The matching polytope has exponential extension complexity

Thomas Rothvoß

Department of Mathematics, MIT

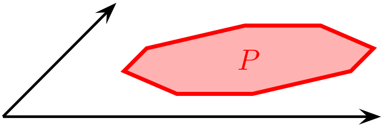
Guwahati, India — Dec 2013

Massachusetts
Institute of
Technology

Extended formulation

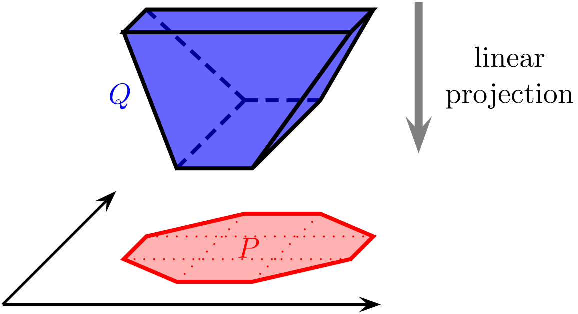
Extended formulation

- Given polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$



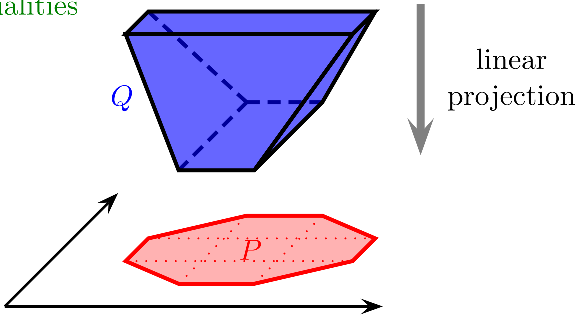
Extended formulation

- Given polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$
- Write $P = \{x \in \mathbb{R}^n \mid \exists y : Bx + Cy \leq d\}$



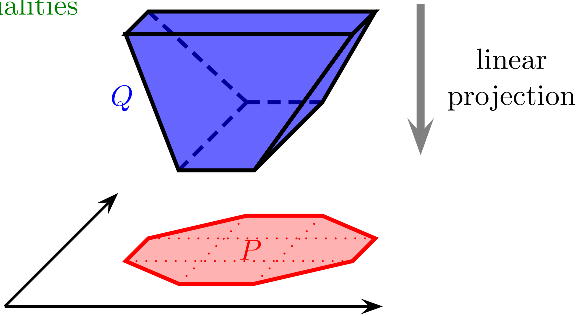
Extended formulation

- Given polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$
  $\rightarrow$ many inequalities
- Write $P = \{x \in \mathbb{R}^n \mid \exists y : Bx + Cy \leq d\}$
  $\rightarrow$ few inequalities



Extended formulation

- Given polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$
  $\rightarrow$ many inequalities
- Write $P = \{x \in \mathbb{R}^n \mid \exists y : Bx + Cy \leq d\}$
  $\rightarrow$ few inequalities



- **Extension complexity:**

$$\mathrm{xc}(P) := \min \left\{ \#\text{facets of } Q \;\middle|\; \begin{array}{c} Q \text{ polyhedron} \\ p \text{ linear map} \\ p(Q) = P \end{array} \right\}$$

What's known?

Compact formulations:

- Spanning Tree Polytope [Kipp Martin '91]
- Perfect Matching in planar graphs [Barahona '93]
- Perfect Matching in bounded genus graphs [Gerards '91]
- $O(n \log n)$-size for Permutahedron [Goemans '10] ($\rightarrow$ **tight**)
- $n^{O(1/\varepsilon)}$-size ε-apx for Knapsack Polytope [Bienstock '08]
- ...

What's known?

Compact formulations:

- SPANNING TREE POLYTOPE [Kipp Martin '91]
- PERFECT MATCHING in planar graphs [Barahona '93]
- PERFECT MATCHING in bounded genus graphs [Gerards '91]
- $O(n \log n)$-size for PERMUTAHEDRON [Goemans '10] ($\rightarrow$ **tight**)
- $n^{O(1/\varepsilon)}$-size ε-apx for KNAPSACK POLYTOPE [Bienstock '08]
- ...

Here: When is the extension complexity **super polynomial**?

Lower bounds

Lower bounds

- No **symmetric** compact form. for TSP [Yannakakis '91]
  Compact formulation for $\log n$ size matchings, but no symmetric one [Kaibel, Pashkovich & Theis '10]

Lower bounds

- No **symmetric** compact form. for TSP [Yannakakis '91]
  Compact formulation for $\log n$ size matchings, but no symmetric one [Kaibel, Pashkovich & Theis '10]
- xc(random 0/1 polytope) $\geq 2^{\Omega(n)}$ [R. '11]

Lower bounds

- No **symmetric** compact form. for TSP [Yannakakis '91]
  Compact formulation for $\log n$ size matchings, but no symmetric one [Kaibel, Pashkovich & Theis '10]
- $\mathrm{xc}(\text{random } 0/1 \text{ polytope}) \geq 2^{\Omega(n)}$ [R. '11]
- **Breakthrough:** $\mathrm{xc}(\mathrm{TSP}) \geq 2^{\Omega(\sqrt{n})}$
  [Fiorini, Massar, Pokutta, Tiwary, de Wolf '12]

Lower bounds

- No **symmetric** compact form. for TSP [Yannakakis '91]
  Compact formulation for $\log n$ size matchings, but no symmetric one [Kaibel, Pashkovich & Theis '10]
- xc(random 0/1 polytope) $\geq 2^{\Omega(n)}$ [R. '11]
- **Breakthrough:** xc(TSP) $\geq 2^{\Omega(\sqrt{n})}$
  [Fiorini, Massar, Pokutta, Tiwary, de Wolf '12]
- $n^{1/2-\varepsilon}$-apx for clique polytope needs super-poly size
  [Braun, Fiorini, Pokutta, Steuer '12]
  Improved to $n^{1-\varepsilon}$ [Braverman, Moitra '13], [Braun, P. '13]

Lower bounds

- No **symmetric** compact form. for TSP [Yannakakis '91]
  Compact formulation for $\log n$ size matchings, but no symmetric one [Kaibel, Pashkovich & Theis '10]
- xc(random 0/1 polytope) $\geq 2^{\Omega(n)}$ [R. '11]
- **Breakthrough:** xc(TSP) $\geq 2^{\Omega(\sqrt{n})}$
  [Fiorini, Massar, Pokutta, Tiwary, de Wolf '12]
- $n^{1/2-\varepsilon}$-apx for clique polytope needs super-poly size
  [Braun, Fiorini, Pokutta, Steuer '12]
  Improved to $n^{1-\varepsilon}$ [Braverman, Moitra '13], [Braun, P. '13]
- $(2-\varepsilon)$-apx LPs for MaxCut have size $n^{\Omega(\log n / \log\log n)}$
  [Chan, Lee, Raghavendra, Steurer '13]

Lower bounds

- No **symmetric** compact form. for TSP [Yannakakis '91]
  Compact formulation for $\log n$ size matchings, but no symmetric one [Kaibel, Pashkovich & Theis '10]
- xc(random 0/1 polytope) $\geq 2^{\Omega(n)}$ [R. '11]
- **Breakthrough:** xc(TSP) $\geq 2^{\Omega(\sqrt{n})}$
  [Fiorini, Massar, Pokutta, Tiwary, de Wolf '12]
- $n^{1/2-\varepsilon}$-apx for clique polytope needs super-poly size
  [Braun, Fiorini, Pokutta, Steuer '12]
  Improved to $n^{1-\varepsilon}$ [Braverman, Moitra '13], [Braun, P. '13]
- $(2-\varepsilon)$-apx LPs for MaxCut have size $n^{\Omega(\log n / \log\log n)}$
  [Chan, Lee, Raghavendra, Steurer '13]

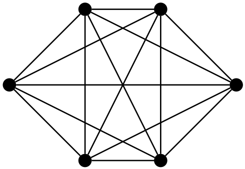
Only **NP**-hard polytopes!!

