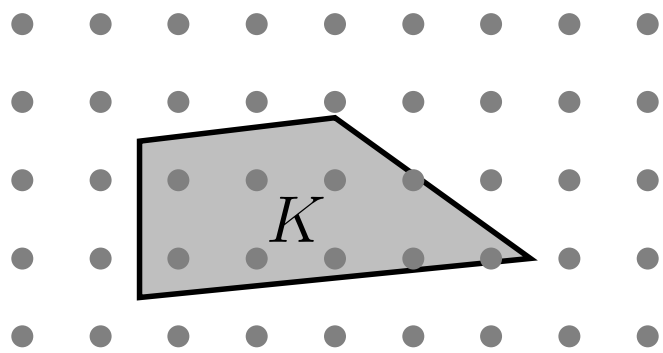

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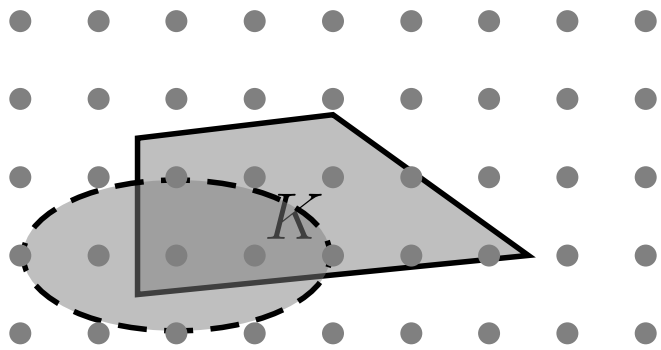

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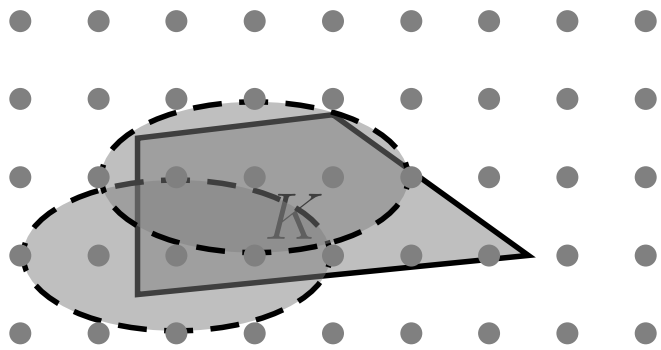

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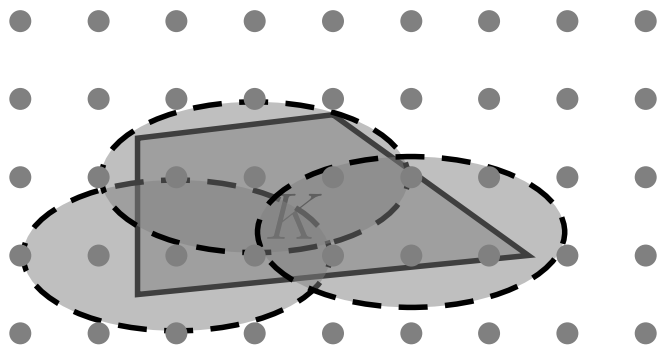

Dadush's lattice point upper bound

Theorem (Dadush 2012)

For full rank lattice $\Lambda \subseteq \mathbb{R}^n$ and convex body $K \subseteq \mathbb{R}^n$ one has

$$|K \cap \Lambda| \leq N := 2^n \max\{\mu(\Lambda, K)^n, 1\} \cdot \frac{Vol_n(K)}{\det(\Lambda)}.$$

Moreover, can compute points in same time.

Proof of moreover part:

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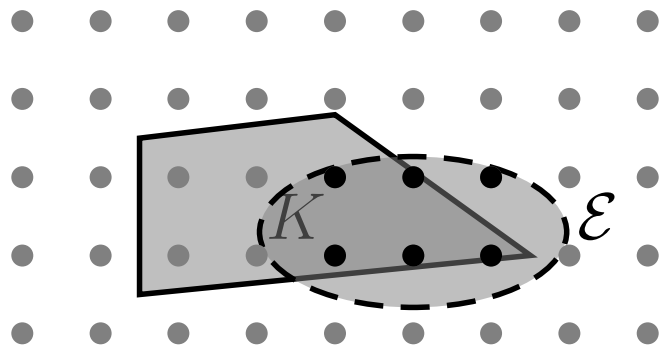

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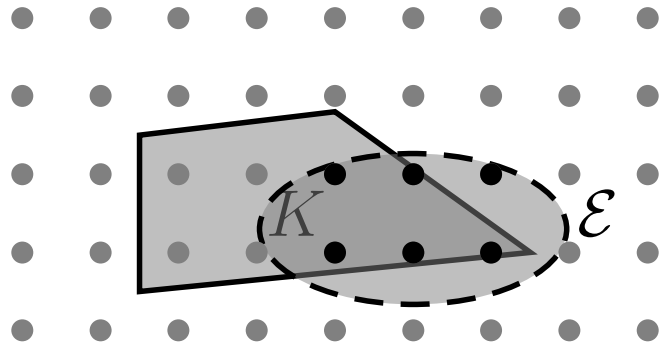
- Hence can enumerate all points in K in time $2^{O(n)} N$.



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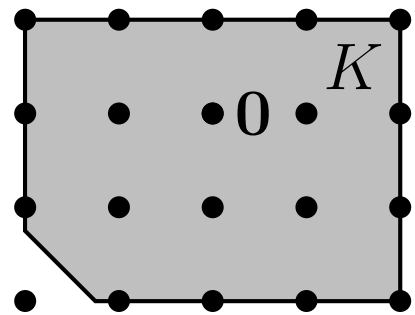
Claim. For any convex body $K \subseteq \mathbb{R}^n$ with $\mu(\mathbb{Z}^n, K) \leq 1$ and any $x \in \mathbb{R}^n$ one has $|K \cap (x + \mathbb{Z}^n)| \leq 2^n \mathrm{Vol}_n(K)$.



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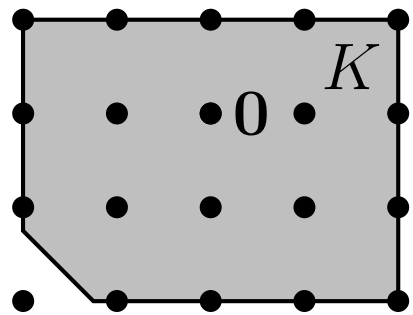

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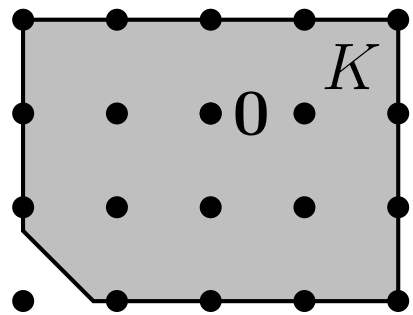

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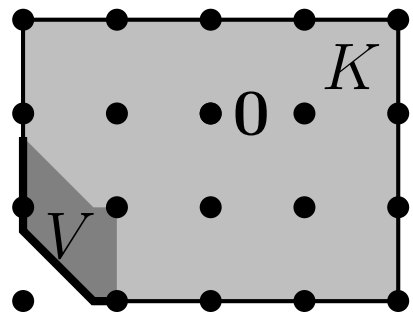

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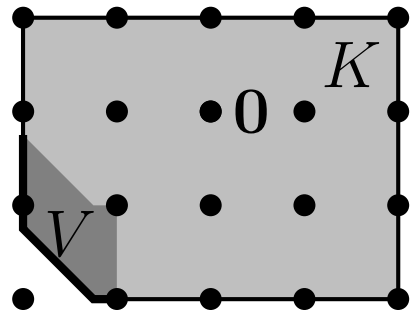

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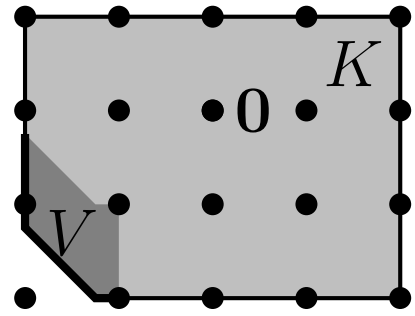

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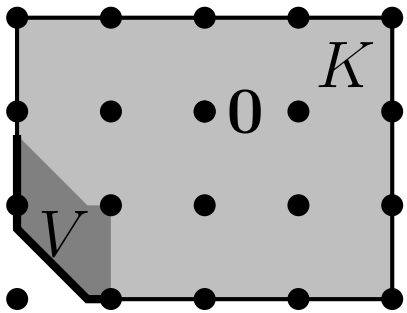

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- For all $x \in \mathbb{R}^n$ one has $(x + \mathbb{Z}^n) \cap K \neq \emptyset \Rightarrow \mathrm{Vol}_n(V) = 1$
- Translates $x + V$ disjoint for $x \in \mathbb{Z}^n$. Hence

$$|K \cap \mathbb{Z}^n| = \sum_{x \in K \cap \mathbb{Z}^n} \underbrace{\mathrm{Vol}_n(x + V)}_{=1} \overset{\text{disj.}}{=} \mathrm{Vol}_n\Big(\bigcup_{x \in K \cap \mathbb{Z}^n} (x + V) \Big) \leq \mathrm{Vol}_n(2K)$$

End of part 1

Open problem 1

Can one even solve Closest Vector problem (or shortest vector) in $\|\cdot\|_2$ in time $2^{O(n)}$ and **polynomial space**?

- In poly space so far only $n^{O(n)}$-time known [Kannan '87]

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Thanks for your attention!

Part 2:
The Reverse Minkowski Theorem

Thomas Rothvoss

Based the work
[Regev, Stephens-Davidowitz 2016]

UNIVERSITY *of*
WASHINGTON

Revisiting Minkowski's Theorem

Theorem (Minkowski's Theorem 1889)

Any full rank lattice $\Lambda \subseteq \mathbb{R}^n$ *with* $det(\Lambda) = 1$ *one has* $\lambda_1(\Lambda) \leq \sqrt{n}$.

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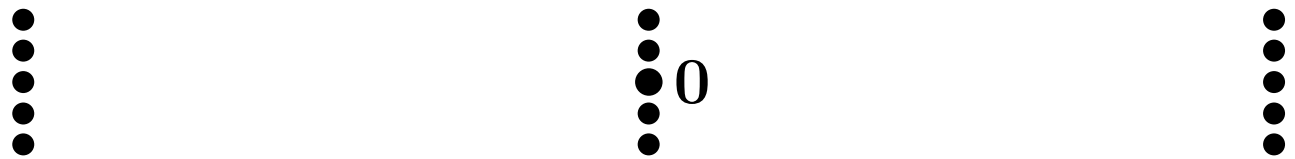

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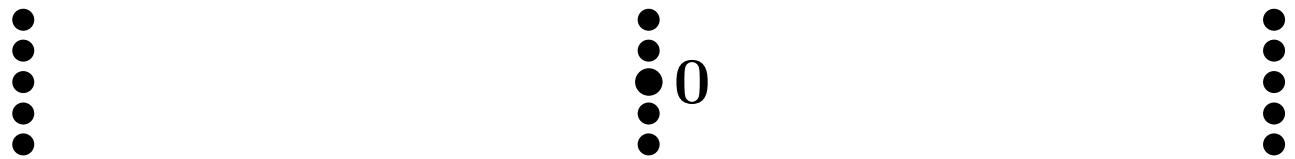


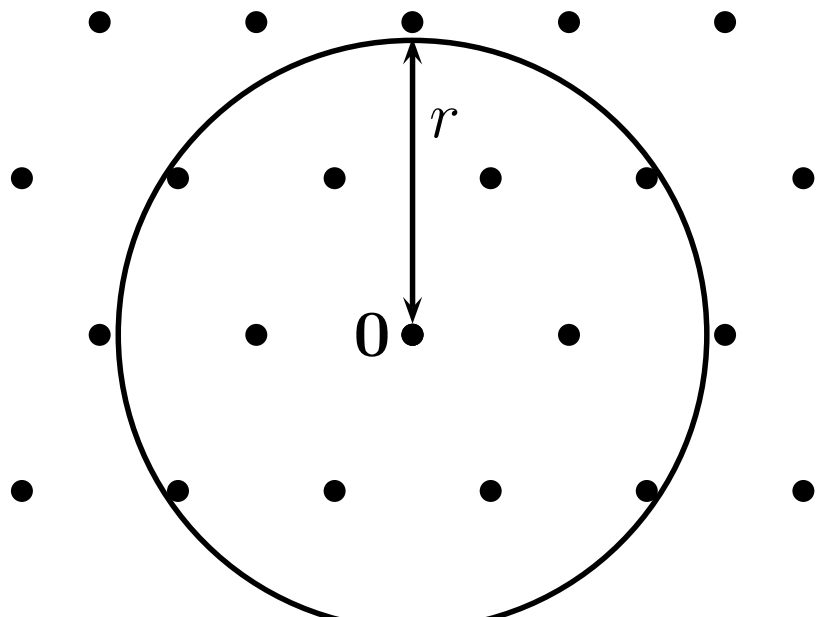
- What if $\det(\Lambda') \geq 1$ for all sublattices $\Lambda' \subseteq \Lambda$?

Warmup

Lemma

Let $\Lambda \subseteq \mathbb{R}^n$ be lattice with $\det(\Lambda') \geq 1 \; \forall \Lambda' \subseteq \Lambda$. For all $r \geq 1$ one has $N := |\Lambda \cap rB_2^n| \leq (3r)^n$

Proof.



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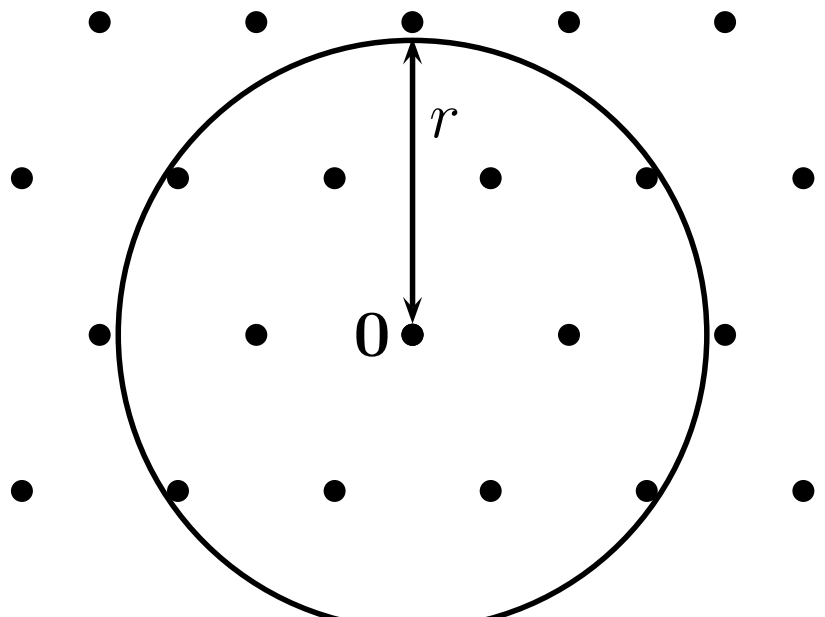

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- Clearly $\lambda_1(\Lambda) \geq 1$.
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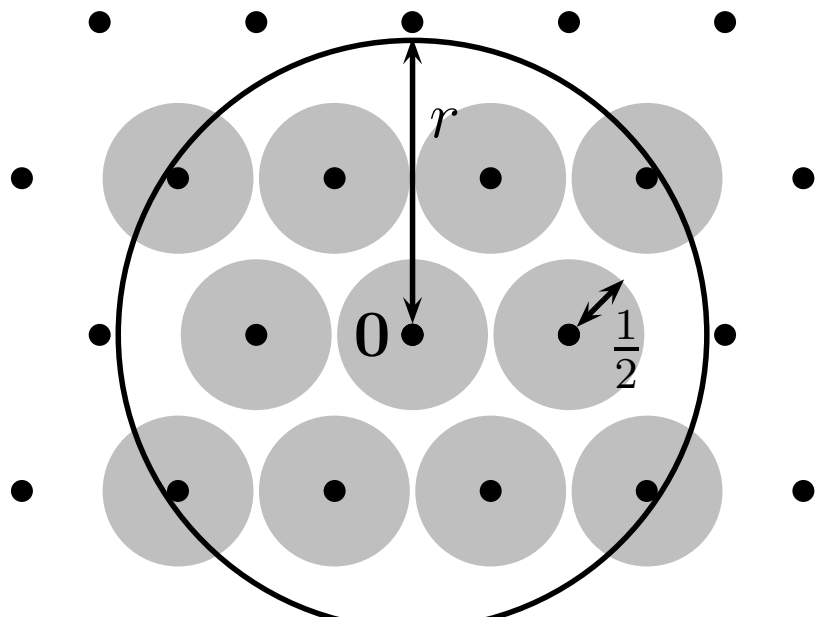

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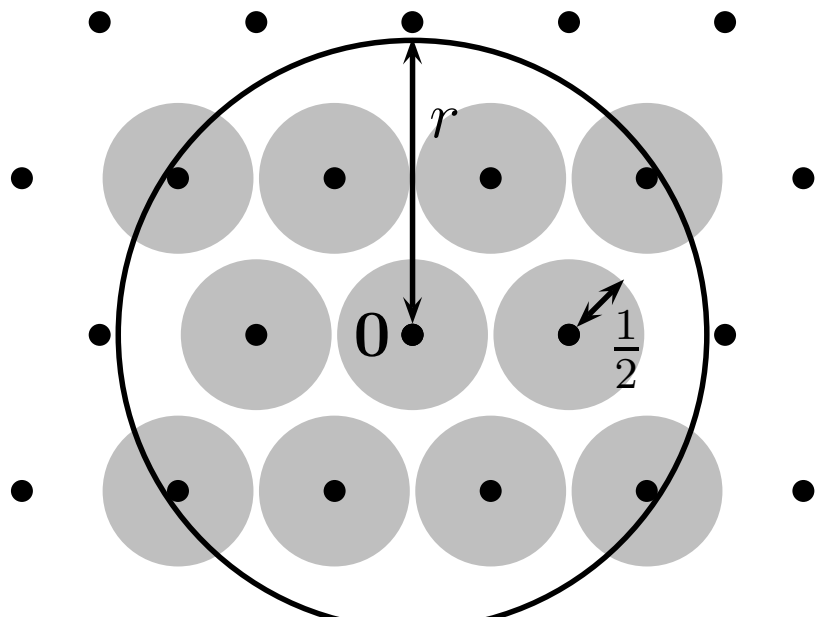
Proof.

- Clearly $\lambda_1(\Lambda) \geq 1$.
- Can pack N disjoint balls of radius $\frac{1}{2}$ into $(r + \frac{1}{2})B_2^n$. So $N \cdot \mathrm{Vol}_n(\frac{1}{2}B_2^n) \leq \mathrm{Vol}_n((r + \frac{1}{2})B_2^n)$. Then $N \leq (3r)^n$. □



The Reverse Minkowski Theorem

Reverse Minkowski Theorem (Regev, Stephens-Da.)

Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice that satisfies $\det(\Lambda') \geq 1$ for all sublattices $\Lambda' \subseteq \Lambda$. Then for $s = \Theta(\log n)$,

$$\rho_{1/s}(\Lambda) = \sum_{x \in \Lambda} \exp(-\pi s^2 \|x\|_2^2) \leq \frac{3}{2}$$

- Means for all $r \geq 1$, $|\Lambda \cap rB_2^n| \leq n^{\Theta(\log(n)) \cdot r^2}$ (i.e. # of points grows **quasi-polynomial** in r)

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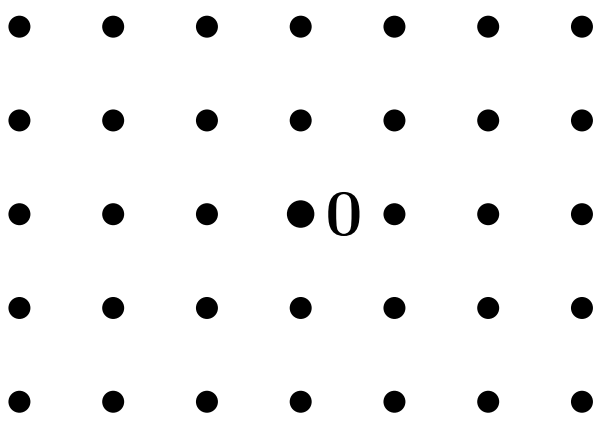
- First conjectured by [Dadush, Regev 2016].
- Conjectured that $s = \Theta(\sqrt{\log(n)})$ is enough which would give $|\Lambda \cap rB_2^n| \leq n^{O(r^2)}$ for $r \geq 1$.

Stable lattice

Definition

A lattice Λ is called **stable** if $\det(\Lambda) = 1$ and $\det(\Lambda') \geq 1$ for all sublattices $\Lambda' \subseteq \Lambda$.

Example: $\mathbb{Z}^n$ is stable.



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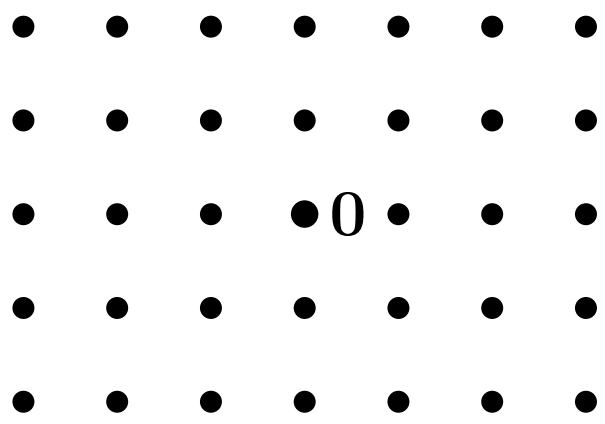
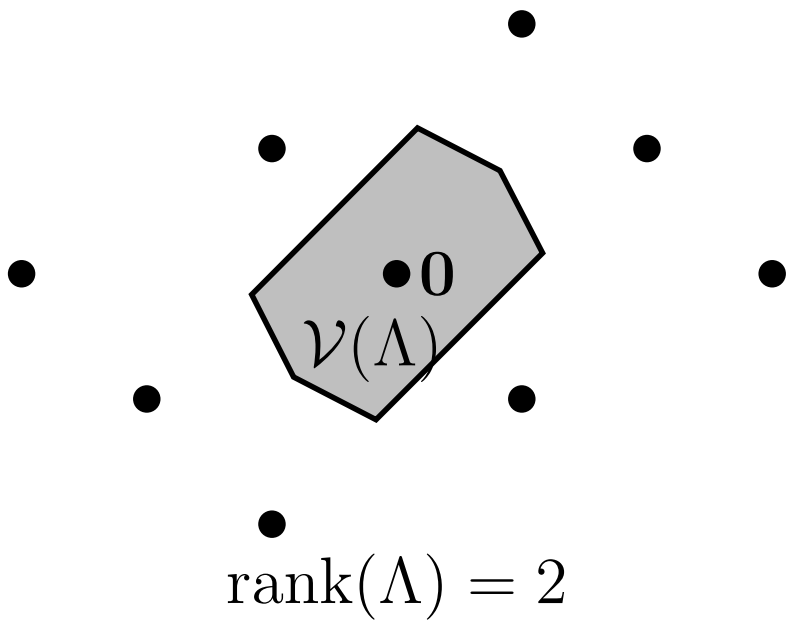


- It suffices to prove Reverse Minkowski for stable lattices (later more on that..).