What about poly-time problems?

Perfect matching polytope

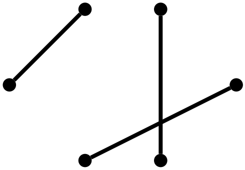
Perfect matching polytope

$G = (V, E)$
(complete)



Perfect matching polytope

$G = (V, E)$
(complete)

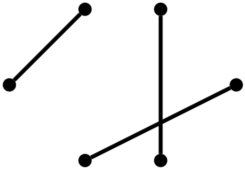


Perfect matching polytope

$G = (V, E)$
(complete)

$$x(\delta(v)) = 1 \quad \forall v \in V$$

$$x_e \geq 0 \quad \forall e \in E$$

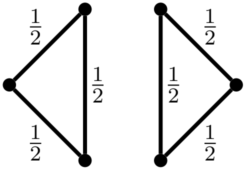


Perfect matching polytope

$G = (V, E)$
(complete)

$$x(\delta(v)) = 1 \quad \forall v \in V$$

$$x_e \geq 0 \quad \forall e \in E$$

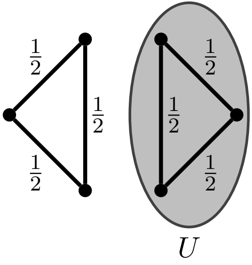


Perfect matching polytope

$$x(\delta(v)) = 1 \quad \forall v \in V$$

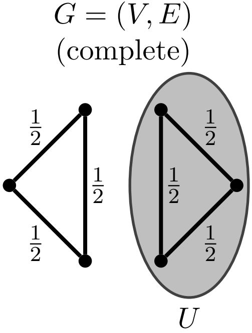
$$x_e \geq 0 \quad \forall e \in E$$

$G = (V, E)$
(complete)



Perfect matching polytope

$$
\begin{aligned}
x(\delta(v)) &= 1 \quad \forall v \in V \\
x(\delta(U)) &\geq 1 \quad \forall U \subseteq V : |U| \text{ odd} \\
x_e &\geq 0 \quad \forall e \in E
\end{aligned}
$$

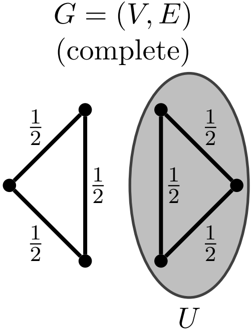


Quick facts:

- Description by [Edmonds '65]

Perfect matching polytope

$$\begin{aligned} x(\delta(v)) &= 1 \quad \forall v \in V \\ x(\delta(U)) &\geq 1 \quad \forall U \subseteq V : |U| \text{ odd} \\ x_e &\geq 0 \quad \forall e \in E \end{aligned}$$

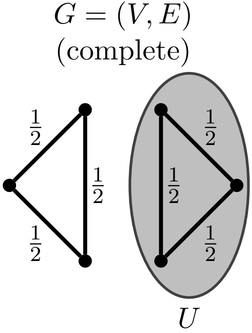


Quick facts:

- Description by [Edmonds '65]
- Can optimize $c^T x$ in strongly poly-time [Edmonds '65]

Perfect matching polytope

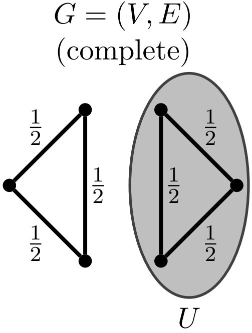
$$\begin{aligned} x(\delta(v)) &= 1 \quad \forall v \in V \\ x(\delta(U)) &\geq 1 \quad \forall U \subseteq V : |U| \text{ odd} \\ x_e &\geq 0 \quad \forall e \in E \end{aligned}$$



Quick facts:

- Description by [Edmonds '65]
- Can optimize $c^T x$ in strongly poly-time [Edmonds '65]
- Separation problem polytime [Padberg, Rao '82]

Perfect matching polytope



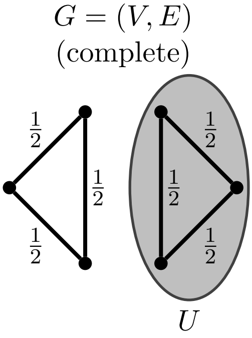
$$\begin{array}{rcll} x(\delta(v)) & = & 1 & \forall v \in V \\ x(\delta(U)) & \geq & 1 & \forall U \subseteq V : |U| \text{ odd} \\ x_e & \geq & 0 & \forall e \in E \end{array}$$

Quick facts:

- Description by [Edmonds '65]
- Can optimize $c^T x$ in strongly poly-time [Edmonds '65]
- Separation problem polytime [Padberg, Rao '82]
- $2^{\Theta(n)}$ facets

Perfect matching polytope

$$\begin{aligned} x(\delta(v)) &= 1 \quad \forall v \in V \\ x(\delta(U)) &\geq 1 \quad \forall U \subseteq V : |U| \text{ odd} \\ x_e &\geq 0 \quad \forall e \in E \end{aligned}$$



Quick facts:

- Description by [Edmonds '65]
- Can optimize $c^T x$ in strongly poly-time [Edmonds '65]
- Separation problem polytime [Padberg, Rao '82]
- $2^{\Theta(n)}$ facets

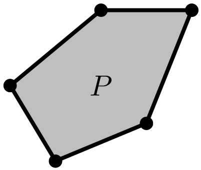
Theorem (R.13)

xc(perfect matching polytope) $\geq 2^{\Omega(n)}$.