The Voronoi cell

- **Recap:** Let $\Lambda \subseteq \mathbb{R}^n$ be lattice. The **Voronoi cell** is

$$\mathcal{V}(\Lambda) = \{x \in \mathbb{R}^n : \|x\|_2 \leq \|x - v\|_2 \; \forall v \in \Lambda \setminus \{\mathbf{0}\}\}$$



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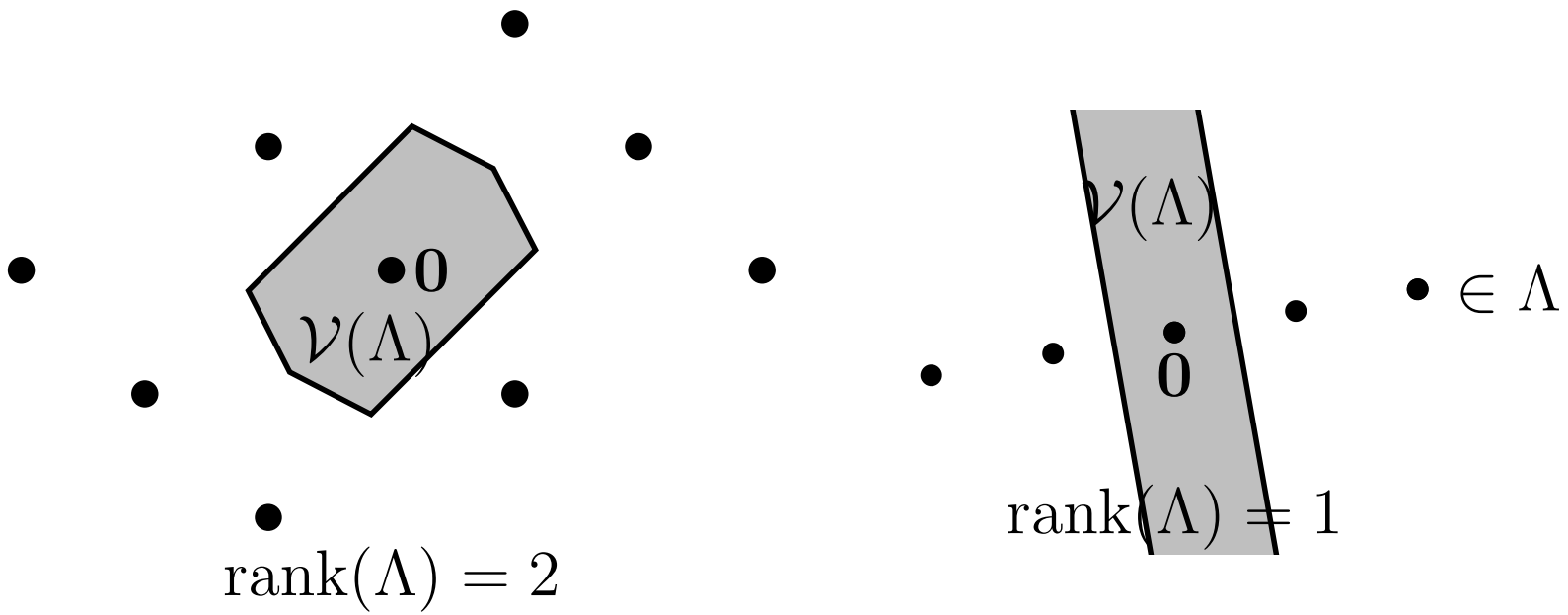

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Lemma

A full rank stable lattice Λ *has* $Vol_n(\mathcal{V}(\Lambda)) = 1$.

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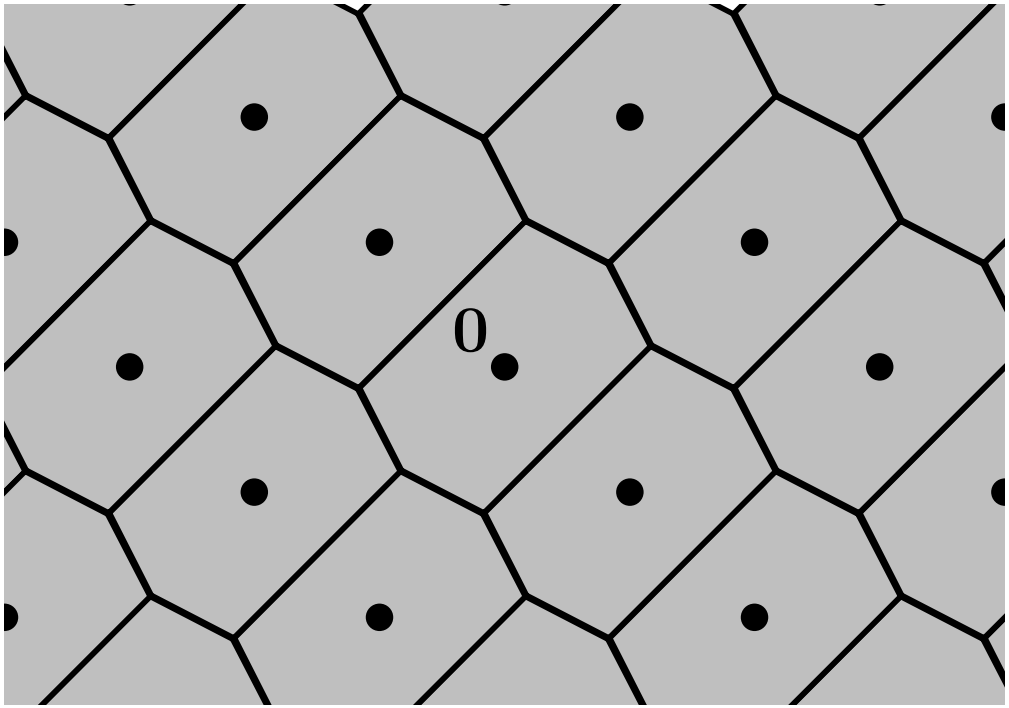
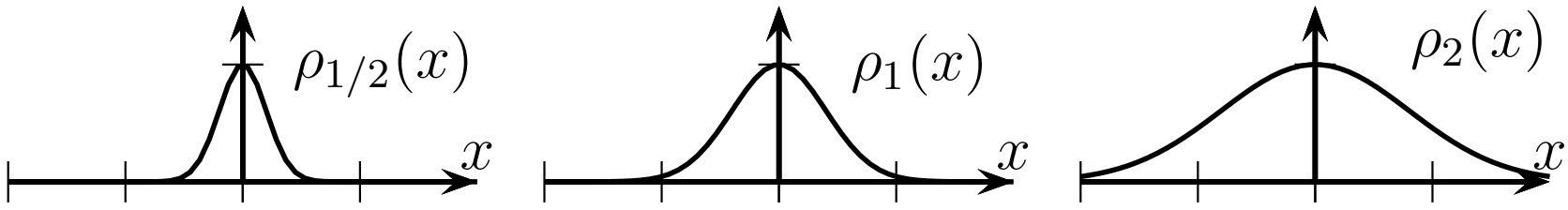
Proof.

- Translates $v + \mathcal{V}(\Lambda)$ tile $\mathbb{R}^n$ and density is $\frac{1}{\det(\Lambda)} = 1$.



Some notation

- **Gaussian density** $\rho_s(x) = \exp(-\pi \|x/s\|_2^2)$



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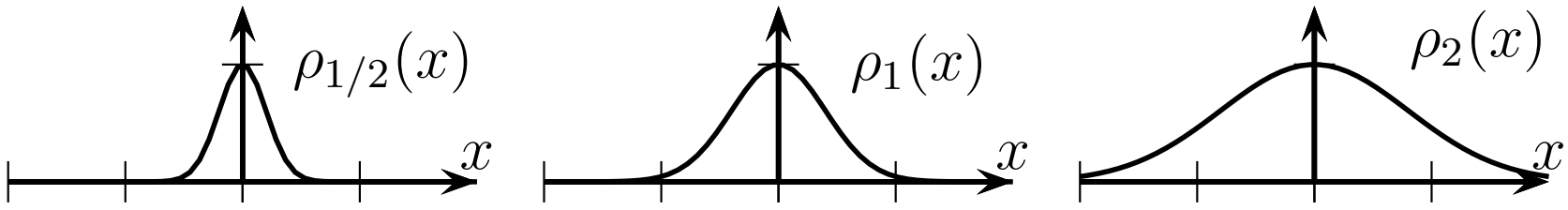

- **Gaussian measure**

$$\gamma_{n,s}(K) = \frac{1}{\rho_s(\mathbb{R}^n)} \int_K \rho_s(x)dx$$

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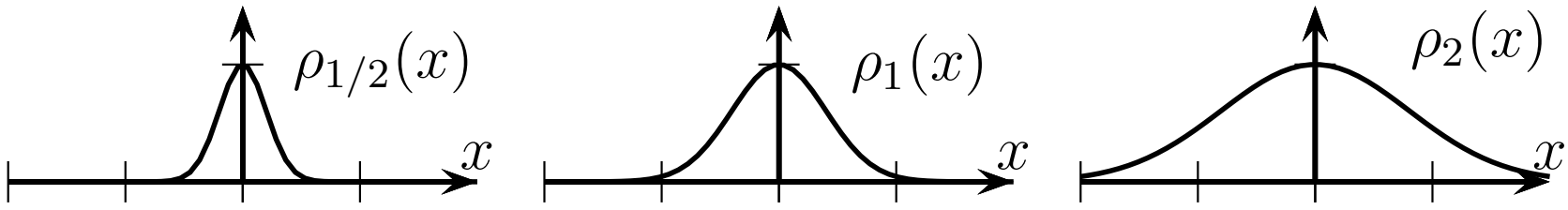

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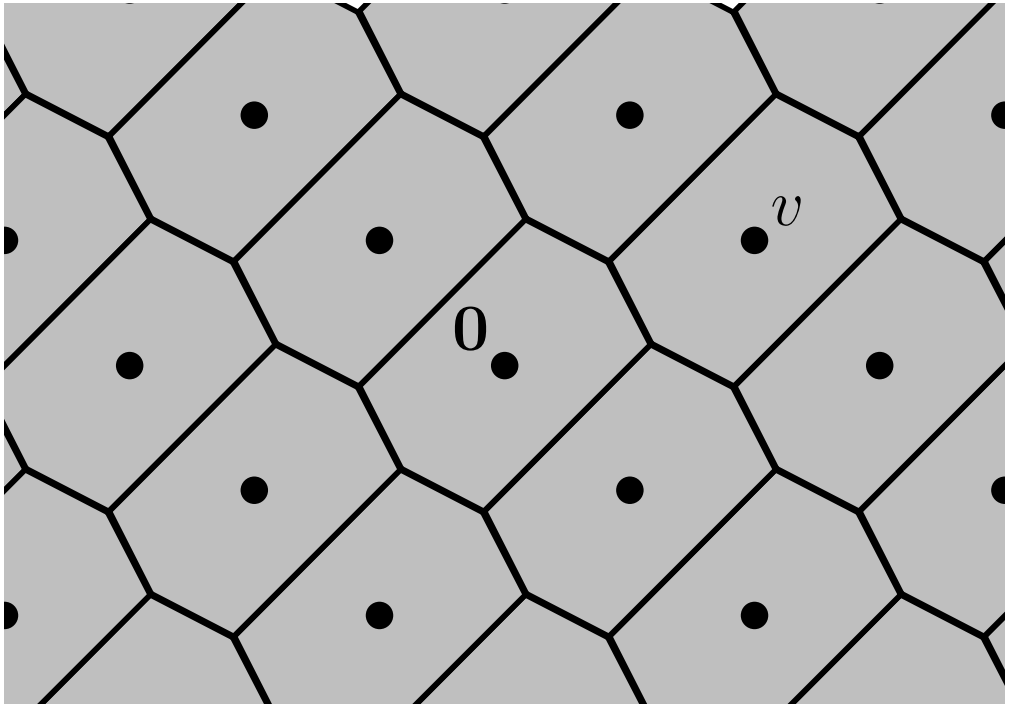
- **Standard gaussian measure**

$$\gamma_n(K) = \frac{1}{(2\pi)^{n/2}} \int_K \exp\Big(-\frac{\|x\|_2^2}{2}\Big)dx = \gamma_{n,\sqrt{2\pi}}(K)$$

Voronoi cell large $\Rightarrow$ $\rho(\Lambda)$ small

Lemma

For any lattice $\Lambda \subseteq \mathbb{R}^n$ and $s > 0$, $\rho_s(\Lambda) \cdot \gamma_{n,s}(\mathcal{V}(\Lambda)) \leq 1$.



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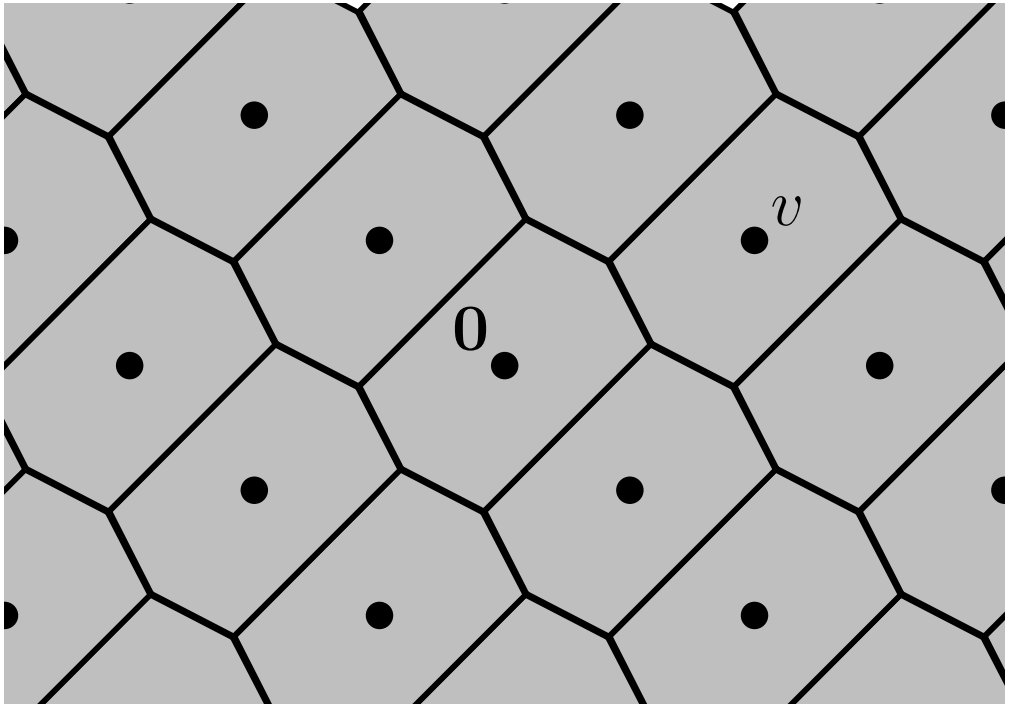

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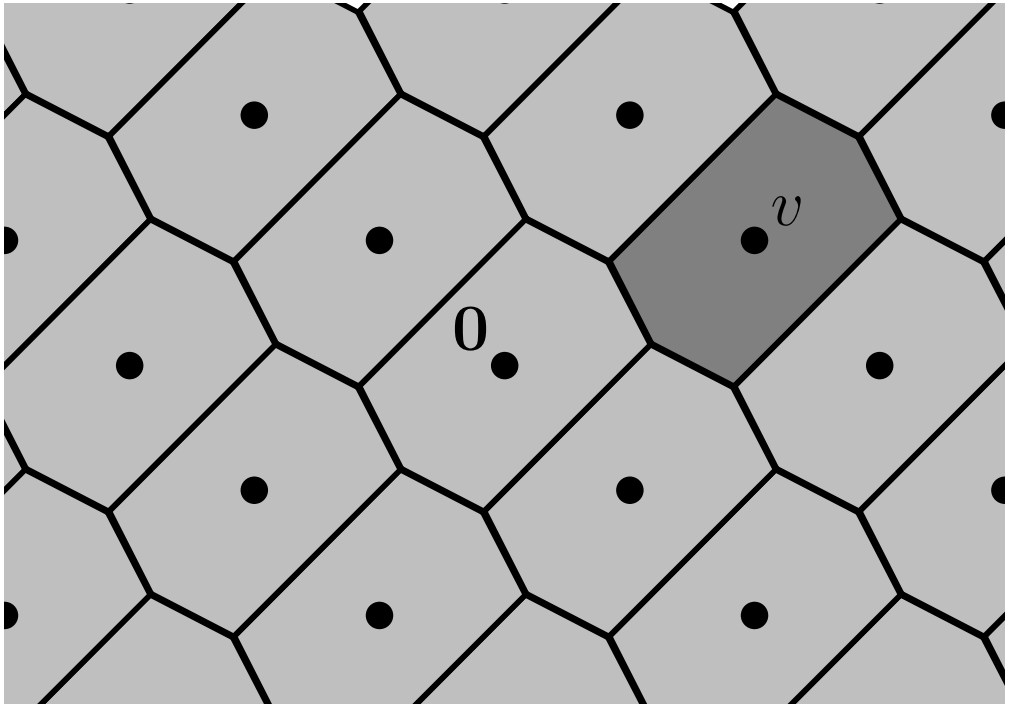

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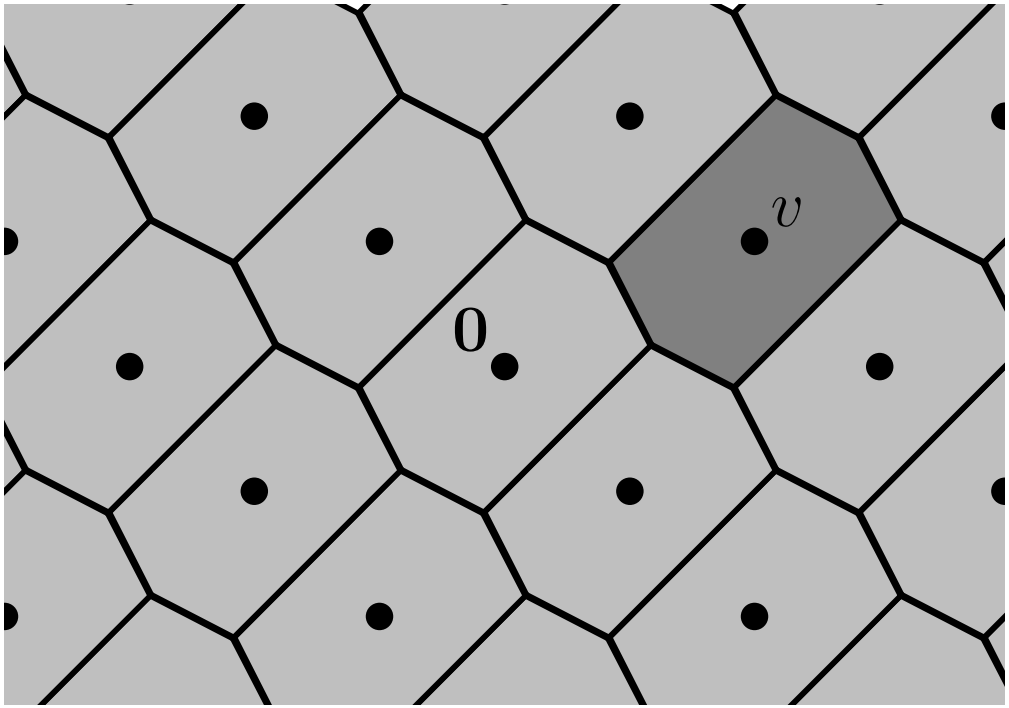

- For $v \in \Lambda$ one has $\gamma_{n,1}(v + \mathcal{V}) \geq e^{-\pi\|v\|_2^2}\gamma_{n,1}(\mathcal{V})$.
- Summing gives $1 = \sum_{v\in\Lambda} \gamma_{n,1}(v + \mathcal{V}) \geq \rho_1(\Lambda) \cdot \gamma_{n,1}(\mathcal{V})$ $\square$

Convex geometry

- Is it true that a symmetric convex body K with $\mathrm{Vol}_n(K) = 1$ has large Gaussian measure (after scaling with $\Theta(\log n)$?
- Maybe, maybe not..

Convex geometry

- Is it true that a symmetric convex body K with $\mathrm{Vol}_n(K) = 1$ has large Gaussian measure (after scaling with $\Theta(\log n)$?
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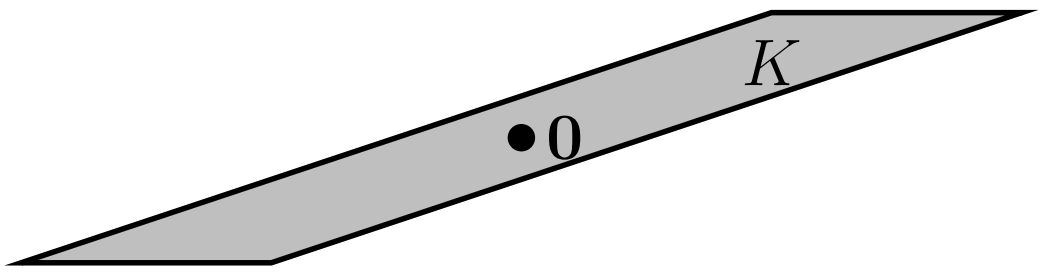

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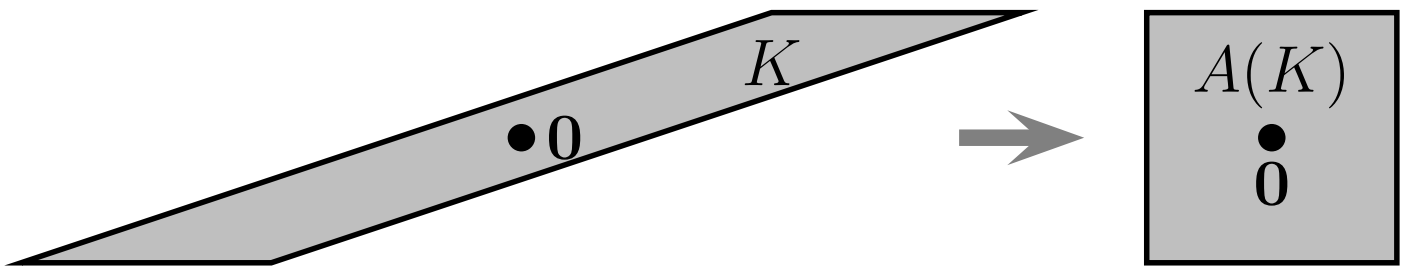


Theorem

For any symmetric convex body $K \subseteq \mathbb{R}^n$ with $Vol_n(K) = 1$ there is a matrix $A \in \mathbb{R}^{n \times n}$ with $|\det(A)| = 1$ so that $\gamma_n(\Theta(\log n) \cdot A(K)) \geq \frac{2}{3}$.

Isotropic position

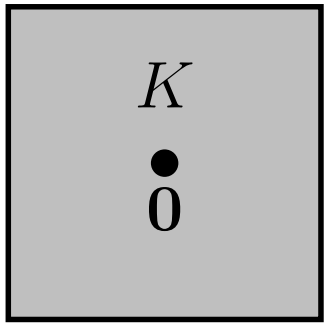
Definition

A symmetric convex body $K \subseteq \mathbb{R}^n$ is in **isotropic s-Gaussian position** if

$$\int_K \rho_s(x) \cdot xx^T dx = \alpha I_n$$

for some $\alpha > 0$.

- Means Gaussian mass is equally spread in all directions.

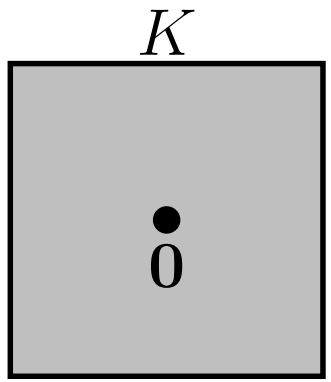


Isotropic position (2)

- If K is in isotropic position, then K maximizes Gaussian measure under volume-preserving rescaling:

Theorem (Bobkov 2011)

Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body and let $s > 0$. If K is in isotropic s-Gaussian position then $\gamma_{n,s}(K) \geq \gamma_{n,s}(A(K))$ for all $A \in \mathbb{R}^{n \times n}$ with $|\det(A)| = 1$.



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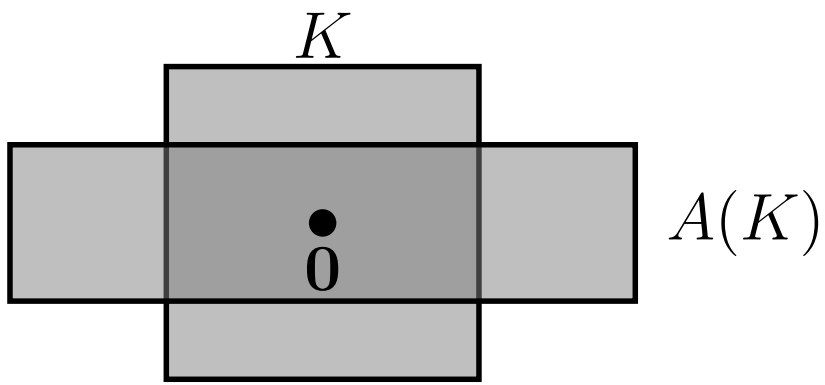

The main technical theorem

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For stable lattice Λ *one has* $\gamma_n(\Theta(\log n) \cdot \mathcal{V}(\Lambda)) \geq \frac{2}{3}$.

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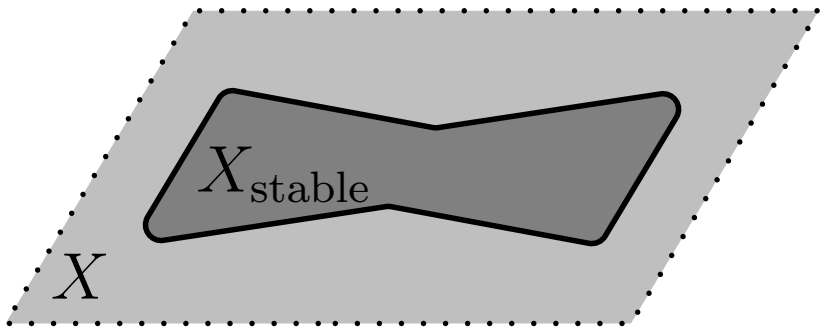

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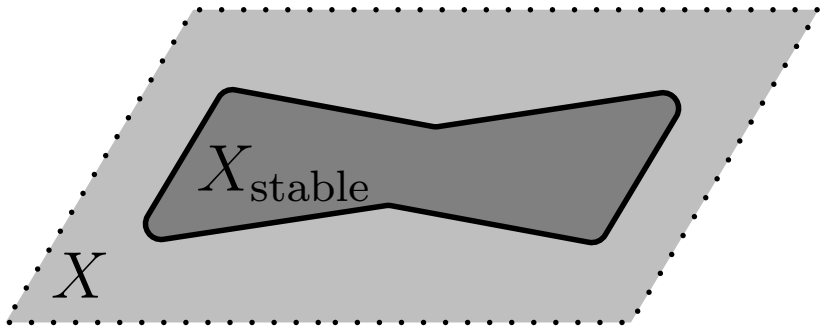

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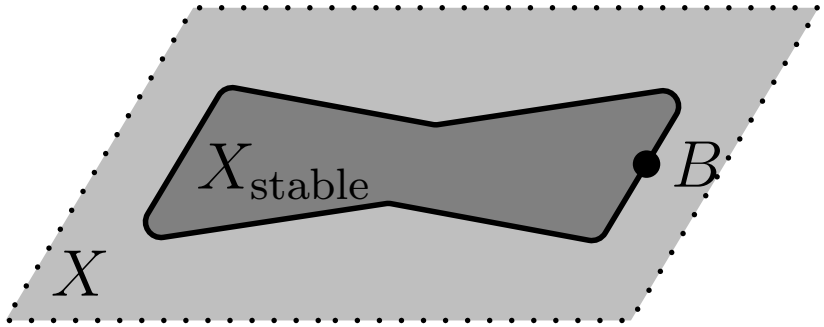

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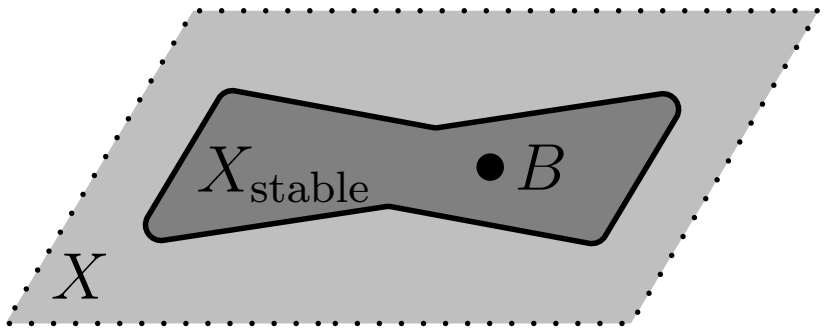

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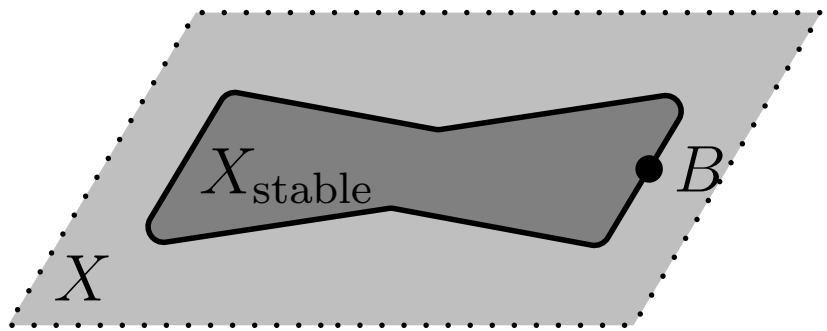

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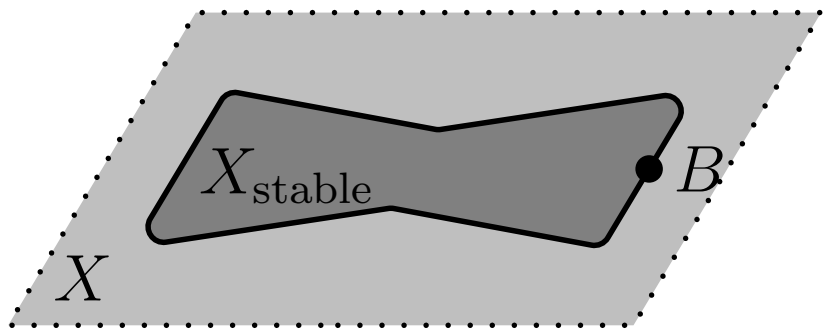

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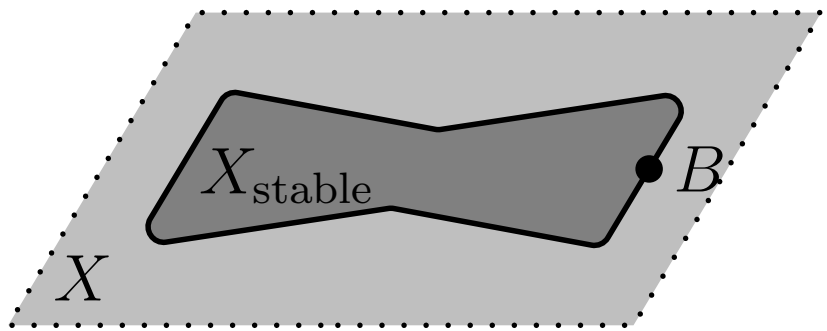
- Λ' also stable. Case incomplete for now... but we made progress..



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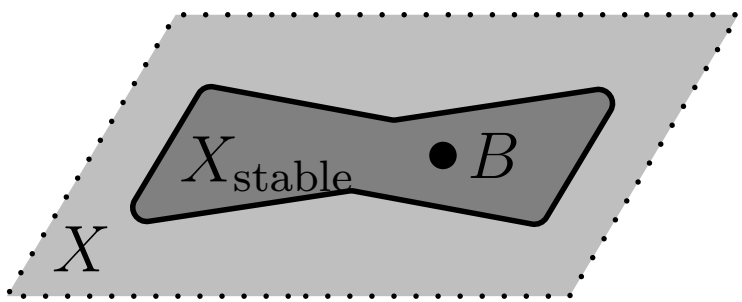

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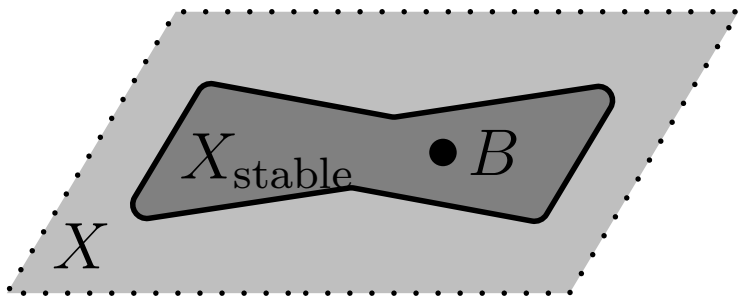

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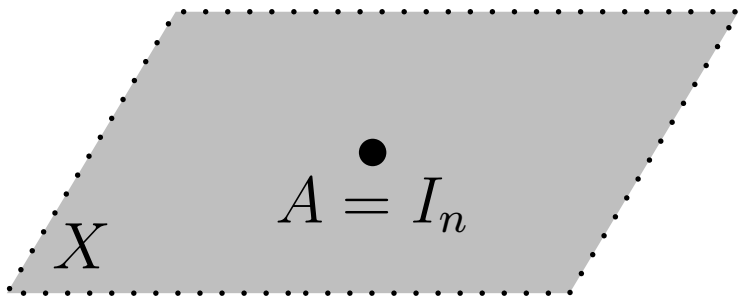

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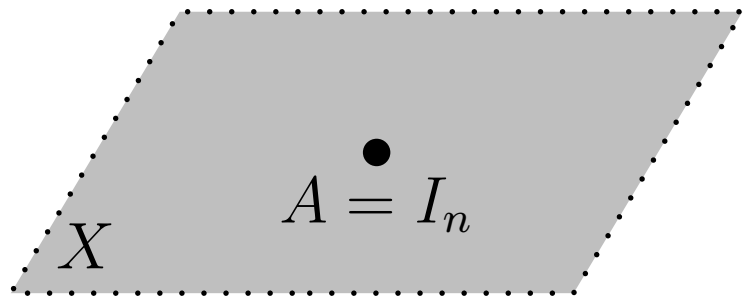

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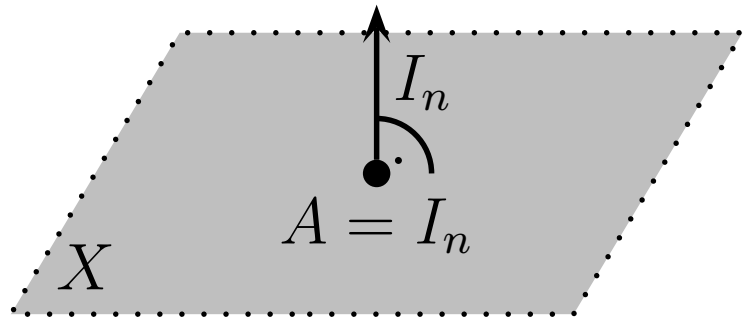

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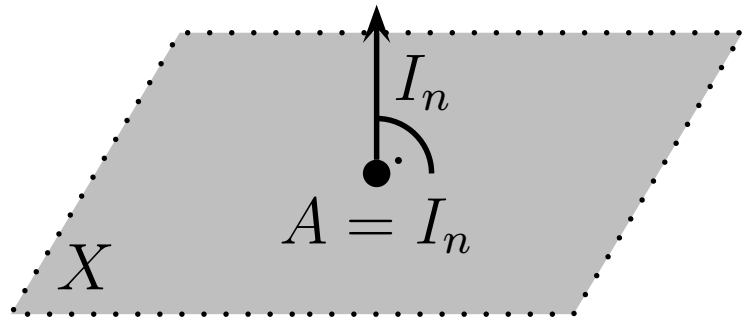

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Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *and lattice* $\Lambda \subseteq \mathbb{R}^n$, *let*

$$G(A) \quad := \quad \frac{1}{|\det(A)|} \int_{\mathcal{V}(A(\Lambda))} f(\|x\|_2^2) dx$$

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Quotient lattices

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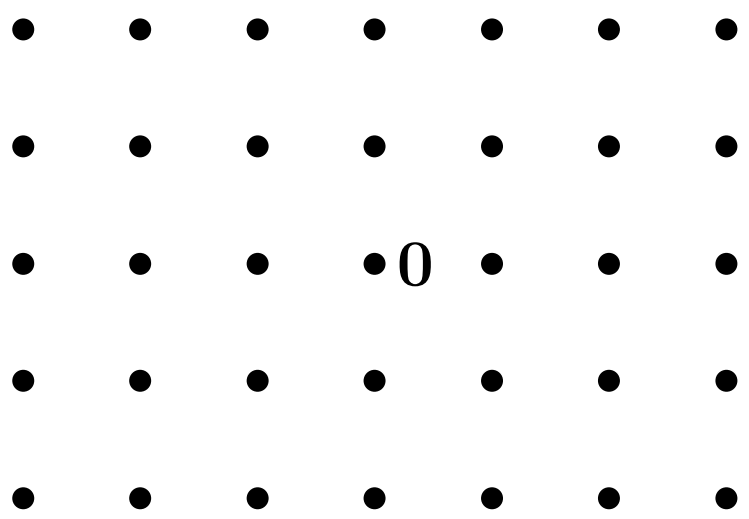
Consider a lattice $\Lambda \subseteq \mathbb{R}^n$ with a primitive sublattice $\Lambda' \subseteq \Lambda$. The **quotient lattice** is $\Lambda/\Lambda' = \Pi_{\mathrm{span}(\Lambda')^\perp}(\Lambda)$.



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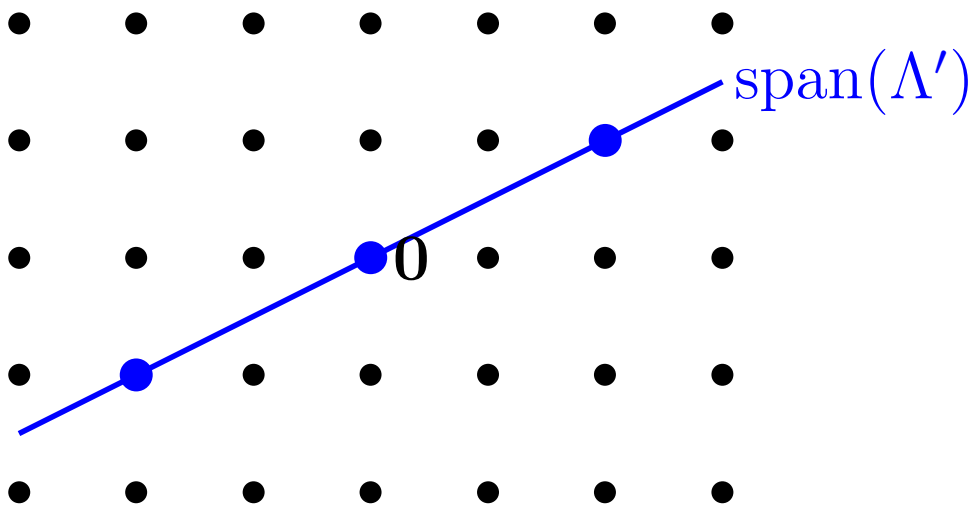

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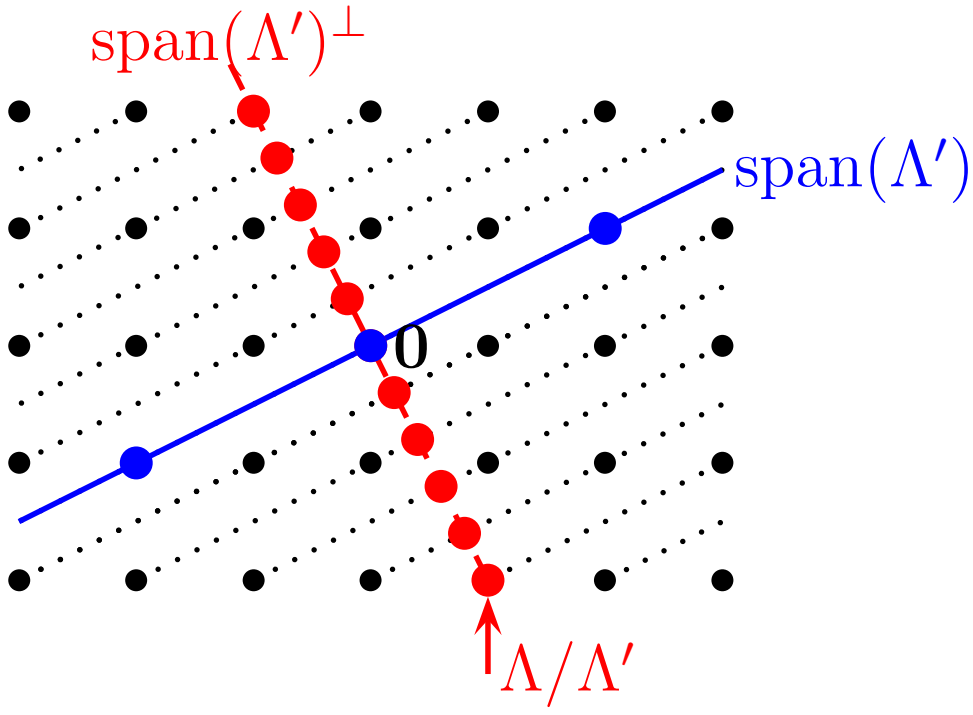

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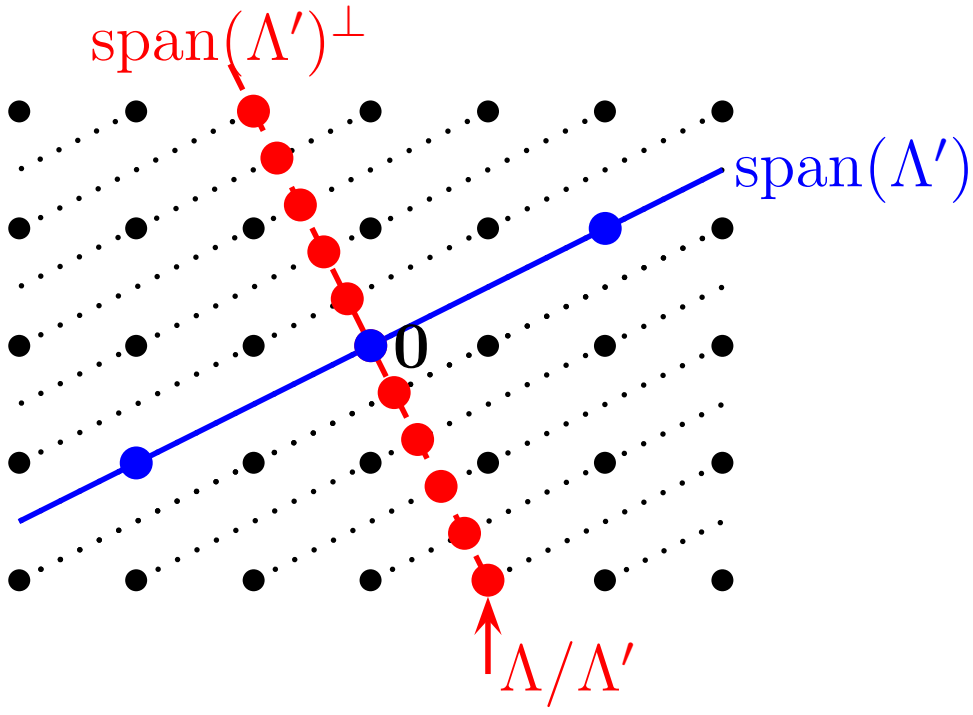


- **Intuition:** We can factor Λ into Λ' and Λ/Λ'

Quotient lattices (2)

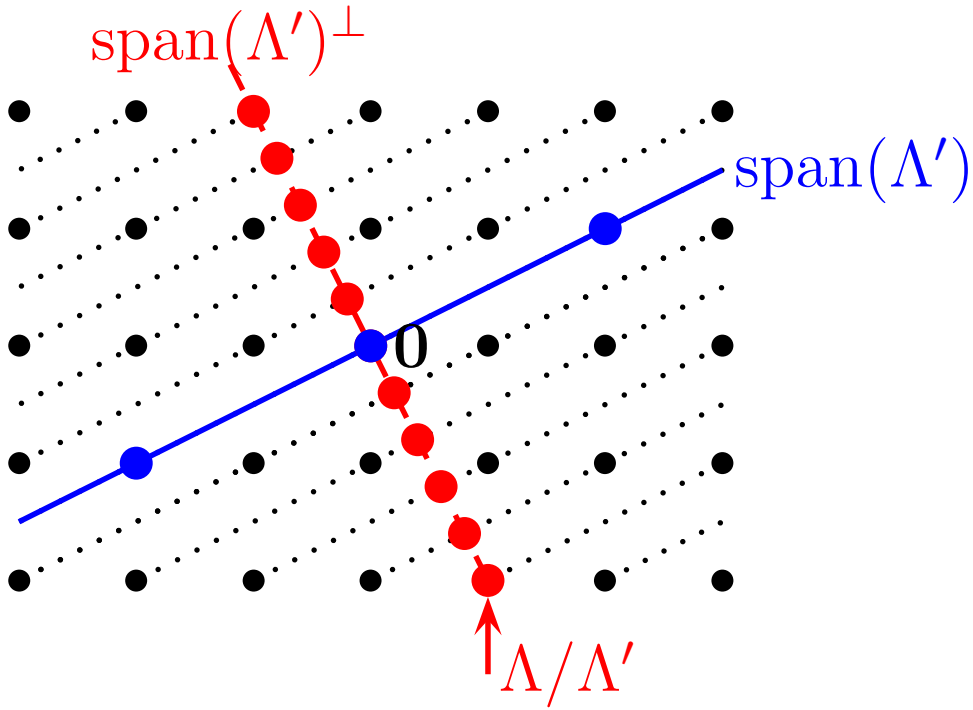
Lemma

For lattice Λ and a primitive sublattice $\Lambda' \subseteq \Lambda$:

(i) $\det(\Lambda) = \det(\Lambda') \cdot \det(\Lambda/\Lambda')$

(ii) *For any* $s > 0$, $\rho_s(\Lambda) \leq \rho_s(\Lambda') \cdot \rho_s(\Lambda/\Lambda')$

(iii) $\gamma_n(\mathcal{V}(\Lambda)) \geq \gamma_n(\mathcal{V}(\Lambda')) \cdot \gamma_n(\mathcal{V}(\Lambda/\Lambda'))$



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&= \exp\left(-\frac{\mathrm{rank}(\Lambda)}{3n}\right) \quad \square
\end{aligned}$$

Proof for non-stable lattices

Corollary

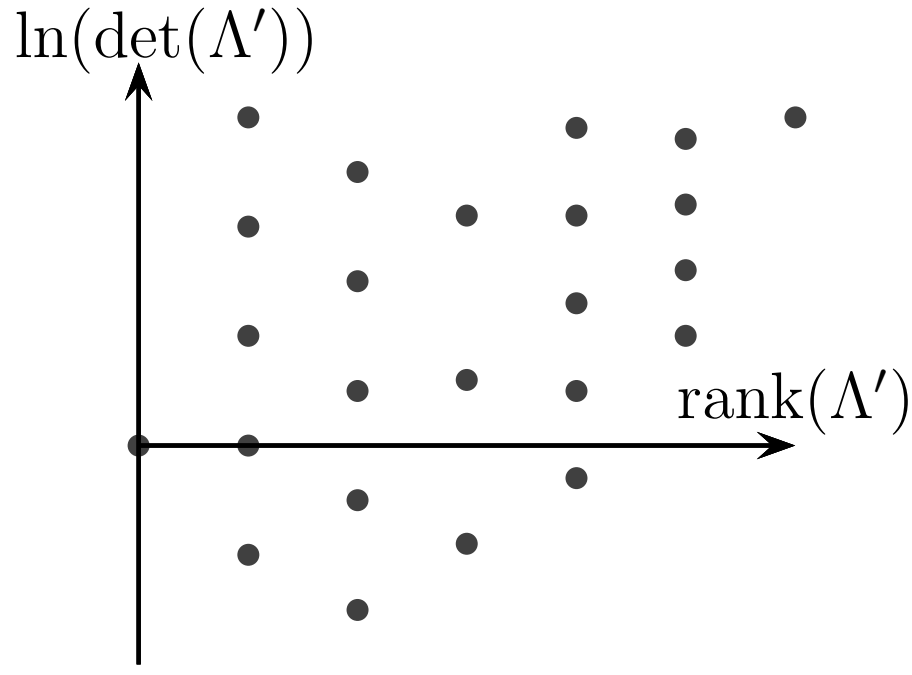
For stable lattice $\Lambda \subseteq \mathbb{R}^n$, $\rho_{1/\Theta(\log n)}(\Lambda) \leq \frac{3}{2}$.

- But what if Λ is not stable (only $\det(\Lambda') \geq 1 \; \forall \Lambda' \subseteq \Lambda$)?

The canonical filtration

- For lattice $\Lambda \subseteq \mathbb{R}^n$, consider **canonical plot**

$$Q := \left\{ \left(\operatorname{rank}(\Lambda'), \ln(\det(\Lambda')) \right) \mid \text{sublattice } \Lambda' \subseteq \Lambda \right\}$$



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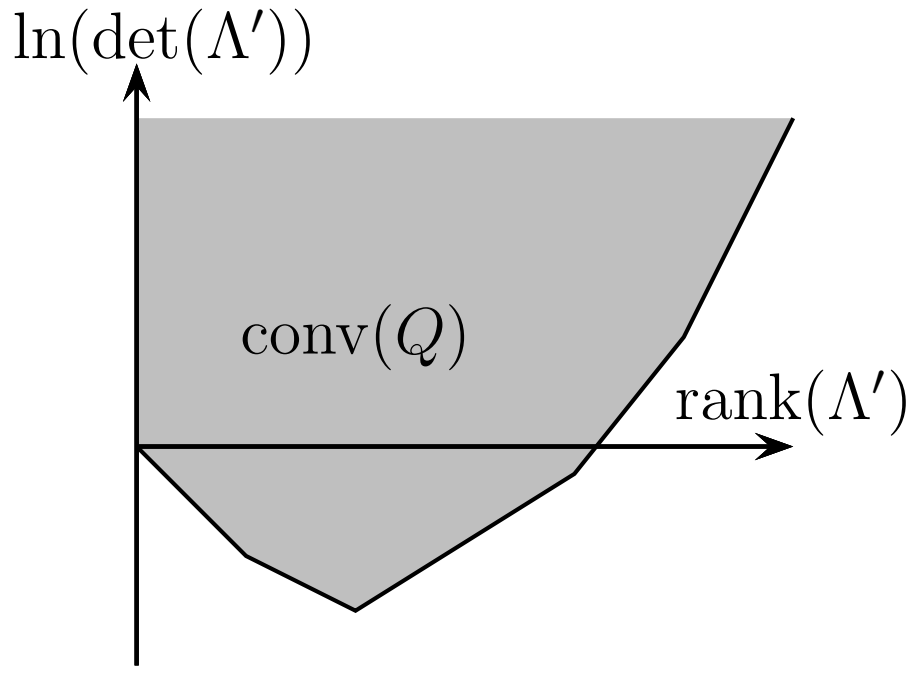

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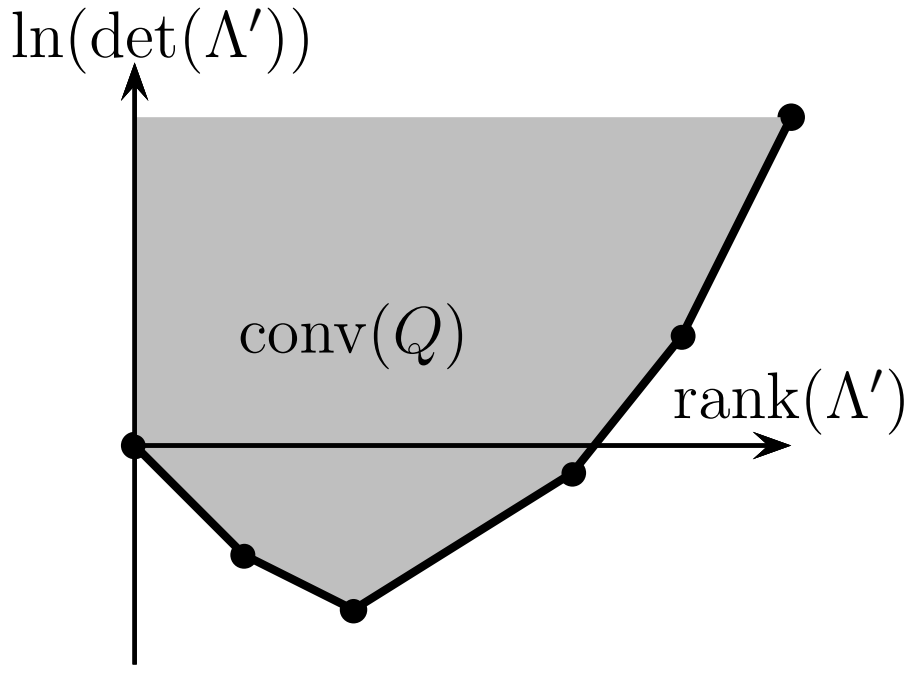
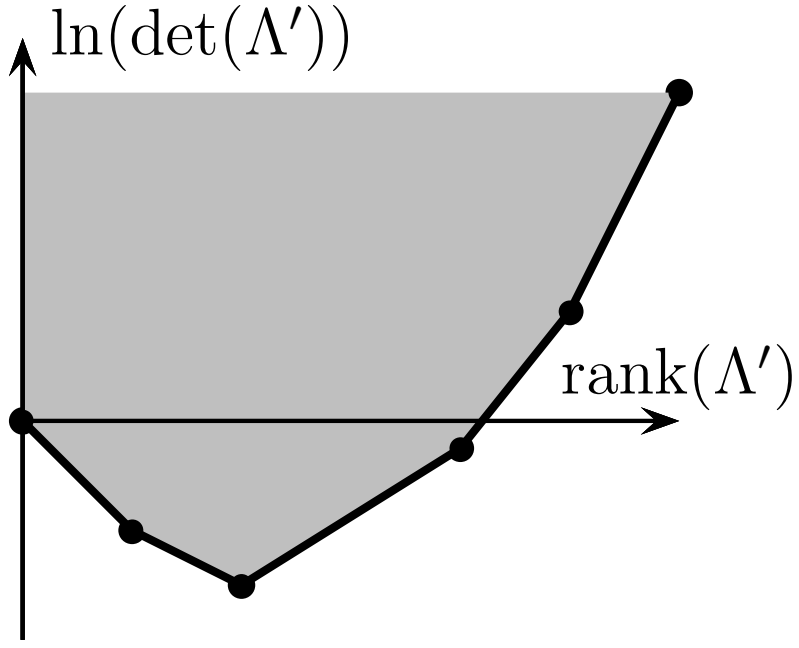
- Lower envelope of $\operatorname{conv}(Q)$ is called **canonical polygon**



The canonical filtration (2)



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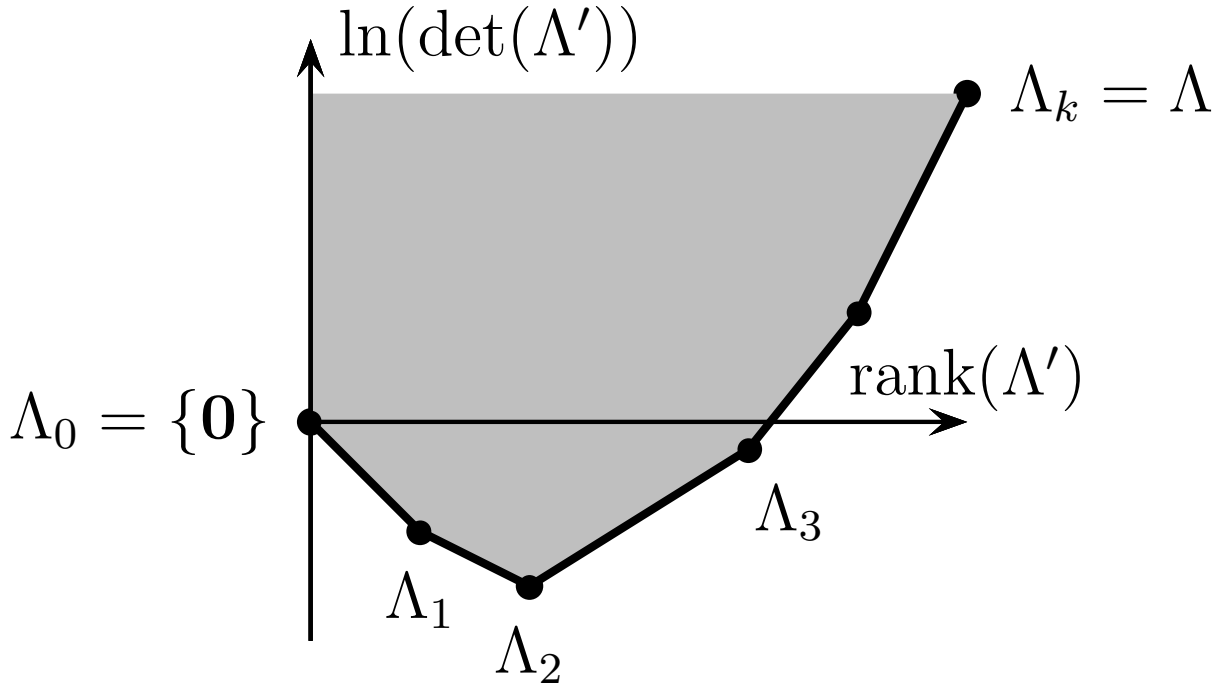

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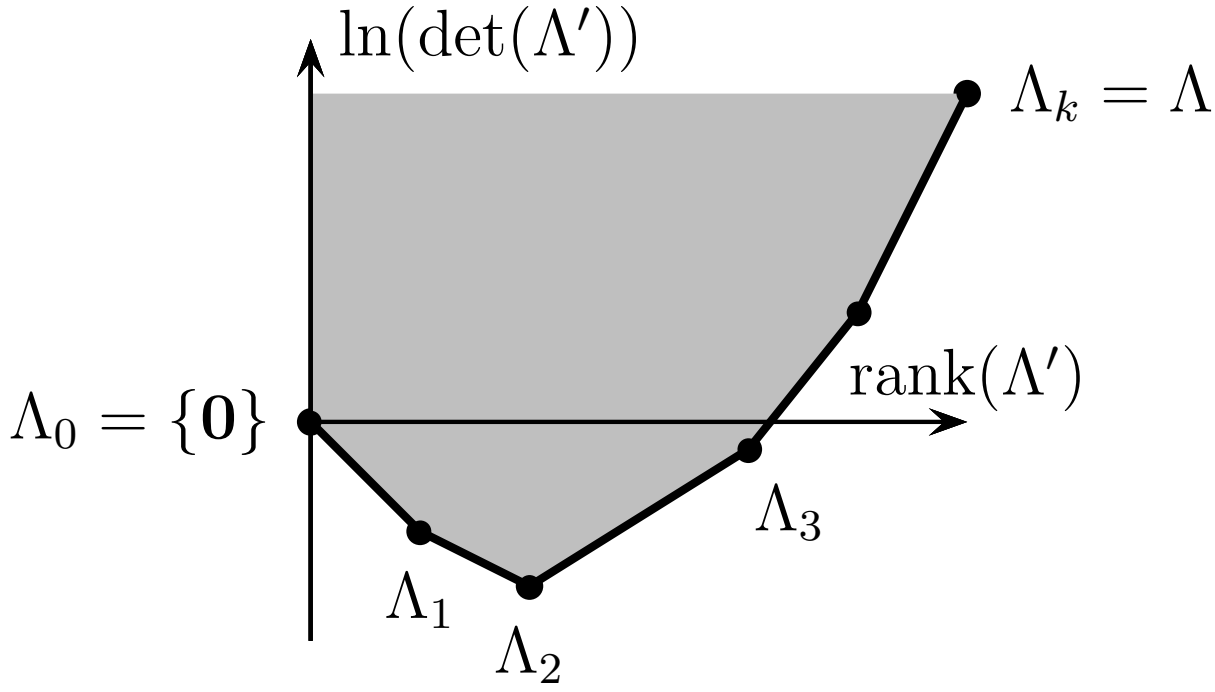

Theorem (Canonical filtration)

(a) *The vertices of the canonical plot form a chain*

$$\{\mathbf{0}\} = \Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_k = \Lambda.$$

(b) $r_i := \det(\Lambda_i/\Lambda_{i-1})^{1/rank(\Lambda_i/\Lambda_{i-1})}$ *satisfy* $r_1 < \ldots < r_k$

(c) *Each* $\frac{1}{r_i}(\Lambda_i/\Lambda_{i-1})$ *is stable.*

Proof of Reverse Minkowski Theorem

Reverse Minkowski Theorem (Regev, Stephens-Da.)

Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice that satisfies $\det(\Lambda') \geq 1$ for all sublattices $\Lambda' \subseteq \Lambda$. Then for $t = \Theta(\log n)$, $\rho_{1/t}(\Lambda) \leq \frac{3}{2}$.

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□

End of part 2

Open problem 1

Is it true that already for $t = \Theta(\sqrt{\log n})$ and any stable lattice Λ one has $\rho_{1/t}(\Lambda) \leq \frac{3}{2}$.

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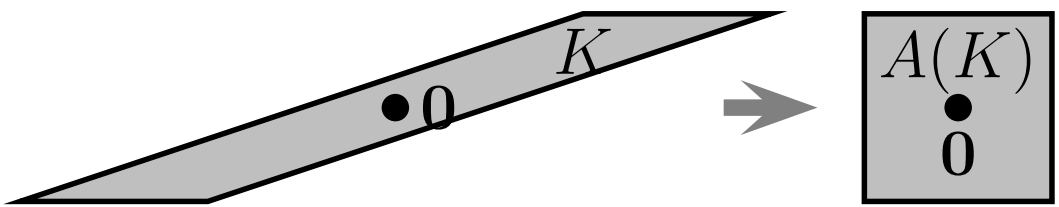

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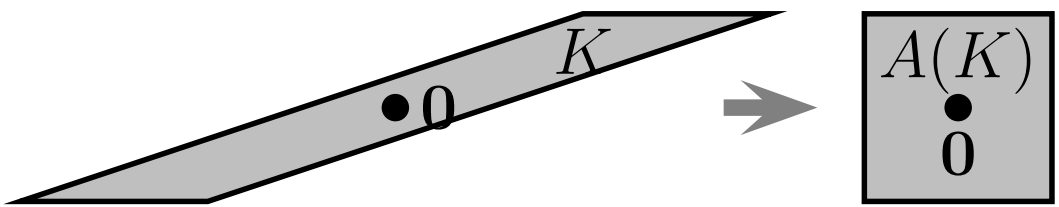

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Thanks for your attention!

Part 3:
The Subspace Flatness Conjecture and Faster Integer Programming

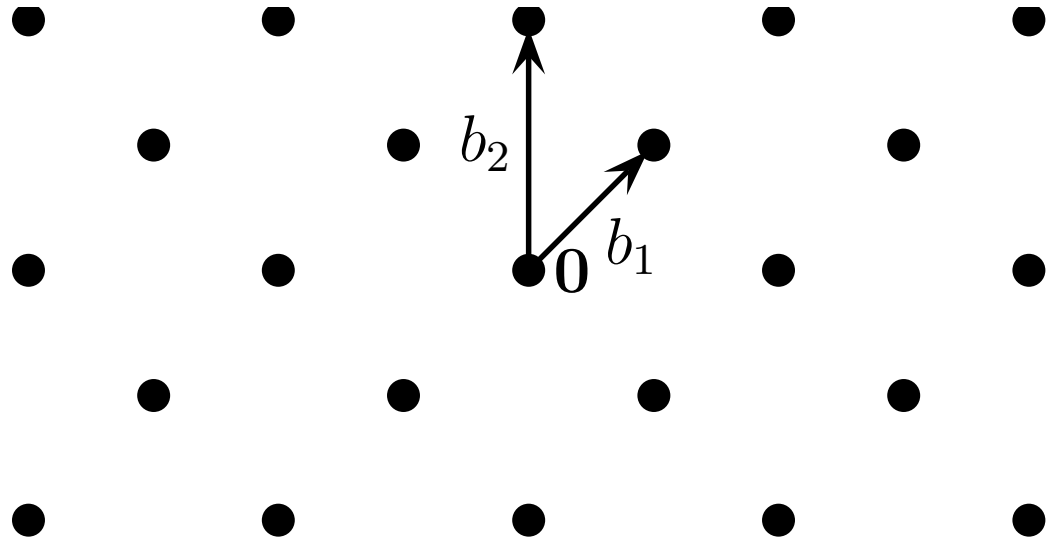
Thomas Rothvoss

Joint work with Victor Reis

UNIVERSITY *of*
WASHINGTON

Covering radius

- Consider a **lattice** $\Lambda = \{Bx : x \in \mathbb{Z}^k\}$ with $B \in \mathbb{R}^{n \times k}$



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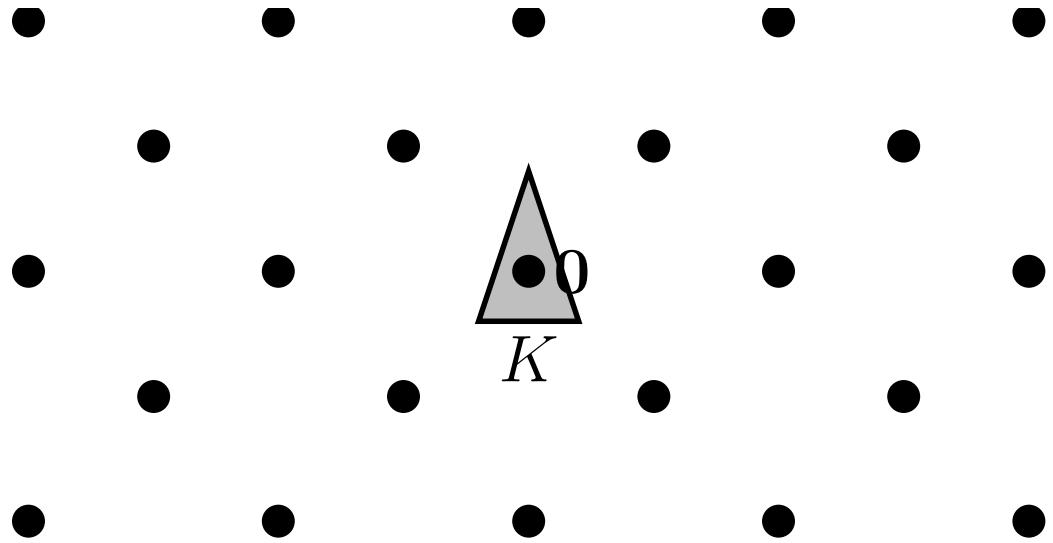

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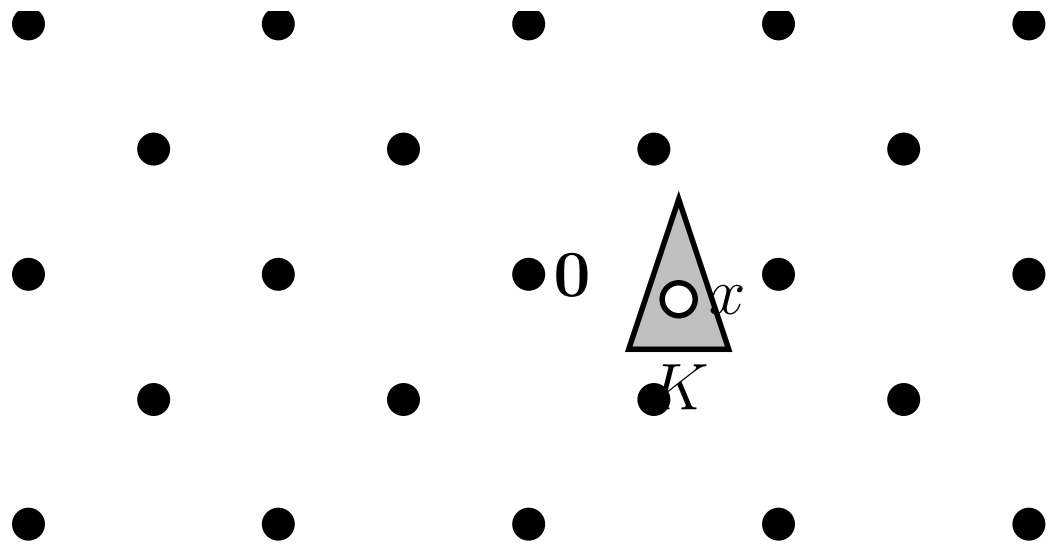