- Previously known: $\text{xc}(P) \geq \Omega(n^2)$

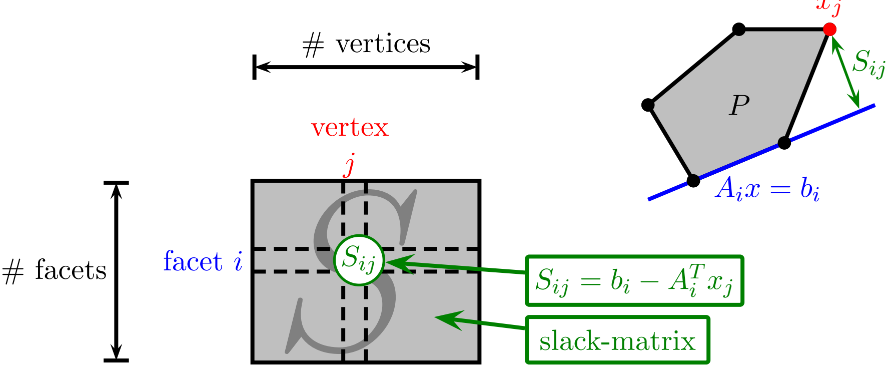
Slack-matrix

Write: $P = \text{conv}(\{x_1, \ldots, x_v\}) = \{x \in \mathbb{R}^n \mid Ax \leq b\}$



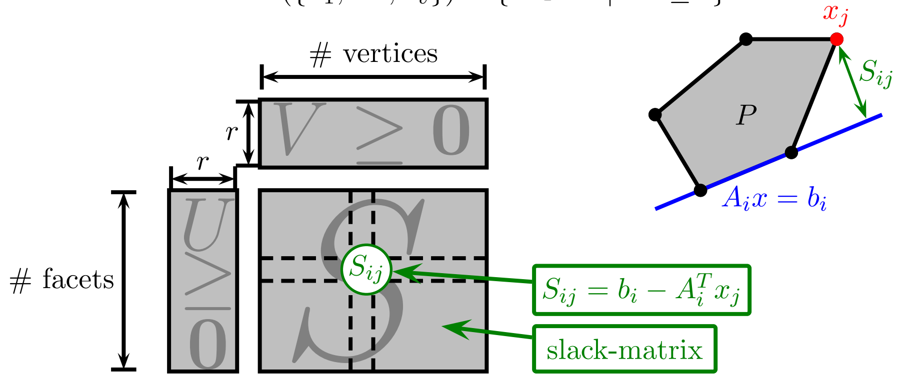
Slack-matrix

Write: $P = \text{conv}(\{x_1, \ldots, x_v\}) = \{x \in \mathbb{R}^n \mid Ax \leq b\}$



Slack-matrix

Write: $P = \text{conv}(\{x_1, \ldots, x_v\}) = \{x \in \mathbb{R}^n \mid Ax \leq b\}$



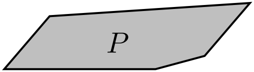
Non-negative rank:

$$\text{rk}_+(S) = \min\{r \mid \exists U \in \mathbb{R}^{f \times r}_{\geq 0}, V \in \mathbb{R}^{r \times v}_{\geq 0} : S = UV\}$$

Yannakakis' Theorem

Theorem (Yannakakis '91)

If S is the **slack-matrix** for $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, then $\mathrm{xc}(P) = \mathrm{rk}_+(S)$.



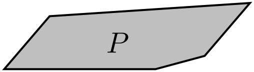
Yannakakis' Theorem

Theorem (Yannakakis '91)

If S is the **slack-matrix** for $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, then $\mathrm{xc}(P) = \mathrm{rk}_+(S)$.

Factorization $S = UV \Rightarrow$ extended formulation:

- Let $P = \{x \in \mathbb{R}^n \mid \exists y \geq \mathbf{0} : Ax + Uy = b\}$



Yannakakis' Theorem

Theorem (Yannakakis '91)

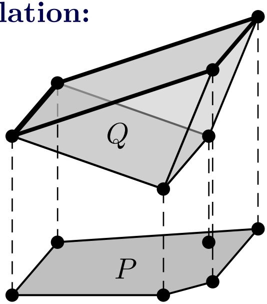
If S is the **slack-matrix** for $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, then $\mathrm{xc}(P) = \mathrm{rk}_+(S)$.

Factorization $S = UV \Rightarrow$ extended formulation:

- Let $P = \{x \in \mathbb{R}^n \mid \exists y \geq \mathbf{0} : Ax + Uy = b\}$

Extended form. $\Rightarrow$ factorization:

- Given an extension $Q = \{(x, y) \mid Bx + Cy \leq d\}$



Yannakakis' Theorem

Theorem (Yannakakis '91)

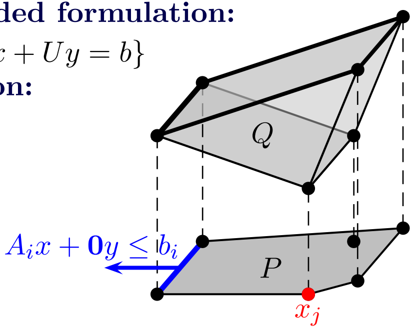
If S is the **slack-matrix** for $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$, then $\mathrm{xc}(P) = \mathrm{rk}_+(S)$.

Factorization $S = UV \Rightarrow$ extended formulation:

- Let $P = \{x \in \mathbb{R}^n \mid \exists y \ge \mathbf{0} : Ax + Uy = b\}$

Extended form. $\Rightarrow$ factorization:

- Given an extension $Q = \{(x, y) \mid Bx + Cy \le d\}$



$\langle u(i), v(j) \rangle = \qquad S_{ij}$

Yannakakis' Theorem

Theorem (Yannakakis '91)

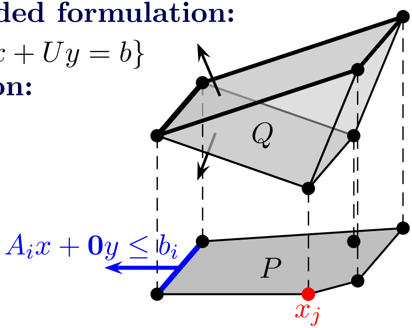
If S is the **slack-matrix** for $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, then $\mathrm{xc}(P) = \mathrm{rk}_+(S)$.

Factorization $S = UV \Rightarrow$ extended formulation:

- Let $P = \{x \in \mathbb{R}^n \mid \exists y \geq \mathbf{0} : Ax + Uy = b\}$

Extended form. $\Rightarrow$ factorization:

- Given an extension $Q = \{(x, y) \mid Bx + Cy \leq d\}$
- For facet i: $u(i) :=$ conic comb of i



$\langle u(i), v(j) \rangle = \qquad S_{ij}$

Yannakakis' Theorem

Theorem (Yannakakis '91)

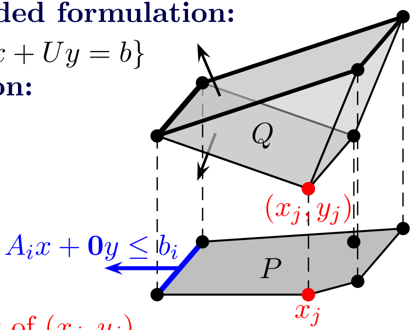
If S is the **slack-matrix** for $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$, then $\mathrm{xc}(P) = \mathrm{rk}_+(S)$.

Factorization $S = UV \Rightarrow$ extended formulation:

- Let $P = \{x \in \mathbb{R}^n \mid \exists y \ge \mathbf{0} : Ax + Uy = b\}$

Extended form. $\Rightarrow$ factorization:

- Given an extension $Q = \{(x, y) \mid Bx + Cy \le d\}$
- For facet i: $u(i) :=$ conic comb of i
- For vertex x_j: $v(j) := d - Bx_j - Cy_j =$ slack of (x_j, y_j)



$$\langle u(i), v(j) \rangle = \qquad S_{ij}$$

Yannakakis' Theorem

Theorem (Yannakakis '91)

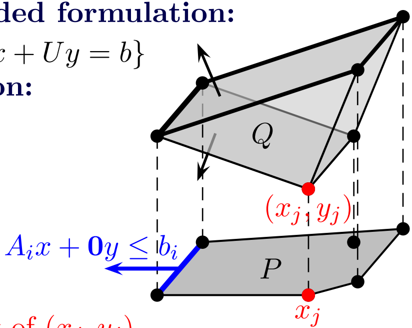
If S is the **slack-matrix** for $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, then $\mathrm{xc}(P) = \mathrm{rk}_+(S)$.