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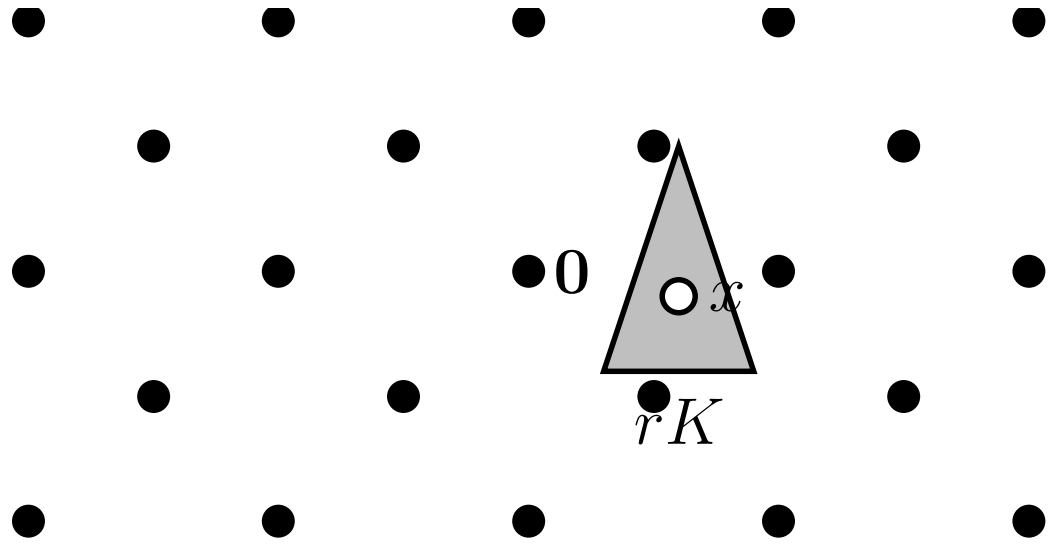

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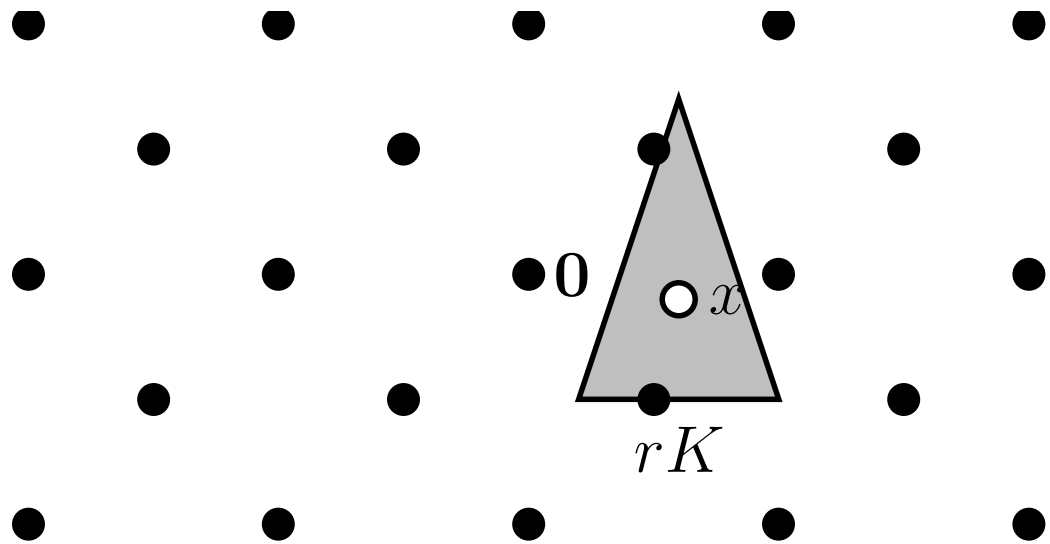
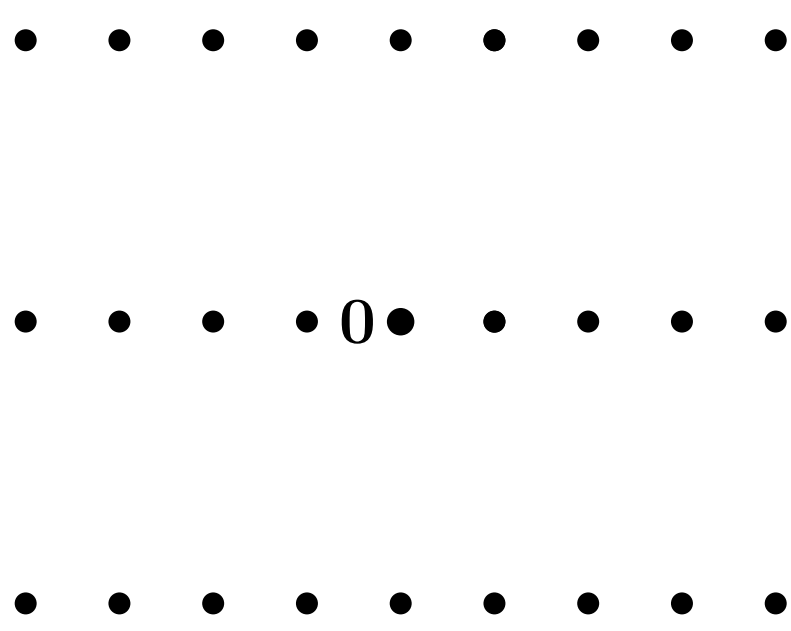


Lower bounds on the covering radius



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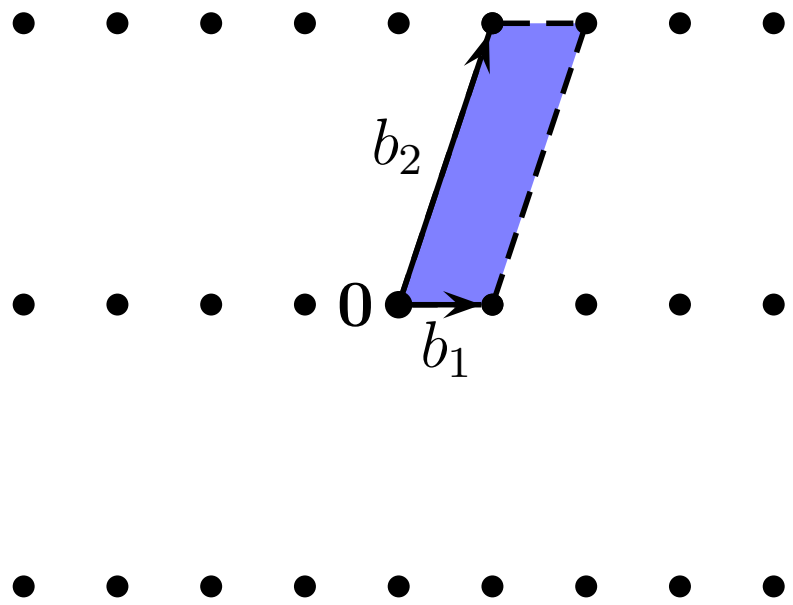

- **Recall:** $\det(\Lambda)$ =volume of fundamental parallelepiped

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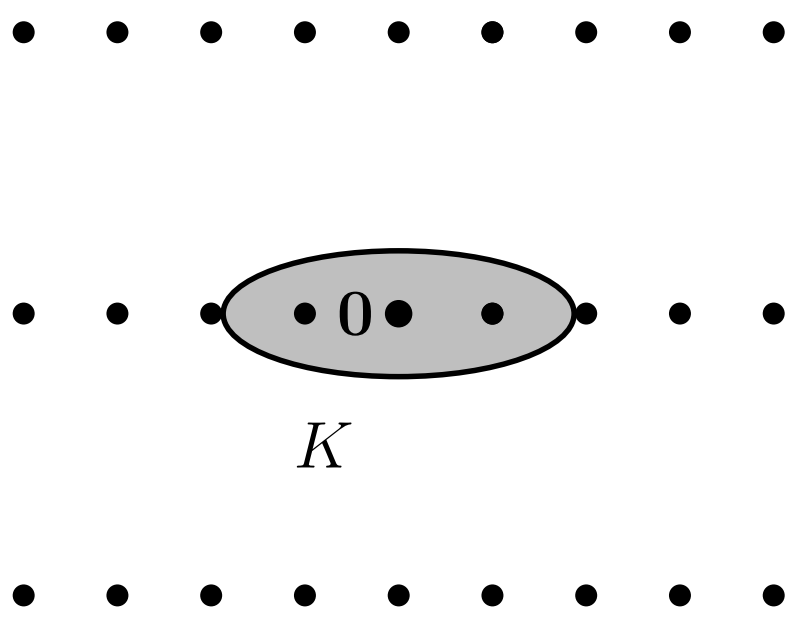

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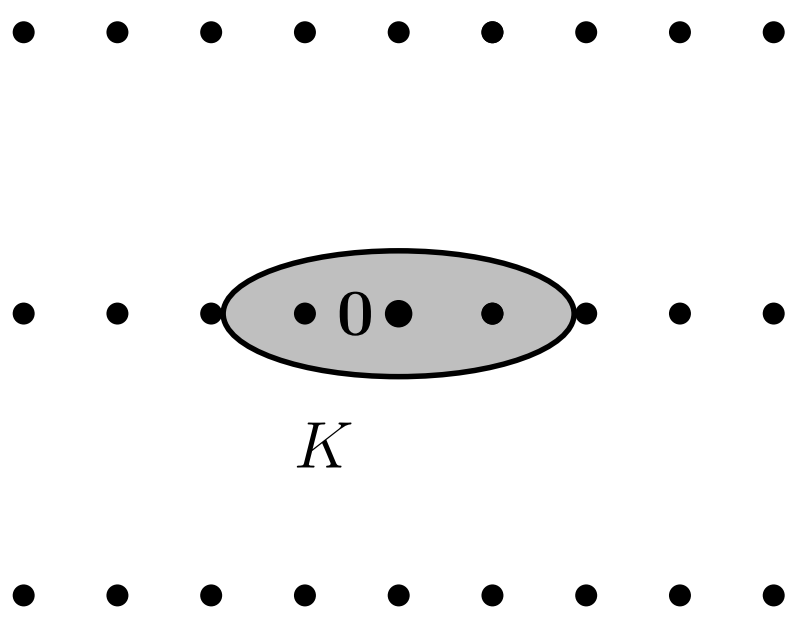

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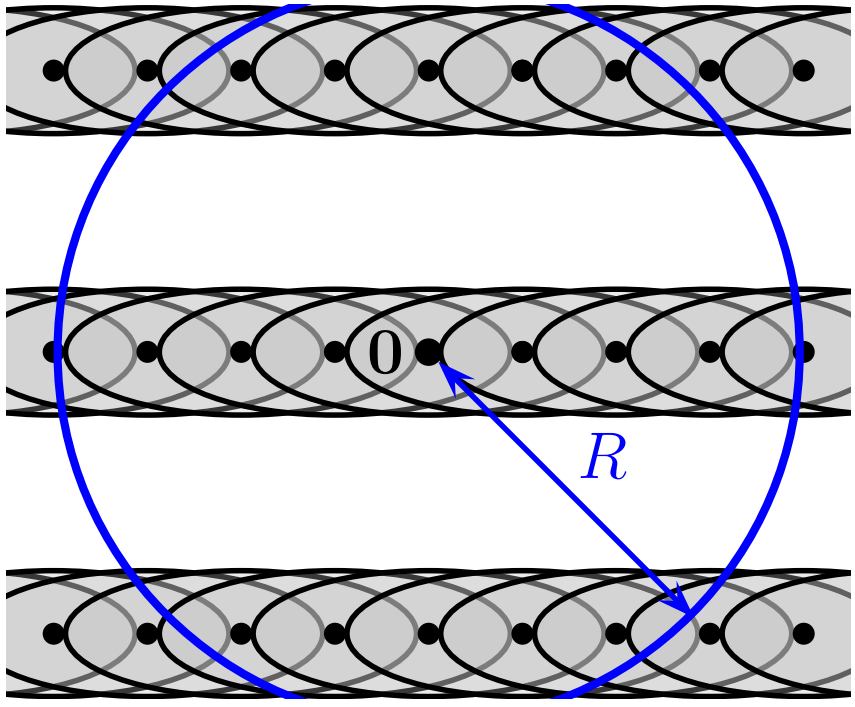

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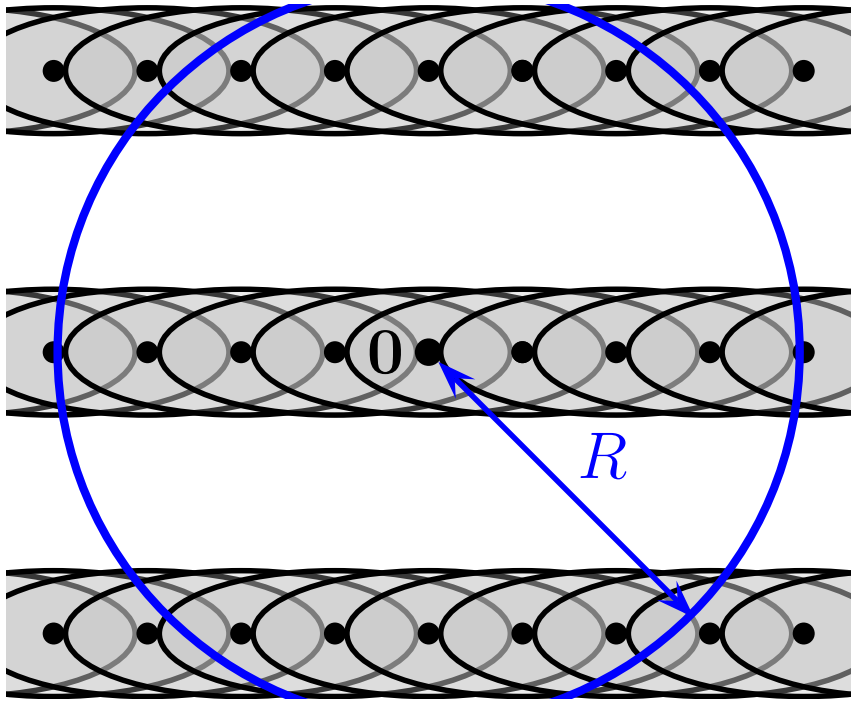

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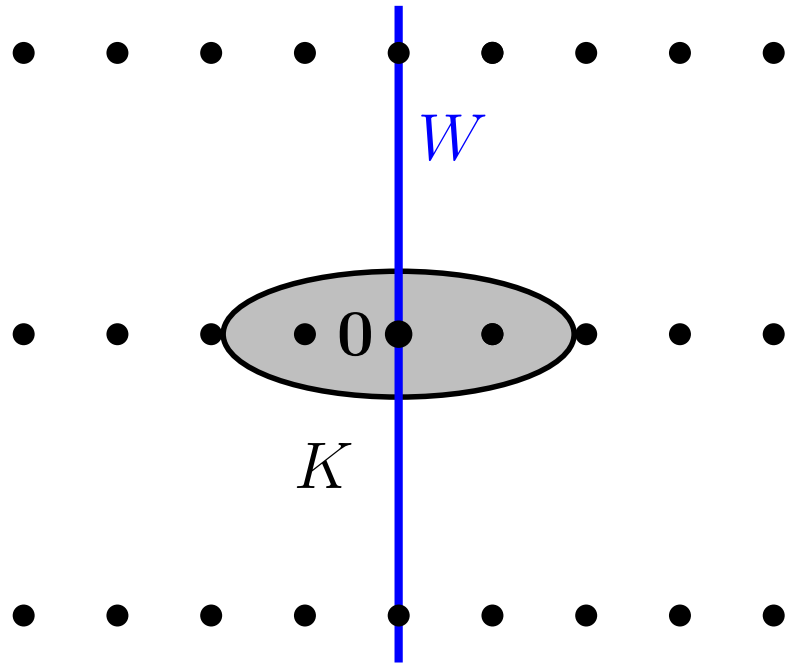

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- For any subspace $\mu(\Lambda, K) \geq \mu(\Pi_W(\Lambda), \Pi_W(K))$

Kannan, Lovász (1988)

- Consider the best volume-based lower bound

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Subspace Flatness Conjecture (Dadush 2012)

For full rank lattice $\Lambda \subseteq \mathbb{R}^n$ and convex body $K \subseteq \mathbb{R}^n$ one has

$$\mu(\Lambda, K) \leq O(\log(n)) \cdot \mu_{KL}(\Lambda, K)$$

- Dadush shows consequences for solving IPs.

Main results

Theorem (Reis, R.'23)

For full rank lattice $\Lambda \subseteq \mathbb{R}^n$ and convex body $K \subseteq \mathbb{R}^n$ one has

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Theorem (Reis, R.'23)

For convex body $K \subseteq \mathbb{R}^n$ one can find a point in $K \cap \mathbb{Z}^n$ in time $(\log n)^{O(n)}$.

Previously best known:

- $2^{O(n^2)}$ [Lenstra '83]
- $n^{O(n)}$ [Kannan '87]
- $2^{O(n)} n^n$ [Dadush '12], [Dadush, Eisenbrand, R. '22]

Main results (2)

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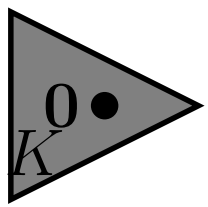

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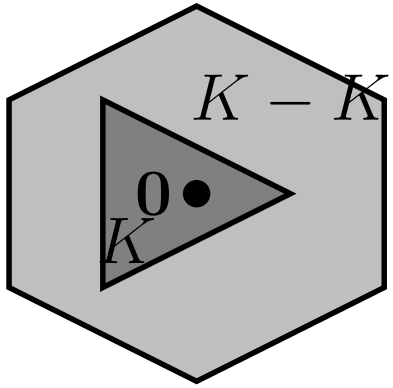

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Flatness constant in dimension n is at most $O(n \log^3(n))$.

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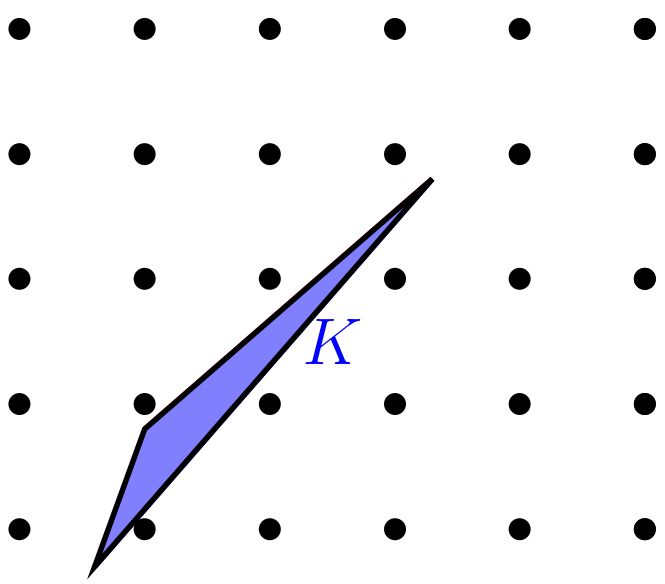

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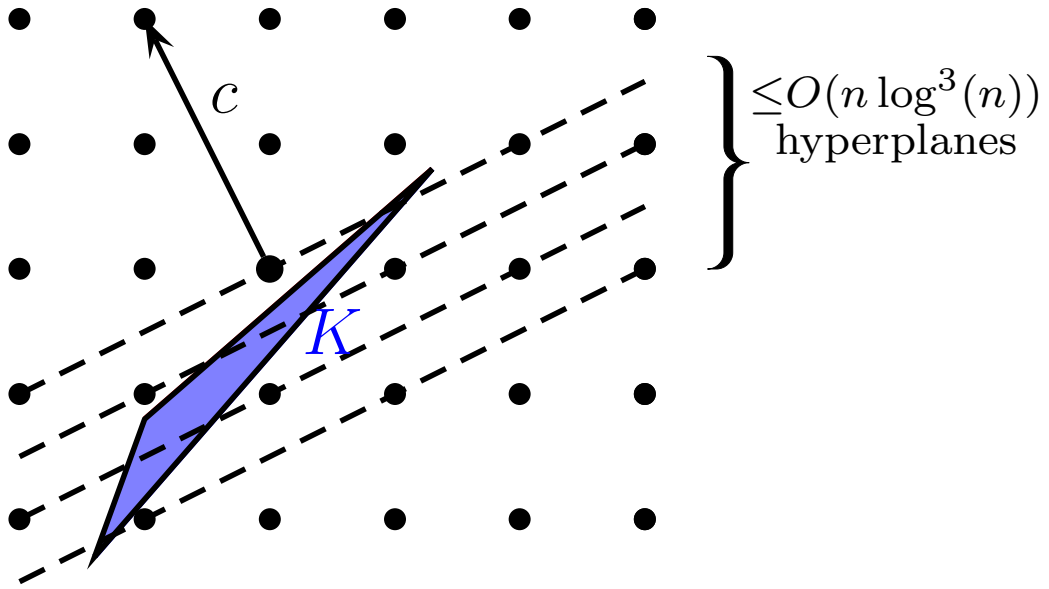
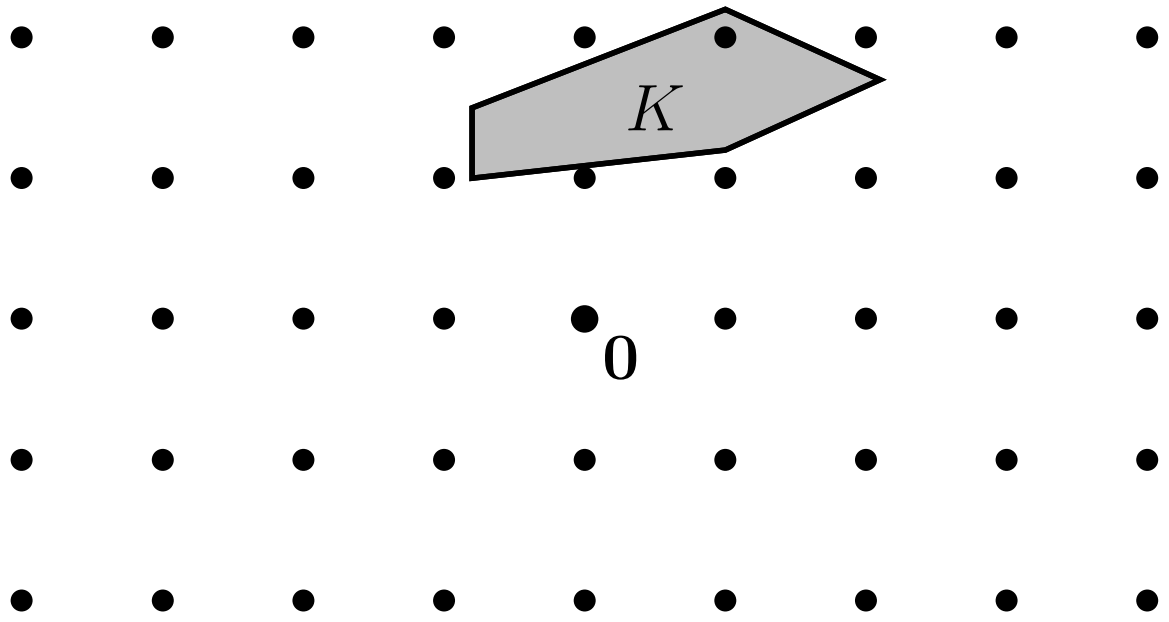


Dadush's algorithm



Input: $K \subseteq \mathbb{R}^n$, lattice Λ

Output: Point in $K \cap \Lambda$

(1) Shrink K so that $\mu(\Lambda, K) \geq 1$

(2) Find subspace W attaining $\mu_{KL}(\Lambda, K)$ approx.