Factorization $S = UV \Rightarrow$ extended formulation:

- Let $P = \{x \in \mathbb{R}^n \mid \exists y \geq \mathbf{0} : Ax + Uy = b\}$

Extended form. $\Rightarrow$ factorization:

- Given an extension $Q = \{(x, y) \mid Bx + Cy \leq d\}$
- For facet i: $u(i) :=$ conic comb of i
- For vertex x_j: $v(j) := d - Bx_j - Cy_j =$ slack of (x_j, y_j)



$$\langle u(i), v(j)\rangle = \underbrace{u(i)^T d}_{=b_i} - \underbrace{u(i)B}_{=A_i} x_j - \underbrace{u(i)C}_{=\mathbf{0}} y_j = S_{ij}$$

Rectangle covering lower bound

Observation

$\mathrm{rk}_+(S) \geq \text{rectangle-covering-number}(S)$.

Rectangle covering lower bound

$$V = \begin{pmatrix} 0 & 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 & 3 \end{pmatrix}$$

$$U = \begin{pmatrix} 3 & 2 \\ 1 & 1 \\ 0 & 2 \\ 0 & 0 \\ 2 & 0 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 4 & 10 & 3 & 5 \\ 0 & 2 & 4 & 1 & 3 \\ 0 & 4 & 4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{pmatrix}$$

Observation

$\mathrm{rk}_+(S) \geq$ rectangle-covering-number(S).

Rectangle covering lower bound

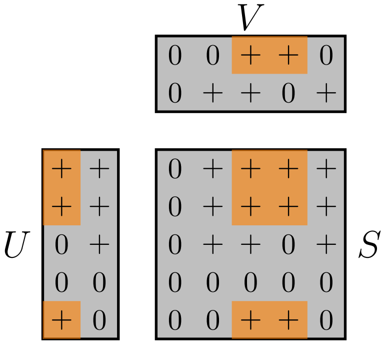
$$V = \begin{pmatrix} 0 & 0 & + & + & 0 \\ 0 & + & + & 0 & + \end{pmatrix}$$

$$U = \begin{pmatrix} + & + \\ + & + \\ 0 & + \\ 0 & 0 \\ + & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & + & + & + & + \\ 0 & + & + & + & + \\ 0 & + & + & 0 & + \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & + & + & 0 \end{pmatrix} = S$$

Observation

$\mathrm{rk}_+(S) \geq$ rectangle-covering-number(S).

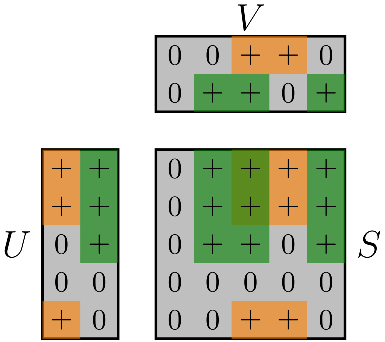
Rectangle covering lower bound



Observation

$\mathrm{rk}_+(S) \geq \text{rectangle-covering-number}(S)$.

Rectangle covering lower bound



Observation

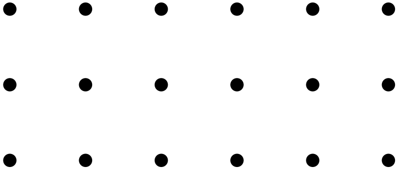
$\mathrm{rk}_+(S) \geq \text{rectangle-covering-number}(S)$.

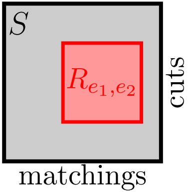
Rectangle covering for matching

- Recall $S_{U,M} = |\delta(U) \cap M| - 1$

Observation

Rect-cov-num(matching polytope) $\leq O(n^4)$.



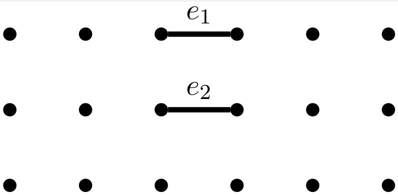


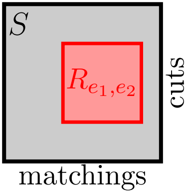
Rectangle covering for matching

- Recall $S_{U,M} = |\delta(U) \cap M| - 1$

Observation

Rect-cov-num(matching polytope) $\leq O(n^4)$.





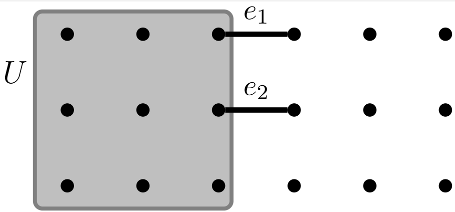
- For $e_1, e_2 \in E$:

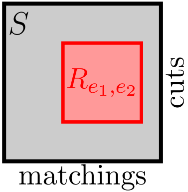
Rectangle covering for matching

- Recall $S_{U,M} = |\delta(U) \cap M| - 1$

Observation

Rect-cov-num(matching polytope) $\leq O(n^4)$.





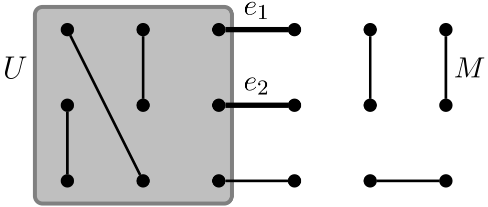
- For $e_1, e_2 \in E$: take $\{U \mid e_1, e_2 \in \delta(U)\}$

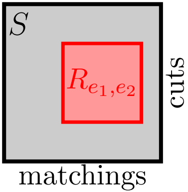
Rectangle covering for matching

- Recall $S_{U,M} = |\delta(U) \cap M| - 1$

Observation

Rect-cov-num(matching polytope) $\leq O(n^4)$.





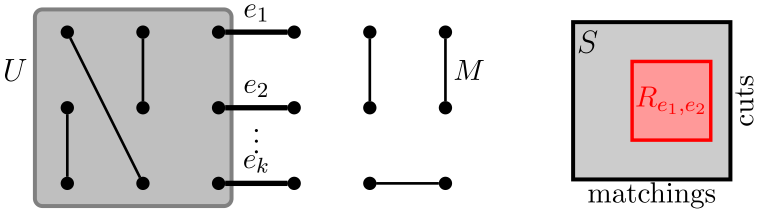
- For $e_1, e_2 \in E$: take $\{U \mid e_1, e_2 \in \delta(U)\} \times \{M \mid e_1, e_2 \in M\}$

Rectangle covering for matching

- Recall $S_{U,M} = |\delta(U) \cap M| - 1$

Observation

Rect-cov-num(matching polytope) $\leq O(n^4)$.



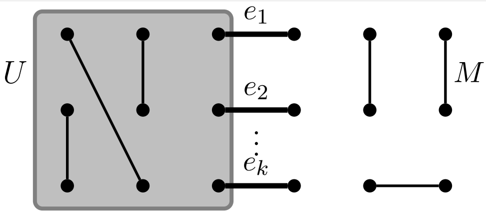
- For $e_1, e_2 \in E$: take $\{U \mid e_1, e_2 \in \delta(U)\} \times \{M \mid e_1, e_2 \in M\}$
- (U, M) with $M \cap \delta(U) = \{e_1, \ldots, e_k\}$ lies in $\binom{k}{2}$ rectangles

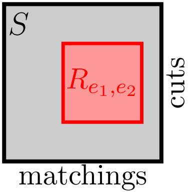
Rectangle covering for matching

- Recall $S_{U,M} = |\delta(U) \cap M| - 1$

Observation

Rect-cov-num(matching polytope) $\leq O(n^4)$.





- For $e_1, e_2 \in E$: take $\{U \mid e_1, e_2 \in \delta(U)\} \times \{M \mid e_1, e_2 \in M\}$
- (U, M) with $M \cap \delta(U) = \{e_1, \ldots, e_k\}$ lies in $\binom{k}{2}$ rectangles

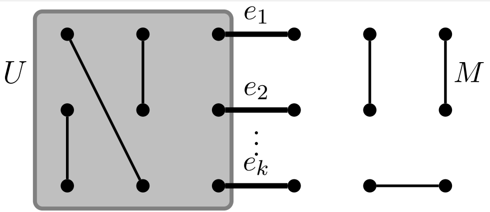
$$S \stackrel{?}{=} \sum_{e_1,e_2} \boxed{R_{e_1,e_2}}$$

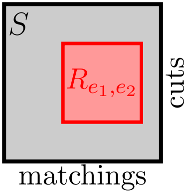
Rectangle covering for matching

- Recall $S_{U,M} = |\delta(U) \cap M| - 1$

Observation

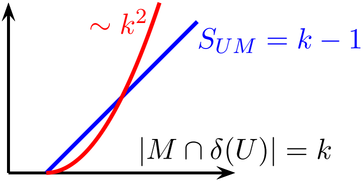
Rect-cov-num(matching polytope) $\leq O(n^4)$.





- For $e_1, e_2 \in E$: take $\{U \mid e_1, e_2 \in \delta(U)\} \times \{M \mid e_1, e_2 \in M\}$
- (U, M) with $M \cap \delta(U) = \{e_1, \dots, e_k\}$ lies in $\binom{k}{2}$ rectangles

$$S \stackrel{?}{=} \sum_{e_1,e_2} R_{e_1,e_2}$$

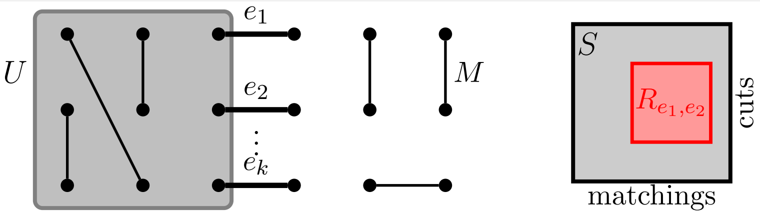


Rectangle covering for matching

- Recall $S_{U,M} = |\delta(U) \cap M| - 1$

Observation

Rect-cov-num(matching polytope) $\leq O(n^4)$.



- For $e_1, e_2 \in E$: take $\{U \mid e_1, e_2 \in \delta(U)\} \times \{M \mid e_1, e_2 \in M\}$
- (U, M) with $M \cap \delta(U) = \{e_1, \ldots, e_k\}$ lies in $\binom{k}{2}$ rectangles

Question

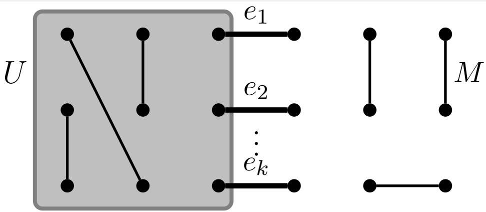
Does every rectangle covering over-cover entries of large slack?

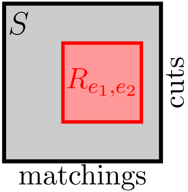
Rectangle covering for matching

- Recall $S_{U,M} = |\delta(U) \cap M| - 1$

Observation

Rect-cov-num(matching polytope) $\leq O(n^4)$.





- For $e_1, e_2 \in E$: take $\{U \mid e_1, e_2 \in \delta(U)\} \times \{M \mid e_1, e_2 \in M\}$
- (U, M) with $M \cap \delta(U) = \{e_1, \ldots, e_k\}$ lies in $\binom{k}{2}$ rectangles

Question

Does every rectangle covering over-cover entries of large slack? **YES!!**

Hyperplane separation lower bound [Fiorini]

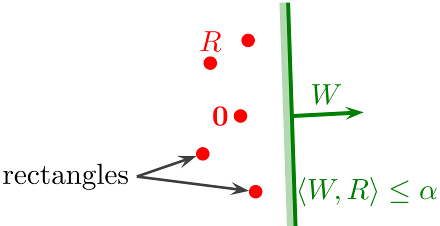
- **Frobenius inner product:** $\langle W, S \rangle := \sum_i \sum_j W_{ij} S_{ij}$

Hyperplane separation lower bound [Fiorini]

- **Frobenius inner product:** $\langle W, S\rangle := \sum_i \sum_j W_{ij} S_{ij}$

Lemma

Pick W: $\langle W, R\rangle \leq \alpha \; \forall$ rectangles R.

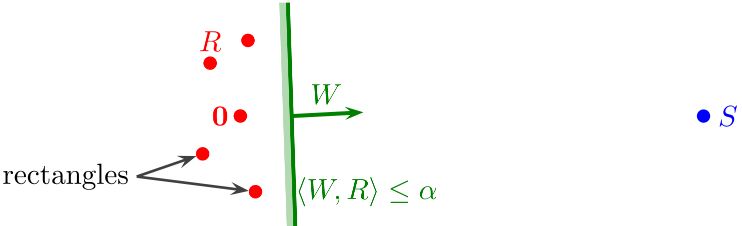


Hyperplane separation lower bound [Fiorini]

- **Frobenius inner product:** $\langle W, S \rangle := \sum_i \sum_j W_{ij} S_{ij}$

Lemma

Pick W: $\langle W, R \rangle \leq \alpha \; \forall$ rectangles R. Then $\mathrm{rk}_+(S) \geq \frac{\langle W, S \rangle}{\|S\|_\infty \cdot \alpha}$



Hyperplane separation lower bound [Fiorini]

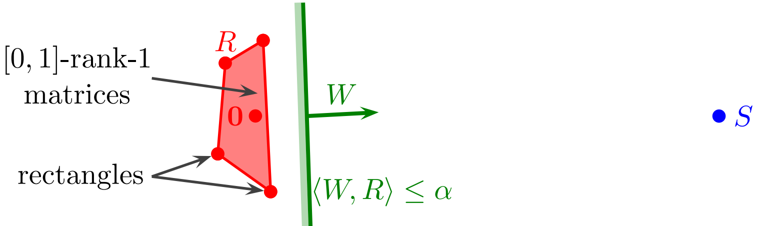
- **Frobenius inner product:** $\langle W, S\rangle := \sum_i \sum_j W_{ij} S_{ij}$