(3) Compute $X := \Pi_W(K) \cap \Pi_W(\Lambda)$

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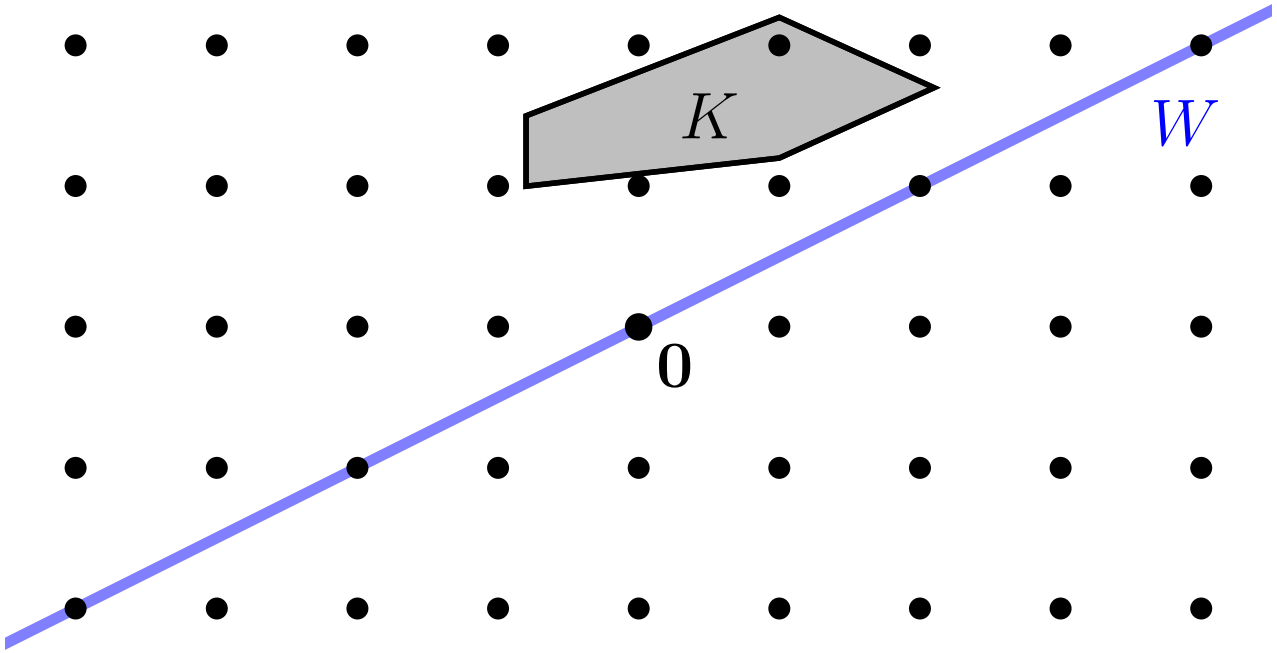

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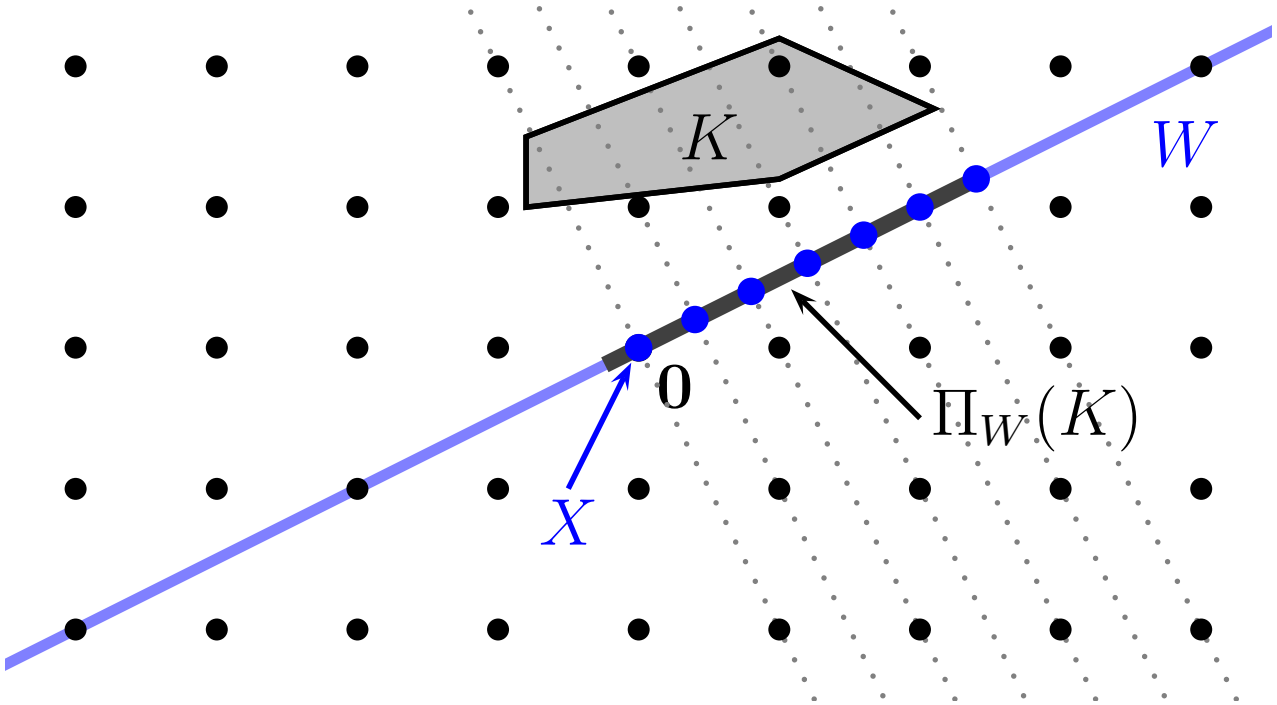

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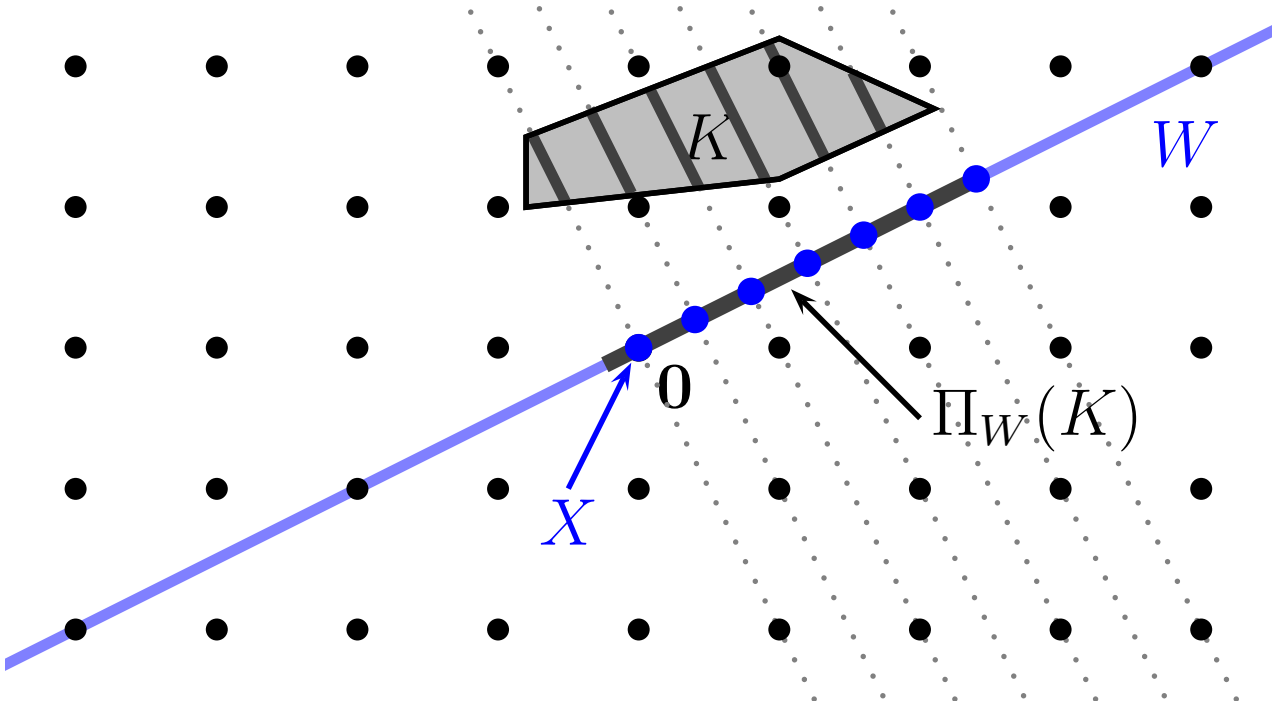

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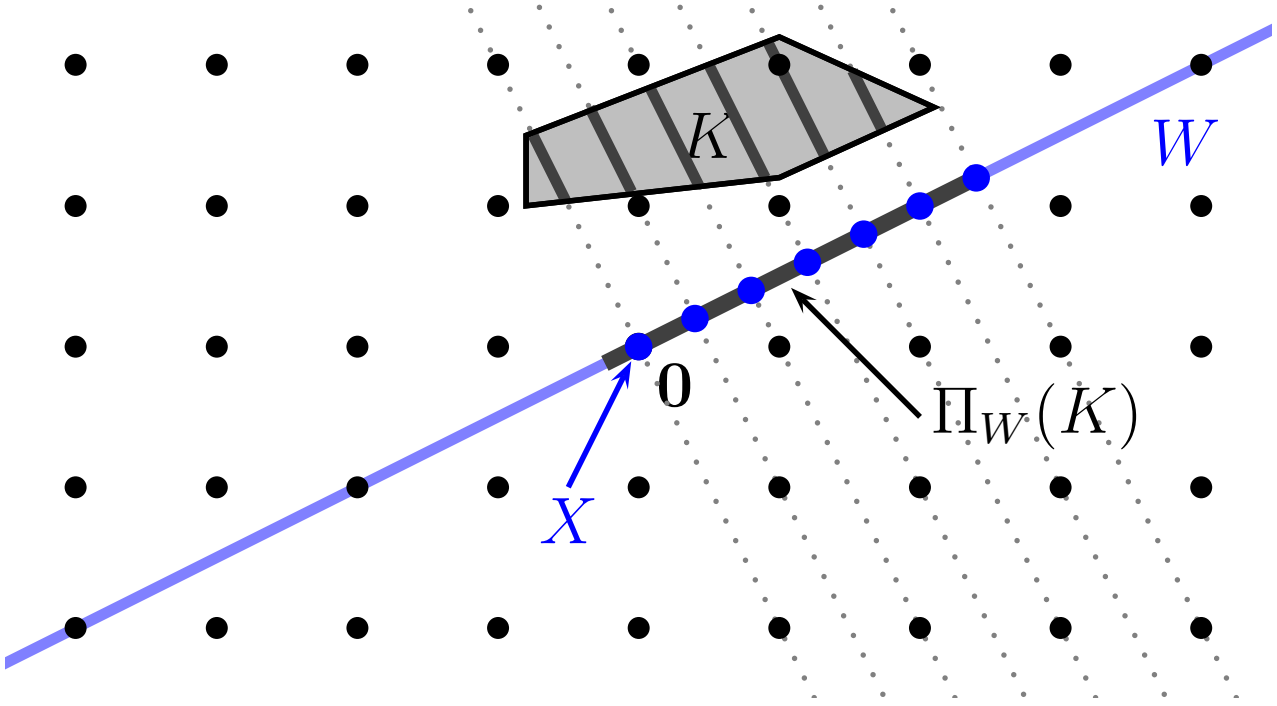
Dadush's algorithm (analysis)



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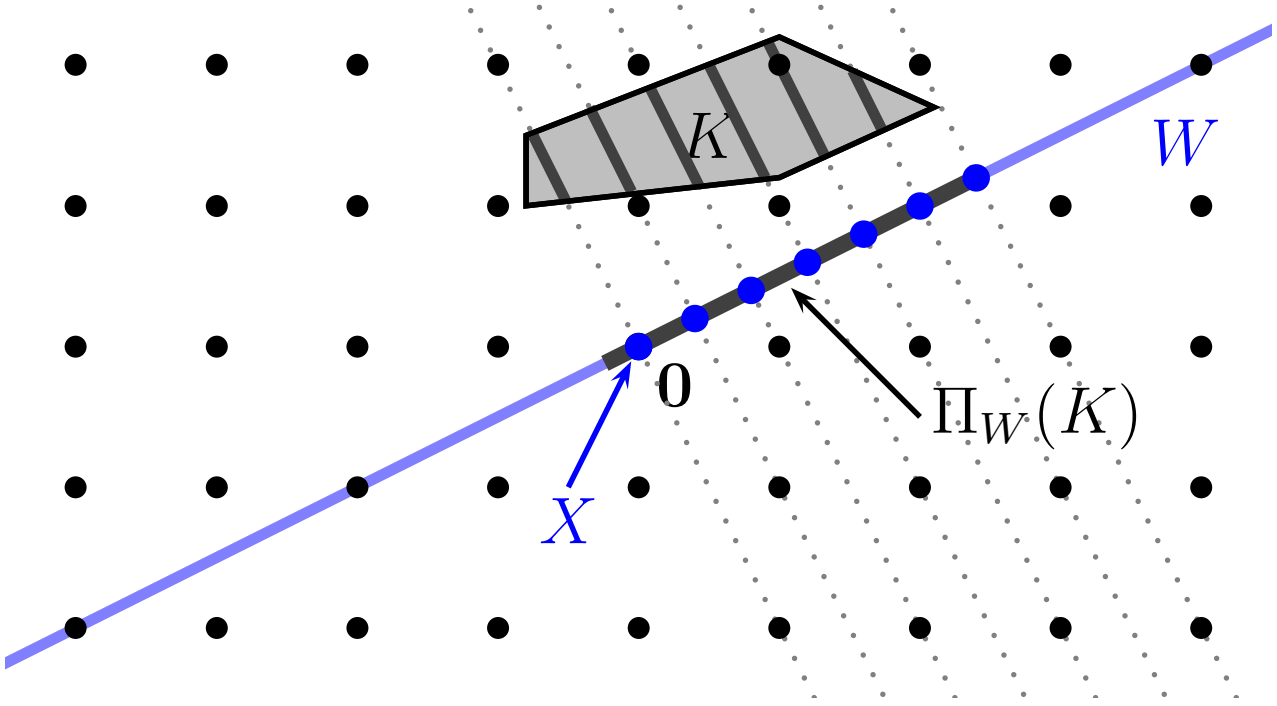

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- Can find W in time $2^{O(n)}$ [Dadush '19]

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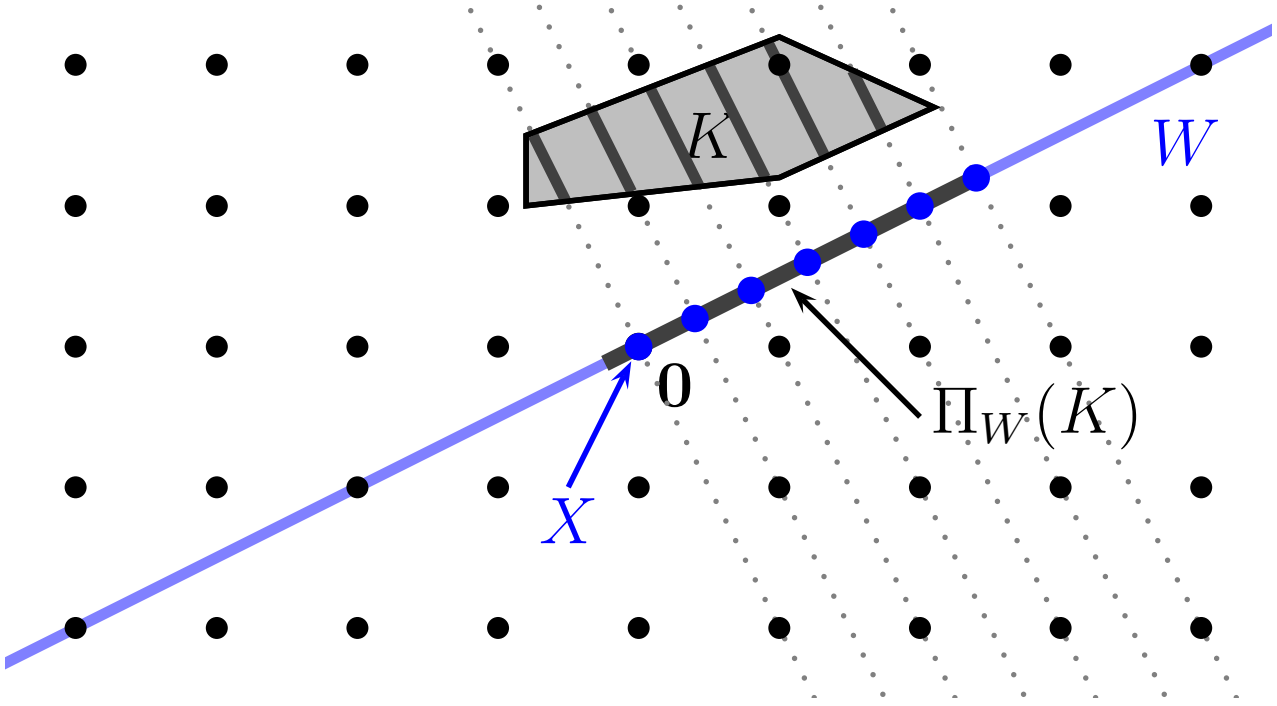

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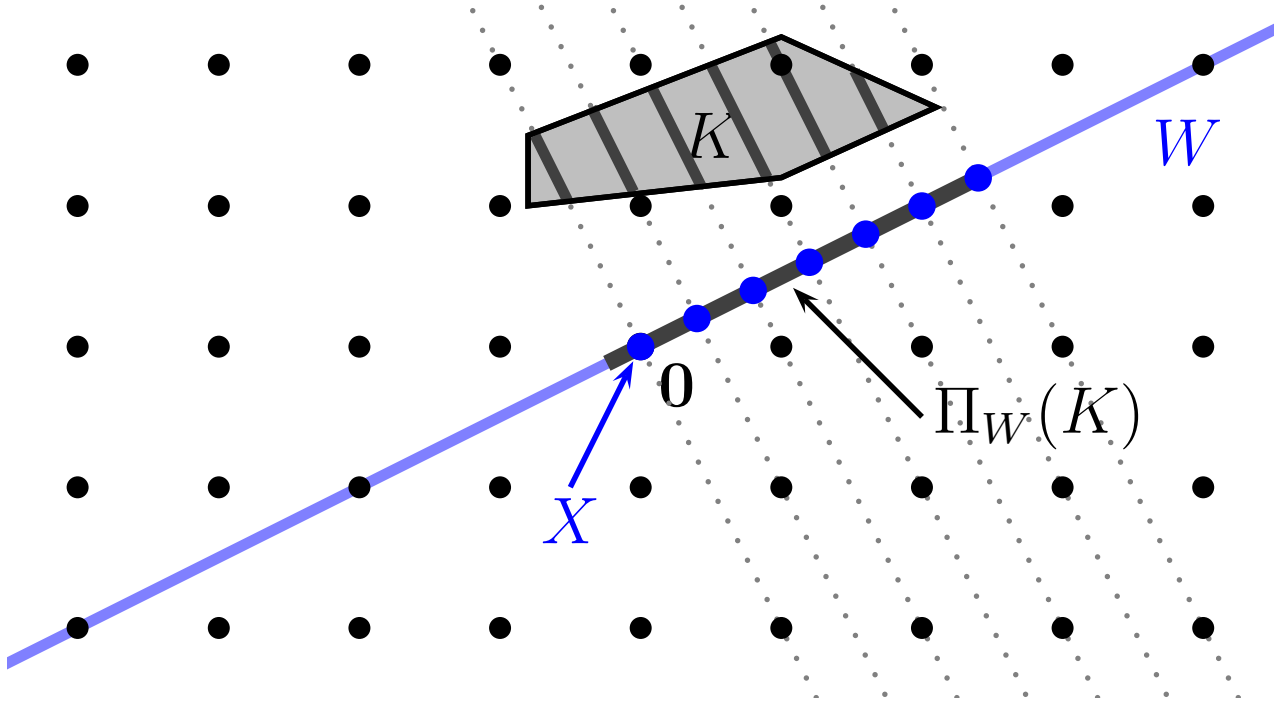

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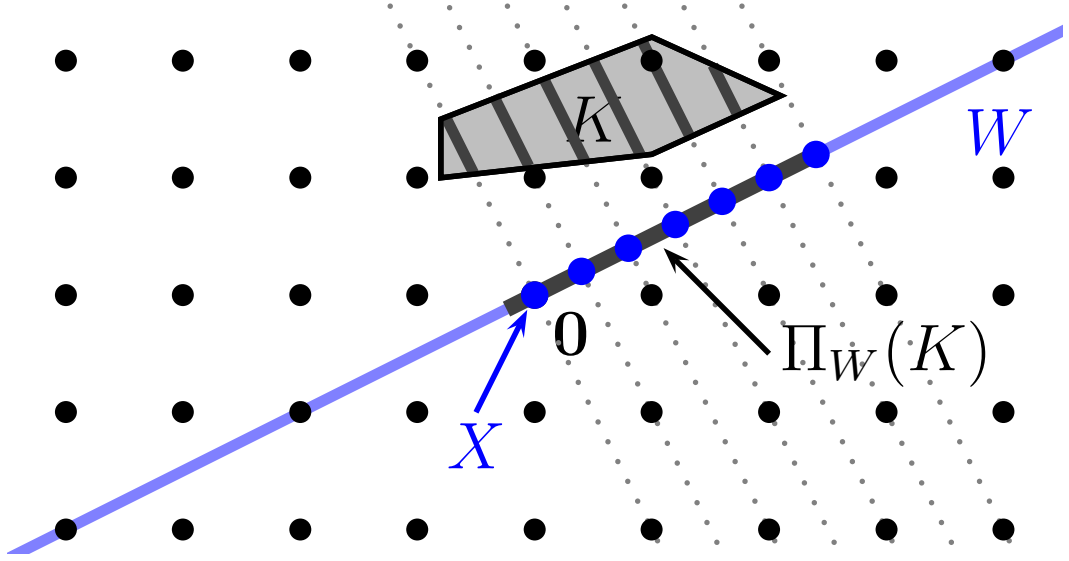
- Can **enumerate** those points in time $O(\log^4 n)^d$
- Recursion

$$T(n) = 2^{O(n)} + \text{polylog}(n)^d \cdot T(n-d) \quad \Rightarrow \quad T(n) \leq \text{polylog}(n)^n$$