Lemma

Pick W: $\langle W, R\rangle \leq \alpha \ \forall$ rectangles R. Then $\mathrm{rk}_+(S) \geq \dfrac{\langle W, S\rangle}{\|S\|_\infty \cdot \alpha}$

- **Proof:** Write $S = \sum_{i=1}^r R_i$ with $\mathrm{rk}_+(R_i) = 1$. Then

$$\langle W, S\rangle = \sum_{i=1}^r \|R_i\|_\infty \cdot \underbrace{\left\langle W, \frac{R_i}{\|R_i\|_\infty}\right\rangle}_{\leq \alpha} \leq \alpha \cdot \sum_{i=1}^r \underbrace{\|R_i\|_\infty}_{\leq \|S\|_\infty} \leq \alpha \cdot r \cdot \|S\|_\infty.$$



Applying the Hyperplane bound

Lemma

Pick W: $\langle W, R\rangle \leq \alpha \; \forall$ rectangles R. Then $\mathrm{rk}_+(S) \geq \dfrac{\langle W, S\rangle}{\|S\|_\infty \cdot \alpha}$

Applying the Hyperplane bound

Lemma

Pick W: $\langle W, R \rangle \leq \alpha \; \forall$ rectangles R. Then $\mathrm{rk}_+(S) \geq \dfrac{\langle W, S \rangle}{\|S\|_\infty \cdot \alpha}$

- Recall $S_{UM} = |\delta(U) \cap M| - 1$

Applying the Hyperplane bound

Lemma

Pick W: $\langle W, R\rangle \leq \alpha\ \forall$ rectangles R. Then $\mathrm{rk}_+(S) \geq \dfrac{\langle W, S\rangle}{\|S\|_\infty \cdot \alpha}$

- Recall $S_{UM} = |\delta(U) \cap M| - 1$
- Abbreviate $Q_\ell := \{(U, M) : |\delta(U) \cap M| = \ell\}$

Applying the Hyperplane bound

Lemma

Pick W: $\langle W, R\rangle \leq \alpha$ $\forall$ rectangles R. Then $\mathrm{rk}_+(S) \geq \frac{\langle W, S\rangle}{\|S\|_\infty \cdot \alpha}$

- Recall $S_{UM} = |\delta(U) \cap M| - 1$
- Abbreviate $Q_\ell := \{(U, M) : |\delta(U) \cap M| = \ell\}$
- Choose

$$W_{U,M} = \begin{cases} \\ \\ 0 & \text{otherwise.} \end{cases}$$

Applying the Hyperplane bound

Lemma

Pick W: $\langle W, R\rangle \leq \alpha \; \forall$ rectangles R. Then $\text{rk}_+(S) \geq \frac{\langle W, S\rangle}{\|S\|_\infty \cdot \alpha}$

- Recall $S_{UM} = |\delta(U) \cap M| - 1$
- Abbreviate $Q_\ell := \{(U, M) : |\delta(U) \cap M| = \ell\}$
- Choose

$$W_{U,M} = \begin{cases} -\infty & |\delta(U) \cap M| = 1 \\ \\ 0 & \text{otherwise.} \end{cases}$$

- Then $\langle W, S\rangle = 0$

Applying the Hyperplane bound

Lemma

Pick W: $\langle W, R\rangle \leq \alpha \; \forall$ rectangles R. Then $\mathrm{rk}_+(S) \geq \frac{\langle W, S\rangle}{\|S\|_\infty \cdot \alpha}$

- Recall $S_{UM} = |\delta(U) \cap M| - 1$
- Abbreviate $Q_\ell := \{(U, M) : |\delta(U) \cap M| = \ell\}$
- Choose

$$W_{U,M} = \begin{cases} -\infty & |\delta(U) \cap M| = 1 \\ \frac{1}{|Q_3|} & |\delta(U) \cap M| = 3 \\ & \\ 0 & \text{otherwise.} \end{cases}$$

- Then $\langle W, S\rangle = 0 + 2$

Applying the Hyperplane bound

Lemma

Pick W: $\langle W, R\rangle \leq \alpha\ \forall$ rectangles R. Then $\mathrm{rk}_+(S) \geq \frac{\langle W, S\rangle}{\|S\|_\infty \cdot \alpha}$

- Recall $S_{UM} = |\delta(U) \cap M| - 1$
- Abbreviate $Q_\ell := \{(U, M) : |\delta(U) \cap M| = \ell\}$
- Choose

$$W_{U,M} = \begin{cases} -\infty & |\delta(U) \cap M| = 1 \\ \frac{1}{|Q_3|} & |\delta(U) \cap M| = 3 \\ -\frac{1}{k-1} \cdot \frac{1}{|Q_k|} & |\delta(U) \cap M| = k \\ 0 & \text{otherwise.} \end{cases}$$

- Then $\langle W, S\rangle = 0 + 2 - 1 = 1$

Applying the Hyperplane bound

Lemma

Pick W: $\langle W, R \rangle \leq \alpha \; \forall$ rectangles R. Then $\text{rk}_+(S) \geq \frac{\langle W, S \rangle}{\|S\|_\infty \cdot \alpha}$

- Recall $S_{UM} = |\delta(U) \cap M| - 1$
- Abbreviate $Q_\ell := \{(U, M) : |\delta(U) \cap M| = \ell\}$
- Choose

$$W_{U,M} = \begin{cases} -\infty & |\delta(U) \cap M| = 1 \\ \frac{1}{|Q_3|} & |\delta(U) \cap M| = 3 \\ -\frac{1}{k-1} \cdot \frac{1}{|Q_k|} & |\delta(U) \cap M| = k \\ 0 & \text{otherwise.} \end{cases}$$

- Then $\langle W, S \rangle = 0 + 2 - 1 = 1$

Lemma

For k large, any rectangle R has $\langle W, R \rangle \leq 2^{-\Omega(n)}$.

Applying the Hyperplane bound (II)

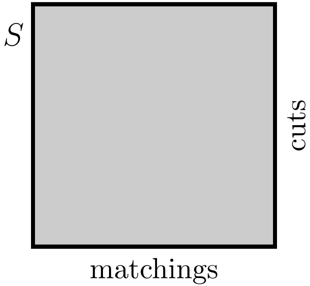
- **Uniform measure:** $\mu_\ell(R) := \frac{|R \cap Q_\ell|}{|Q_\ell|}$

Applying the Hyperplane bound (II)

- **Uniform measure:** $\mu_\ell(R) := \frac{|R \cap Q_\ell|}{|Q_\ell|}$

Main lemma

$\mu_1(R) = 0 \implies \mu_3(R) \leq O(\frac{1}{k^2}) \cdot \mu_k(R) + 2^{-\Omega(n)}$

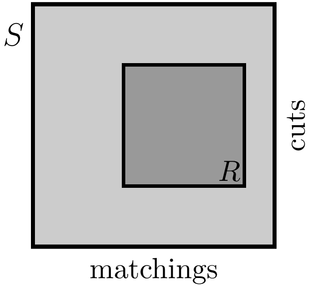


Applying the Hyperplane bound (II)

- **Uniform measure:** $\mu_\ell(R) := \frac{|R \cap Q_\ell|}{|Q_\ell|}$

Main lemma

$\mu_1(R) = 0 \implies \mu_3(R) \leq O(\frac{1}{k^2}) \cdot \mu_k(R) + 2^{-\Omega(n)}$

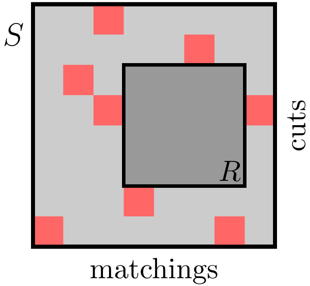


Applying the Hyperplane bound (II)

- **Uniform measure:** $\mu_\ell(R) := \frac{|R \cap Q_\ell|}{|Q_\ell|}$

Main lemma

$\mu_1(R) = 0 \implies \mu_3(R) \leq O(\frac{1}{k^2}) \cdot \mu_k(R) + 2^{-\Omega(n)}$

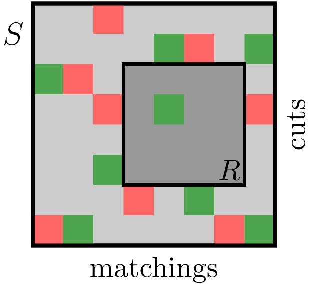


Applying the Hyperplane bound (II)

- **Uniform measure:** $\mu_\ell(R) := \frac{|R \cap Q_\ell|}{|Q_\ell|}$

Main lemma

$\mu_1(R) = 0 \implies \mu_3(R) \le O(\frac{1}{k^2}) \cdot \mu_k(R) + 2^{-\Omega(n)}$

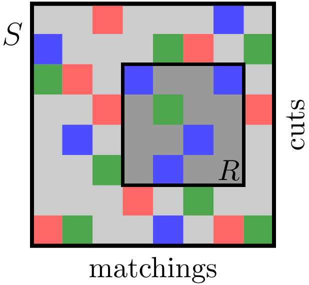


Applying the Hyperplane bound (II)

- **Uniform measure:** $\mu_\ell(R) := \frac{|R \cap Q_\ell|}{|Q_\ell|}$

Main lemma

$\mu_1(R) = 0 \implies \mu_3(R) \leq O(\frac{1}{k^2}) \cdot \mu_k(R) + 2^{-\Omega(n)}$

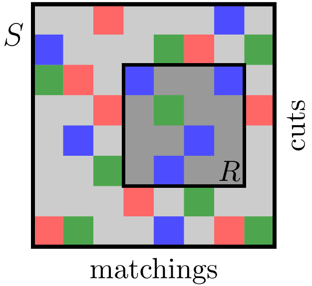


Applying the Hyperplane bound (II)

- **Uniform measure:** $\mu_\ell(R) := \frac{|R \cap Q_\ell|}{|Q_\ell|}$

Main lemma

$\mu_1(R) = 0 \implies \mu_3(R) \leq O(\frac{1}{k^2}) \cdot \mu_k(R) + 2^{-\Omega(n)}$



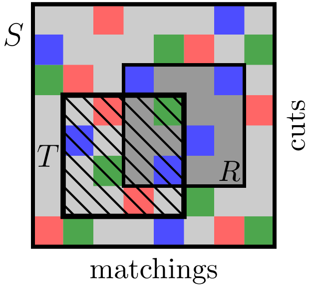
- **Technique:** Partition scheme [Razborov '91]

Applying the Hyperplane bound (II)

- **Uniform measure:** $\mu_\ell(R) := \frac{|R \cap Q_\ell|}{|Q_\ell|}$

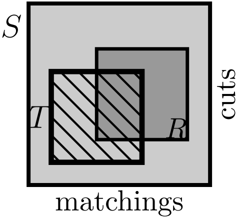
Main lemma

$\mu_1(R) = 0 \implies \mu_3(R) \leq O(\frac{1}{k^2}) \cdot \mu_k(R) + 2^{-\Omega(n)}$

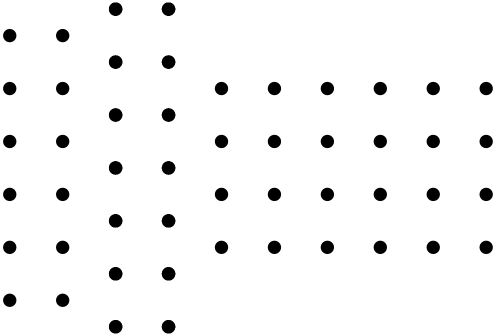


- **Technique:** Partition scheme [Razborov '91]