Dadush's algorithm (analysis) (2)



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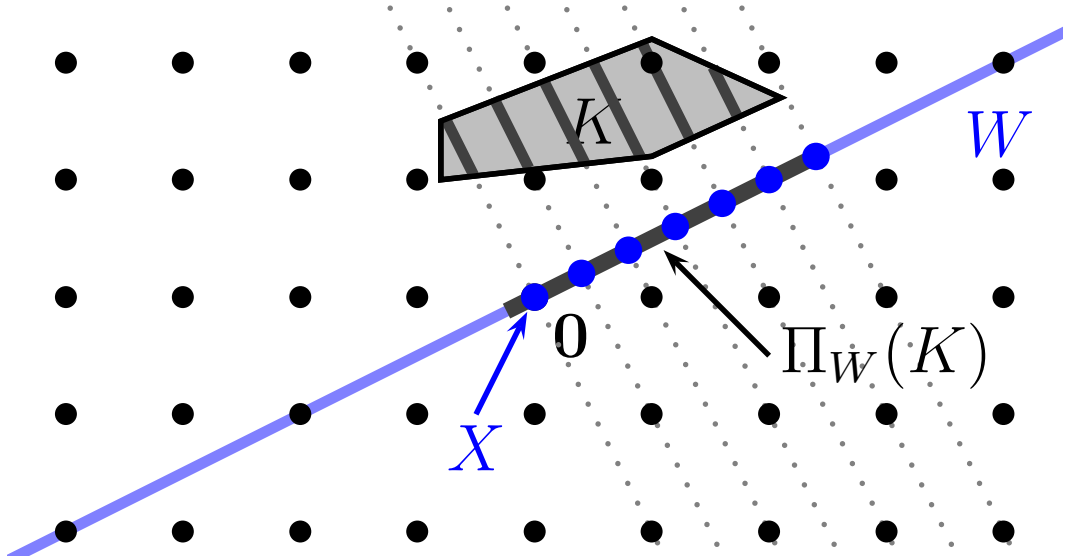

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- Then $(\frac{\mathrm{Vol}_d(\Pi_W(K))}{\det(\Pi_W(\Lambda))})^{1/d} \leq \frac{O(\log^4(n))}{\mu(\Lambda,K)}$
- In any case $(*) \leq O(\log n)^{4d}$

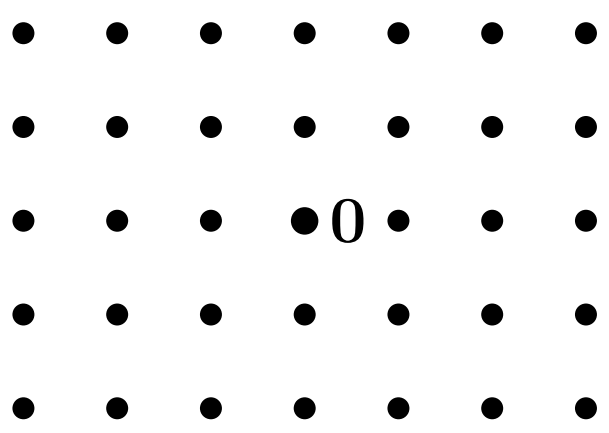
Sketch of Main proof

$$\mu(\Lambda, K) \leq O(\log^3 n) \cdot \mu_{KL}(\Lambda, K)$$

Recap: Stable lattices

Definition

A lattice Λ is called **stable** if $\det(\Lambda) = 1$ and $\det(\Lambda') \geq 1$ for all sublattices $\Lambda' \subseteq \Lambda$.



Example: $\mathbb{Z}^n$ is stable

ℓ-position

- For a symmetric convex body $K \subseteq \mathbb{R}^n$,

$$\ell_K = \mathop{\mathbb{E}}_{x \sim N(0, I_n)} [\|x\|_K^2]^{1/2}$$

- **Intuitively:** $\ell_K =$ "average thinness" of K

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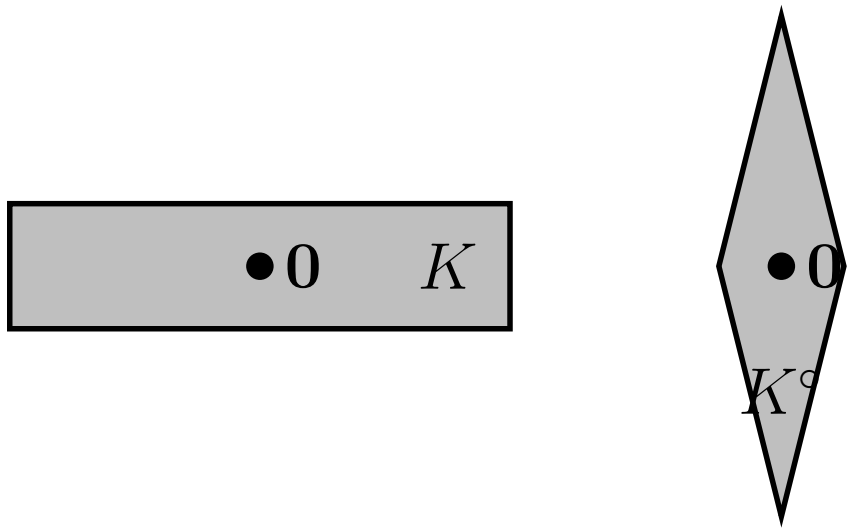
- **Intuitively:** ℓ_K = "average thinness" of K
- **Polar** is $K^\circ = \{x \in \mathbb{R}^n \mid \langle x, y \rangle \leq 1 \ \forall y \in K\}$
- Possible that ℓ_K and ℓ_{K° arbitrarily large



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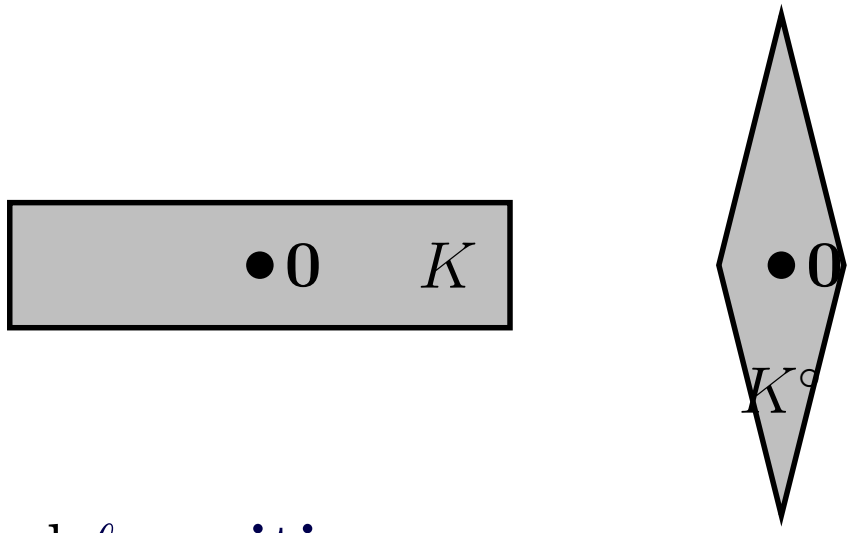
For any symmetric convex body $K \subseteq \mathbb{R}^n$, there is an invertible linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ so that $\ell_{T(K)} \cdot \ell_{(T(K))^\circ} \leq O(n \log n)$.



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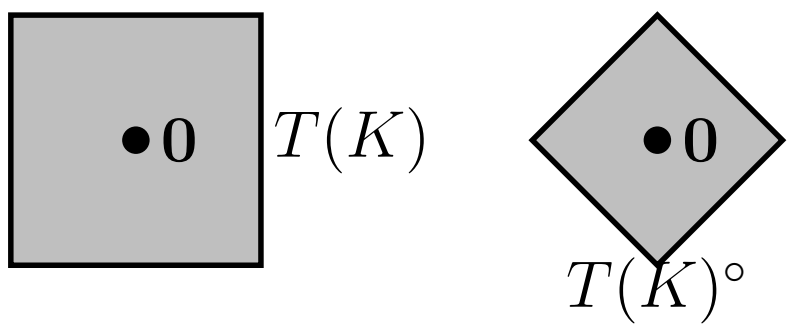

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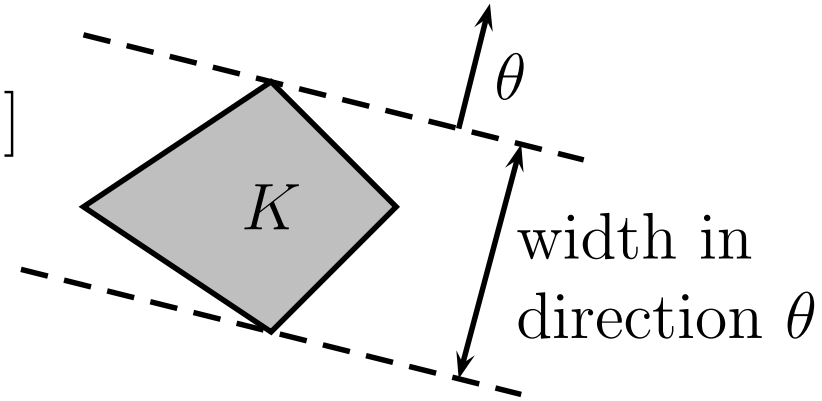
Urysohn Inequality

- The **mean width** of a convex body K is

$$w(K) := \mathop{\mathbb{E}}_{\theta \sim S^{n-1}} [\max\{\langle \theta, x - y \rangle : x, y \in K\}]$$



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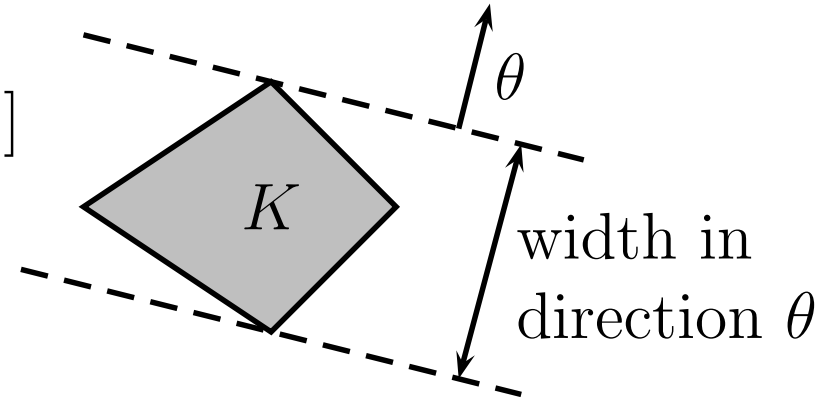

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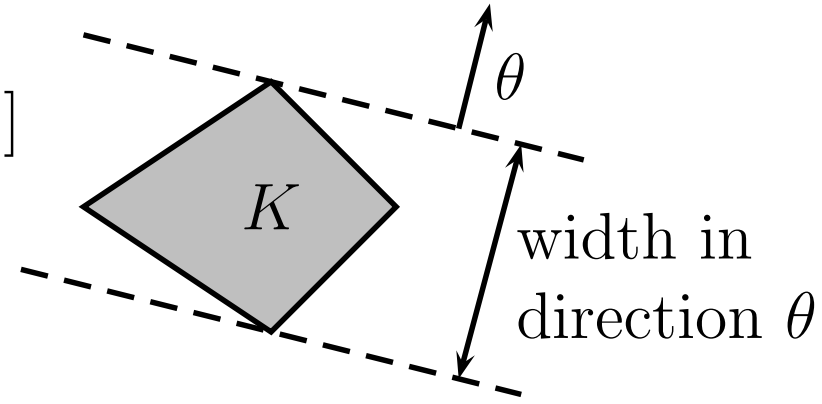

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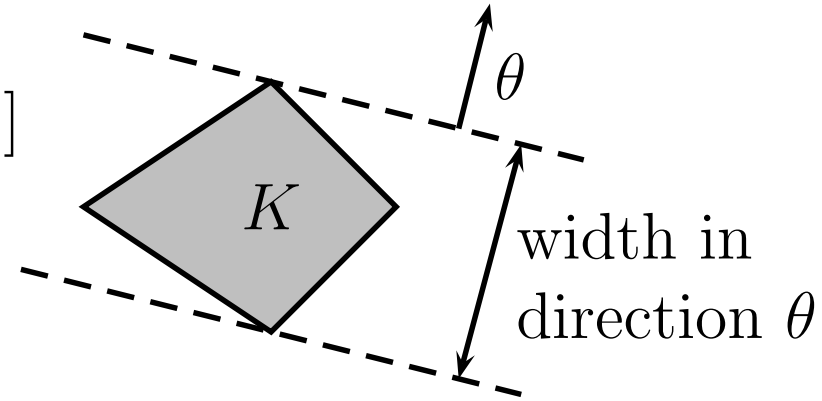

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- For symmetric convex body K, $\ell_{K^\circ} \asymp \sqrt{n} \cdot w(K)$ and so $\mathrm{Vol}_n(K)^{1/n} \lesssim \frac{\ell_{K^\circ}}{n}$

Main theorem for stable + ℓ-position

Proposition

Let K be a symmetric convex body and let Λ be **stable lattice**.

(a) If K in **ℓ-position** then $\mu(\Lambda, K) \leq O(\log^2 n) \cdot \mu_{KL}(\Lambda, K)$.

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Lemma (Variant of Poisson Summ. Formula)

For full rank lattice $\Lambda \subseteq \mathbb{R}^n$, $u \in \mathbb{R}^n$ and $t > 0$ one has

$$\left| \frac{\rho_t(\Lambda + u)}{t^n \det(\Lambda^*)} - 1 \right| \leq \rho_{1/t}(\Lambda^* \setminus \{\mathbf{0}\}).$$

Main theorem for stable + ℓ-position (2)

- We summarize
  - (i) Then $\rho_t((u+\Lambda) \setminus t \cdot K) \leq o(1) \cdot \rho_t(\Lambda)$ [Banaszczyk 1996]
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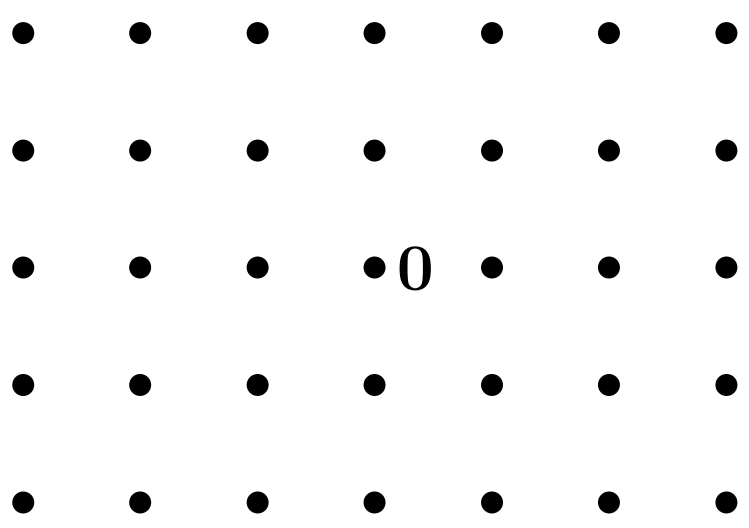
Consider a lattice $\Lambda \subseteq \mathbb{R}^n$ with a primitive sublattice $\Lambda' \subseteq \Lambda$. The **quotient lattice** is $\Lambda/\Lambda' = \Pi_{\mathrm{span}(\Lambda')^\perp}(\Lambda)$.



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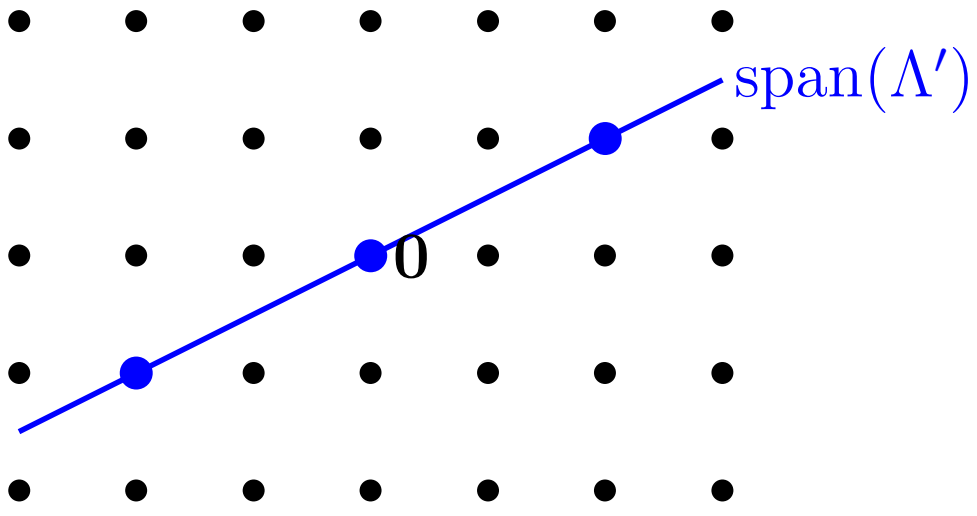

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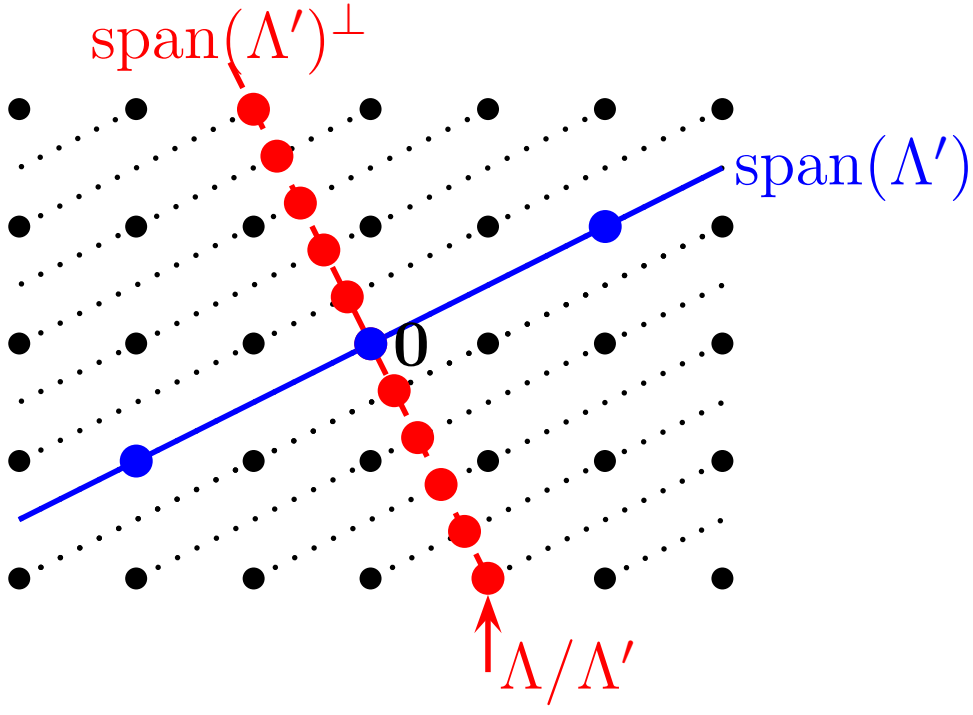

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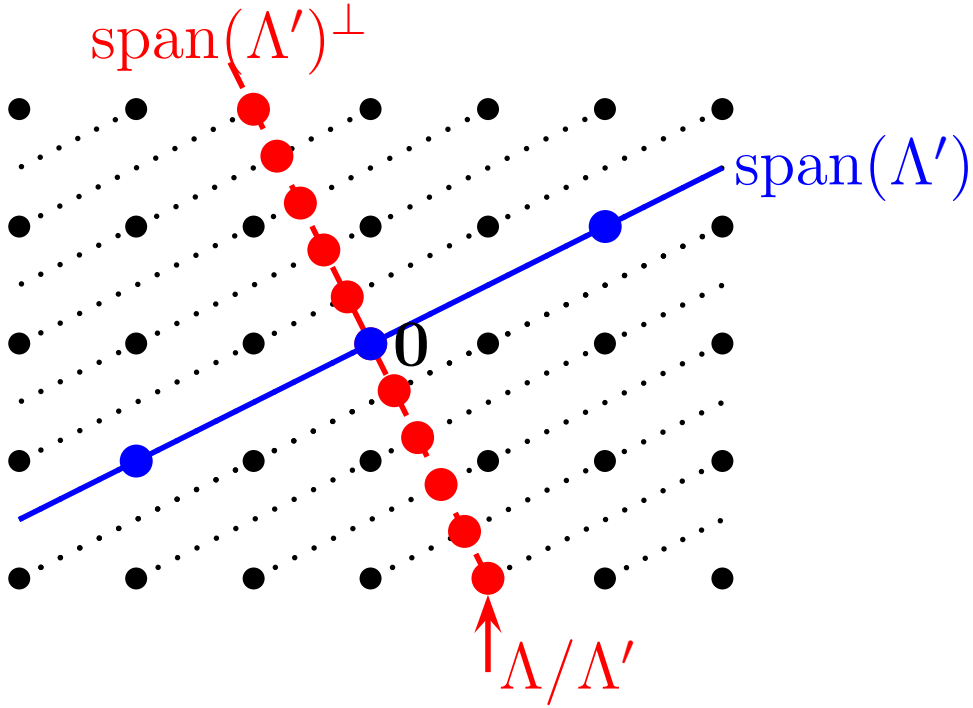
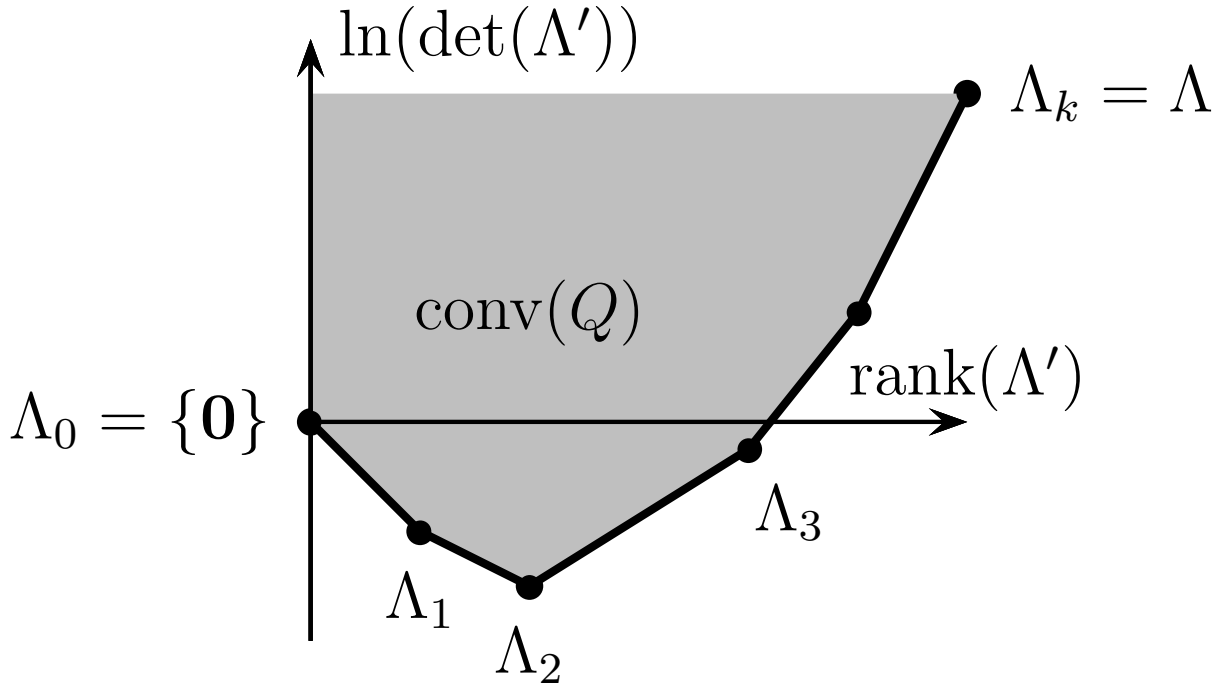


- **Intuition:** We can factor Λ into Λ' and Λ/Λ'
- For example $\det(\Lambda) = \det(\Lambda') \cdot \det(\Lambda/\Lambda')$.

Recap: The canonical filtration



Theorem (Canonical filtration)

(a) *The vertices of the canonical plot form a chain*

$$\{\mathbf{0}\} = \Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_k = \Lambda.$$

(b) $r_i := \det(\Lambda_i/\Lambda_{i-1})^{1/rank(\Lambda_i/\Lambda_{i-1})}$ *satisfy* $r_1 < \ldots < r_k$

(c) *Each* $\frac{1}{r_i}(\Lambda_i/\Lambda_{i-1})$ *is stable.*

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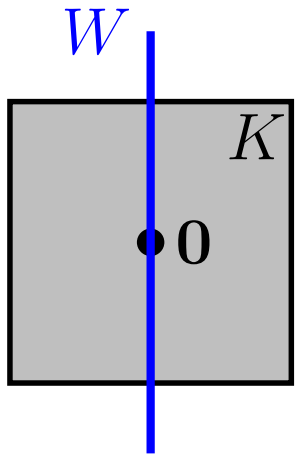

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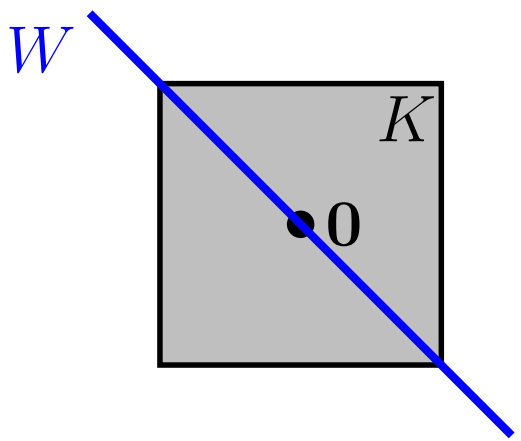

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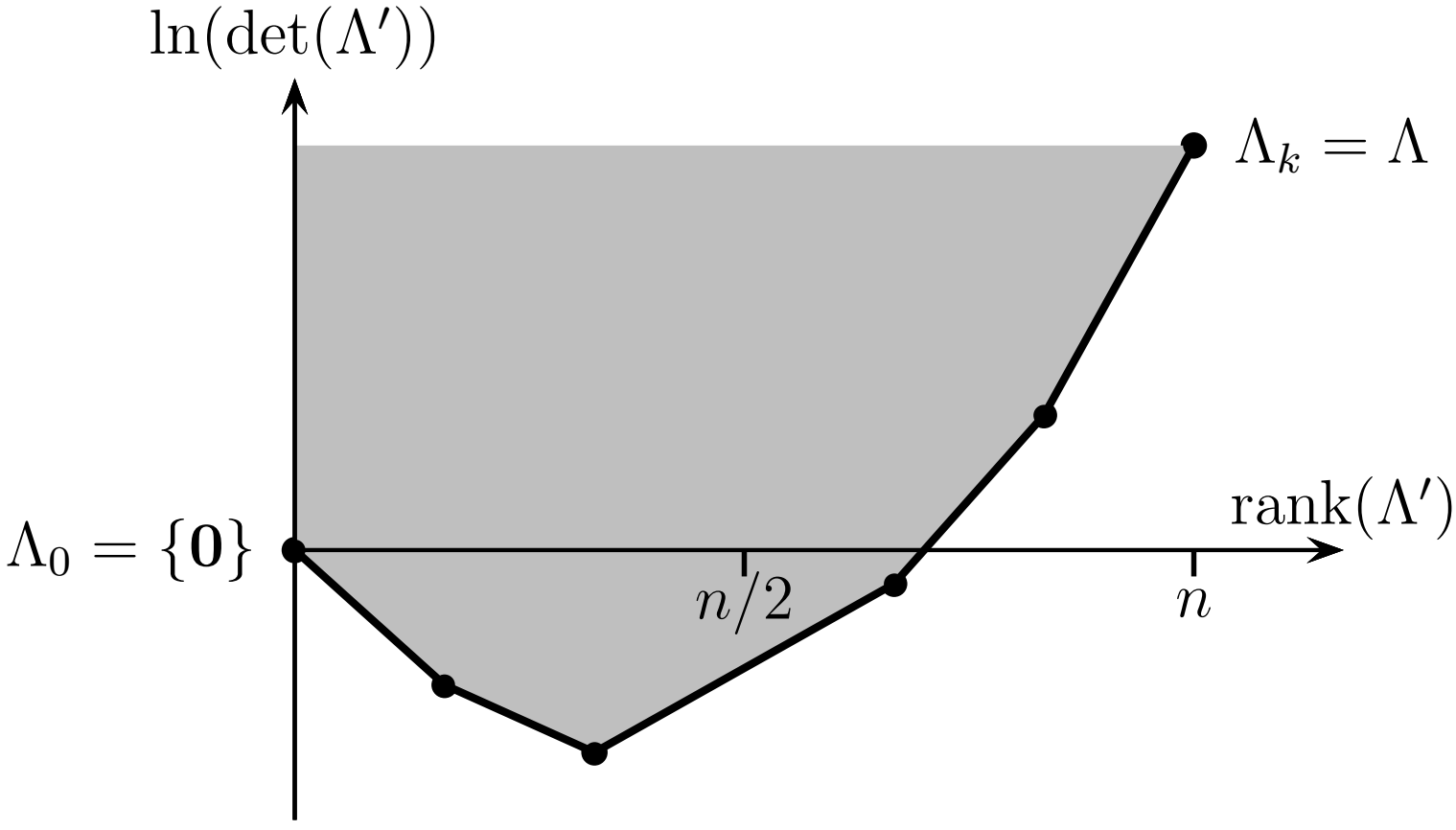

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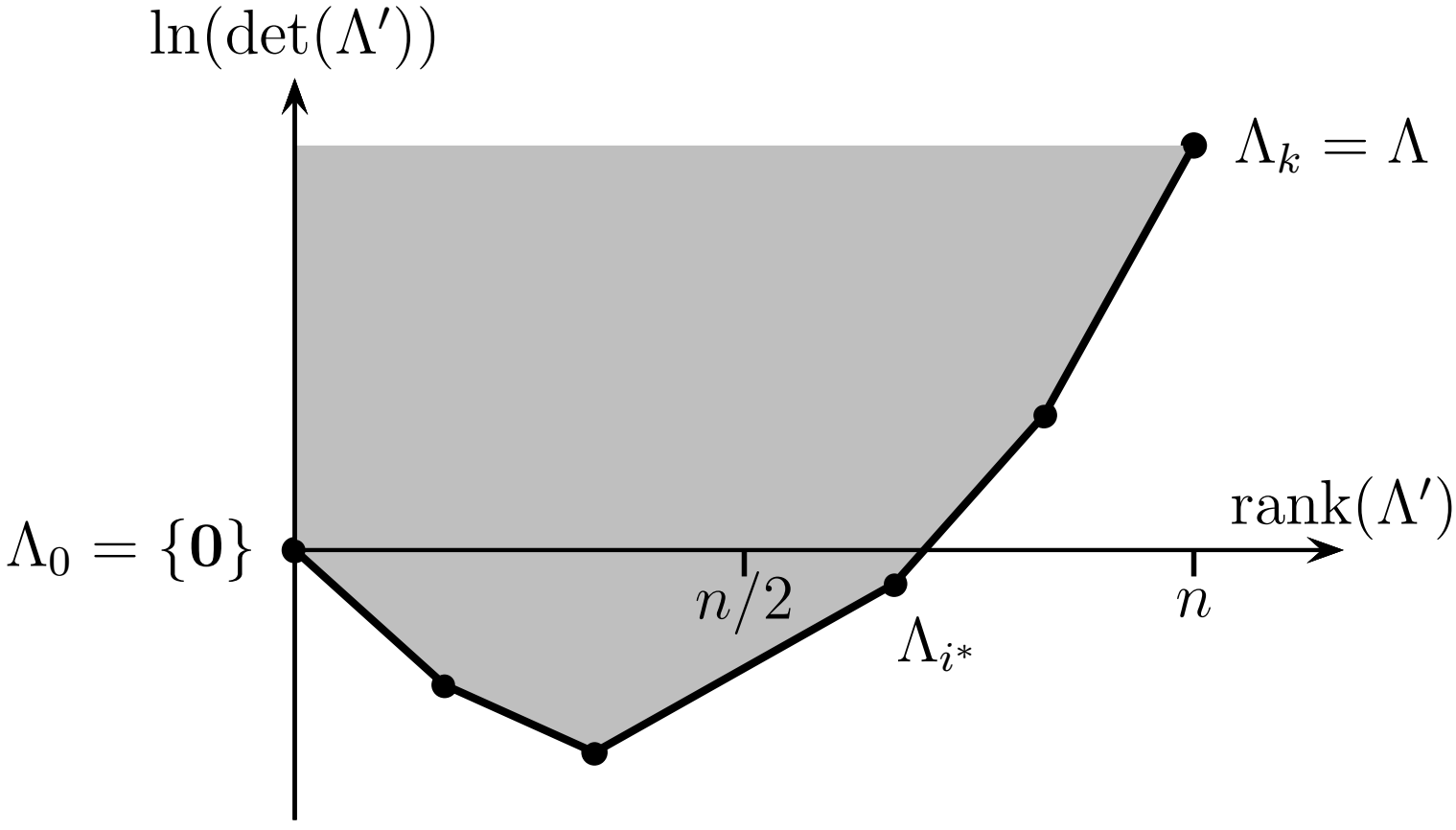

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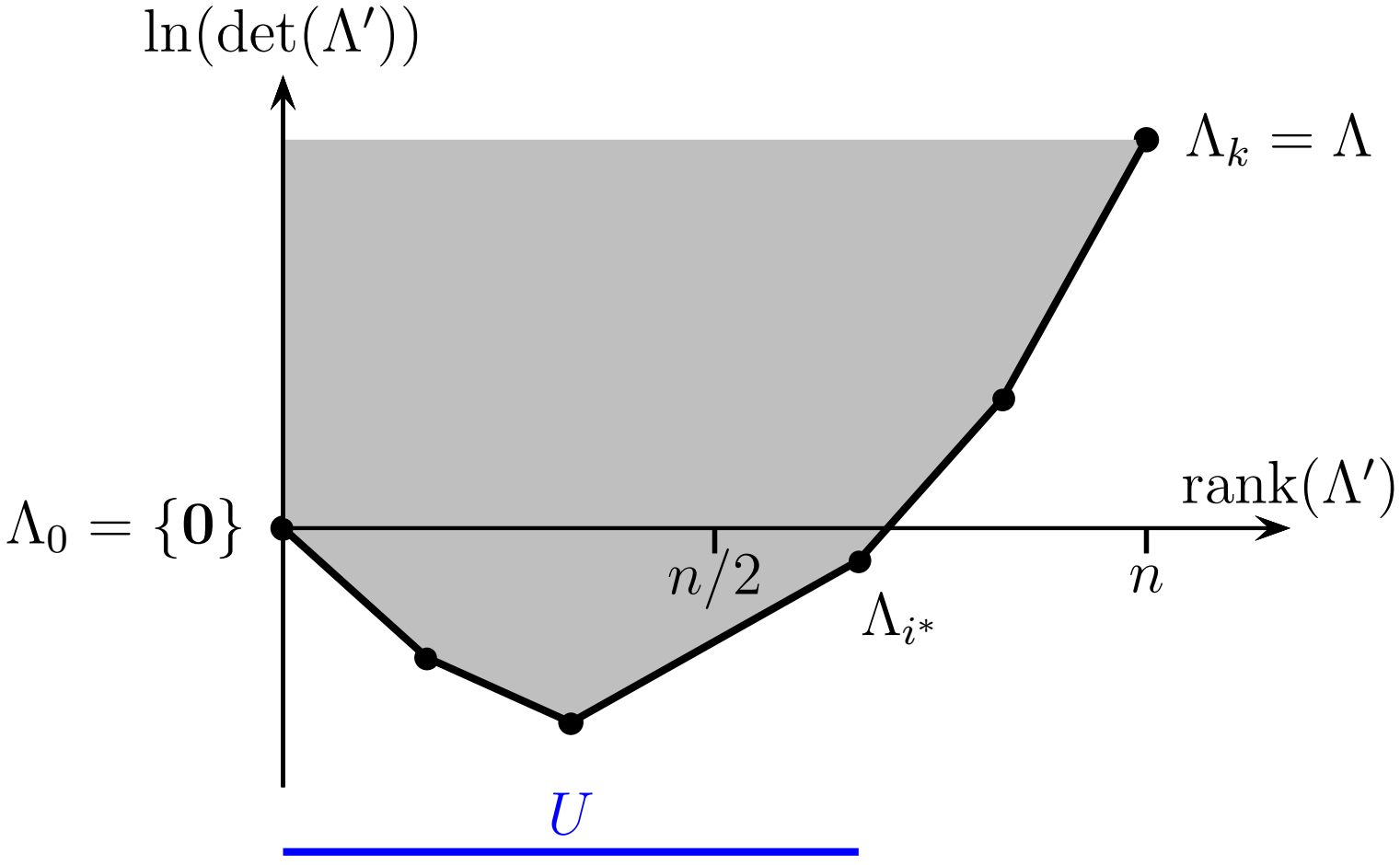

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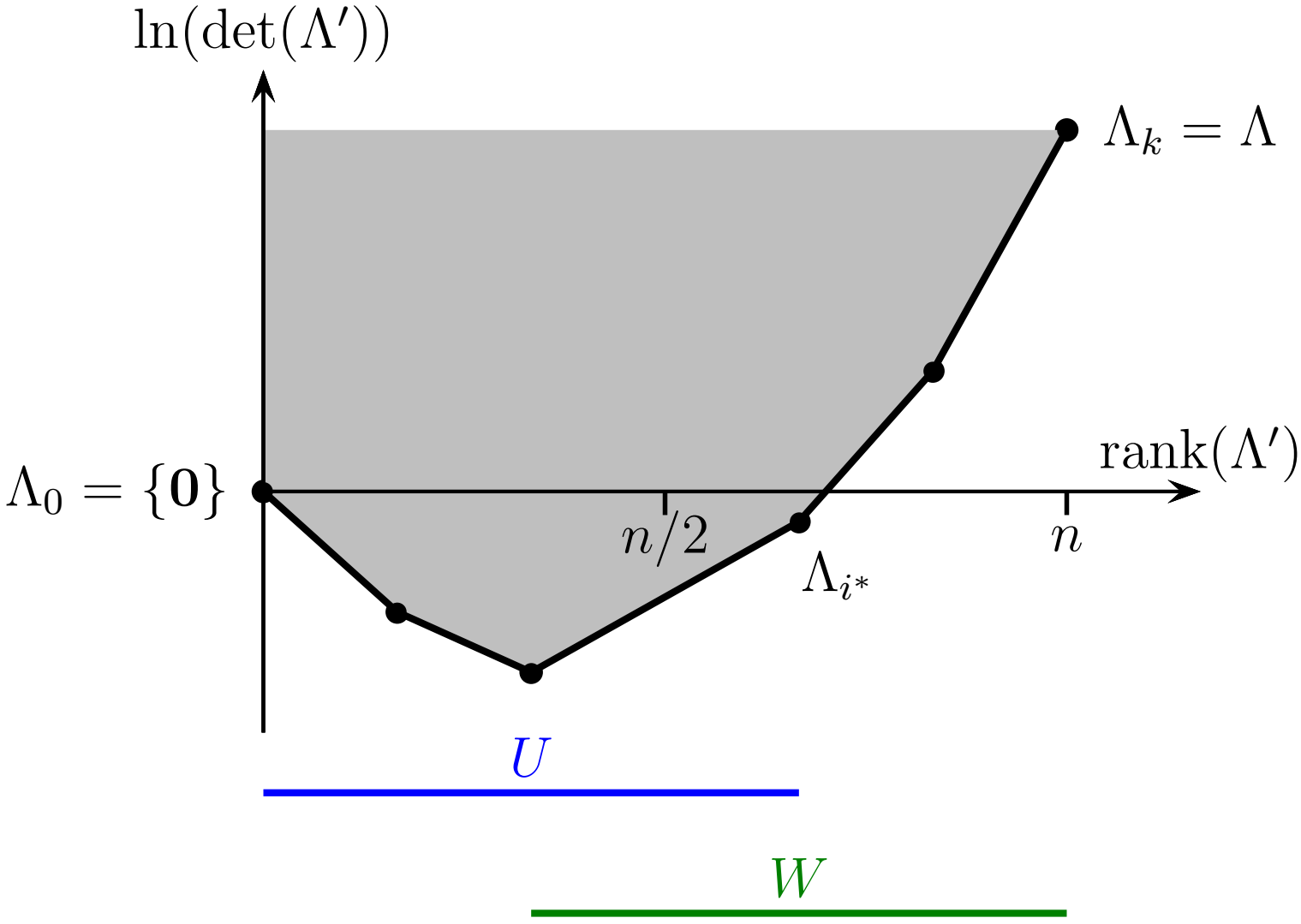
$\ln(\det(\Lambda'))$
$\Lambda_k = \Lambda$
$\Lambda_0 = \{\mathbf{0}\}$
$\mathrm{rank}(\Lambda')$
$n/2$
n
Λ_{i*}
U
W

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- Recall: $\frac{1}{r_i}(\Lambda_i/\Lambda_{i-1})$ stable
- Use monotonicity of ℓ-value: $\ell_{K\cap F} \leq \ell_K$ for subspace F (potentially huge loss!!)

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Putting everything together

$$\mu(\Lambda, K) \lesssim \log(n) \cdot r_{i^*} \cdot \ell_K \lesssim \log^2(n) \cdot \mu_{KL}(\Lambda, K) \quad \square$$

The asymmetric case

Proposition

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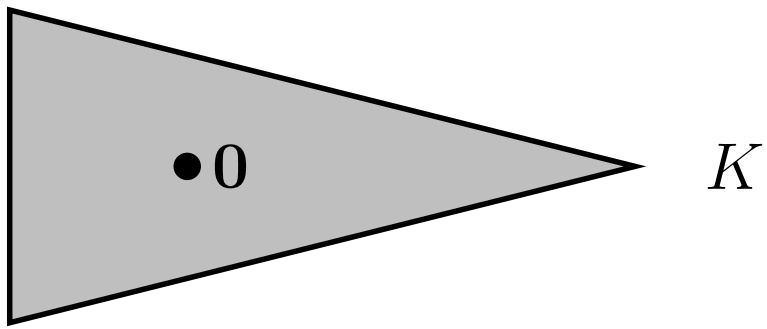
- Assume barycenter is at $\mathbf{0}$



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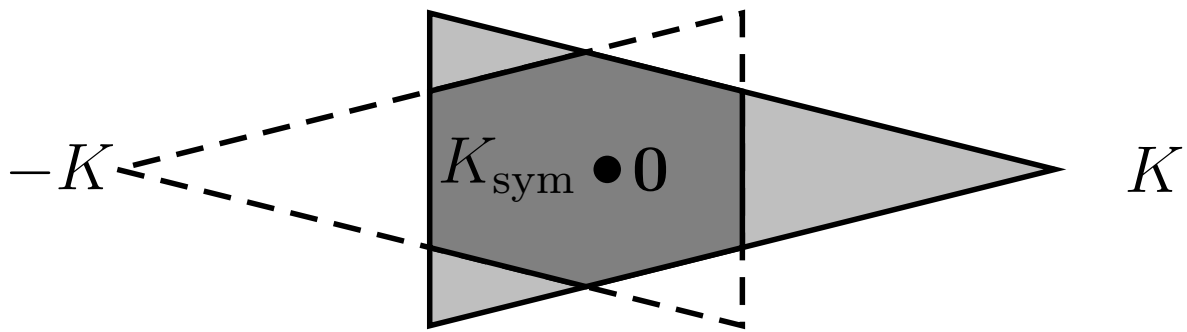
Proposition

Let $K \subseteq \mathbb{R}^n$ be ~~symmetric~~ and convex, let $\Lambda \subseteq \mathbb{R}^n$. $\exists$ subspace $U : \dim(U) \geq \frac{n}{2}$ and $\mu(\Lambda \cap U, K \cap U) \leq O(\log^2 n) \cdot \mu_{KL}(\Lambda, K)$.

- Assume barycenter is at $\mathbf{0}$
- Let $K_{\text{sym}} := K \cap -K$ (**inner symmetrizer**)



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Proposition (Vritsiou '23)

Let $K \subseteq \mathbb{R}^n$ be a convex body with barycenter at $\mathbf{0}$. Let $F \subseteq \mathbb{R}^n$ be a d-dimensional subspace. Then

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- Works for us since we have $d \geq \frac{n}{2}$!

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End of part 3

Open problem 1

Can one solve every **n-variable integer program** in time $2^{O(n)}$?

- Right now, no candidate pathway known!

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Is there a **certificate** for $K \cap \mathbb{Z}^n = \emptyset$ that can be verified in time $2^{O(n)}$?

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Thanks for your attention!