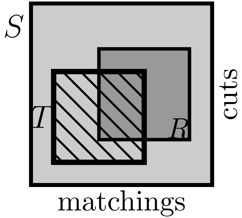
Partitions



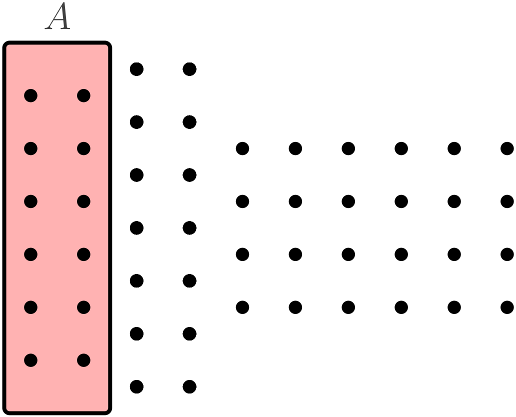
- Partition $T = (A, C, D, B)$



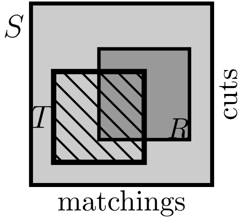
Partitions



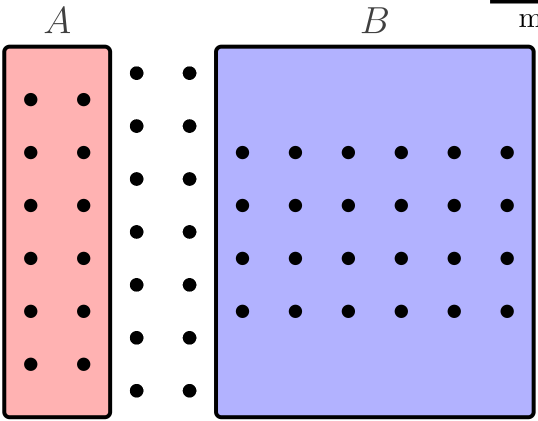
- Partition $T = (A, C, D, B)$



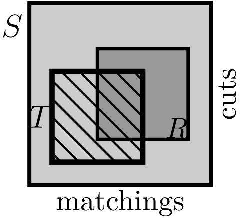
Partitions



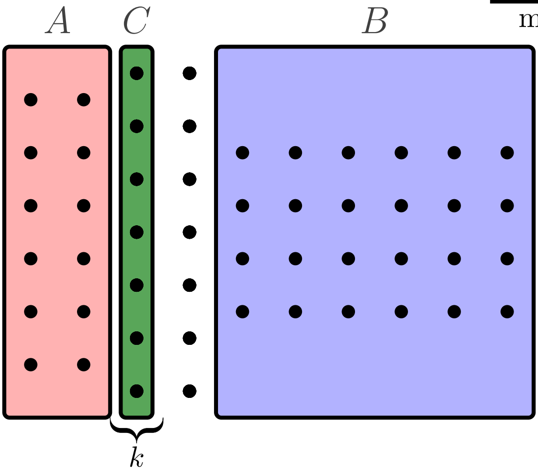
- Partition $T = (A, C, D, B)$



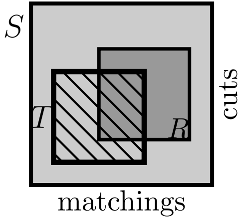
Partitions



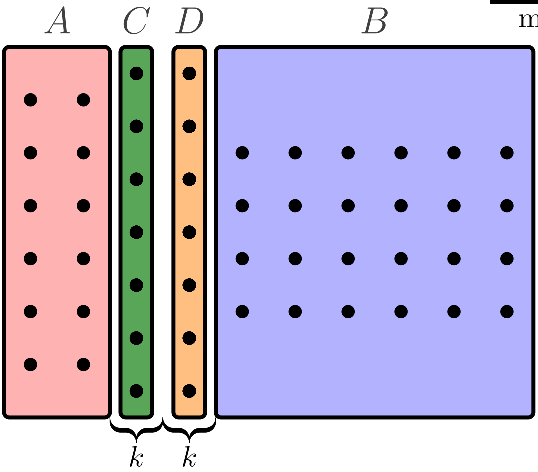
- Partition $T = (A, C, D, B)$



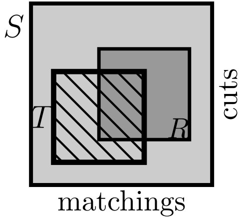
Partitions



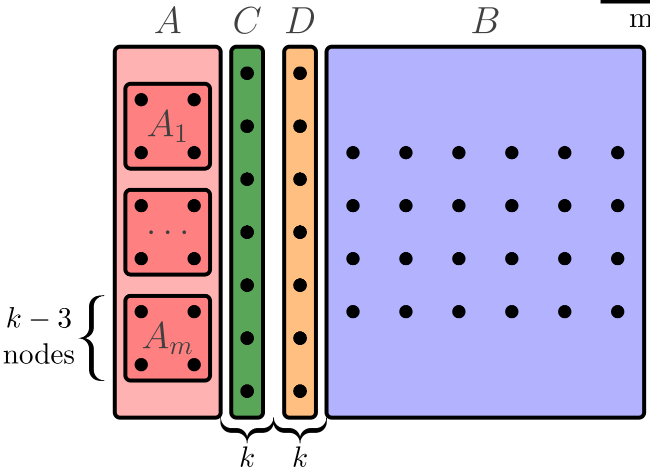
- Partition $T = (A, C, D, B)$



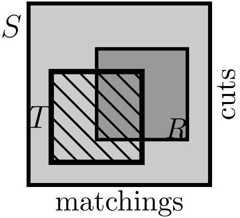
Partitions



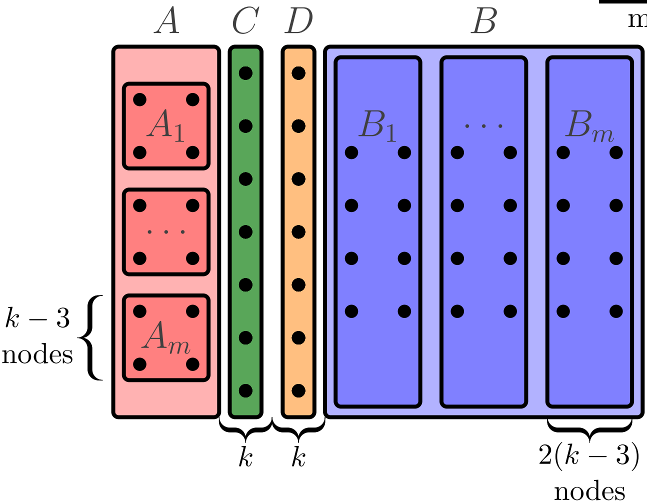
- Partition $T = (A, C, D, B)$



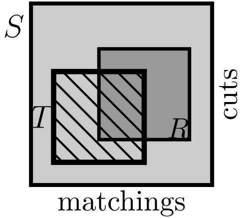
Partitions



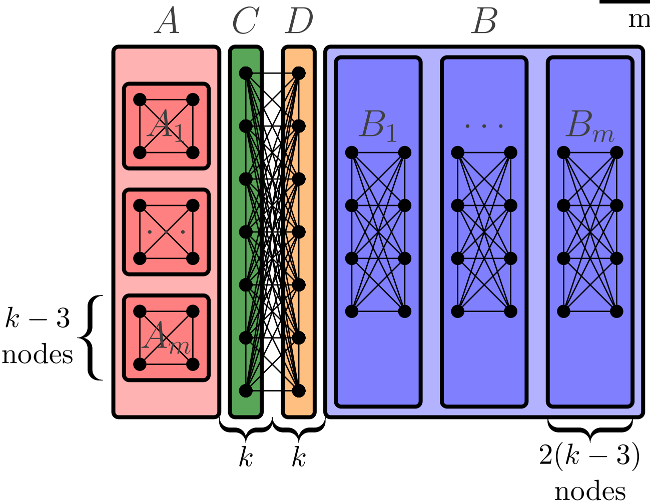
- Partition $T = (A, C, D, B)$



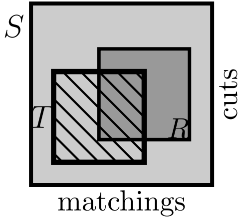
Partitions



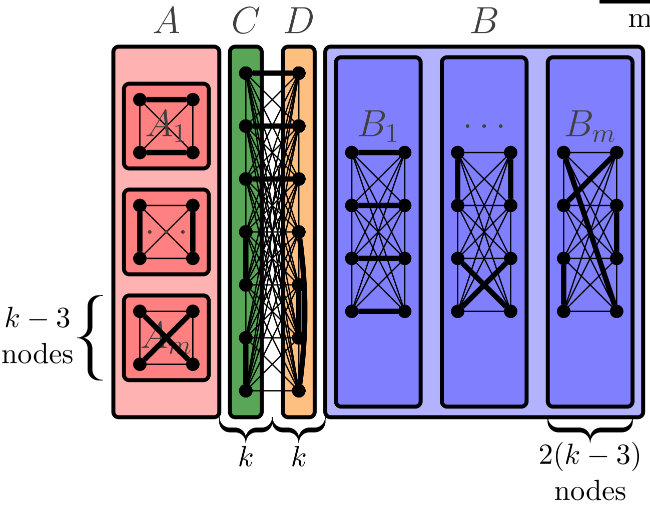
- Partition $T = (A, C, D, B)$
- Edges $E(T)$



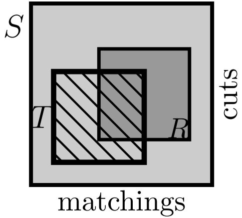
Partitions



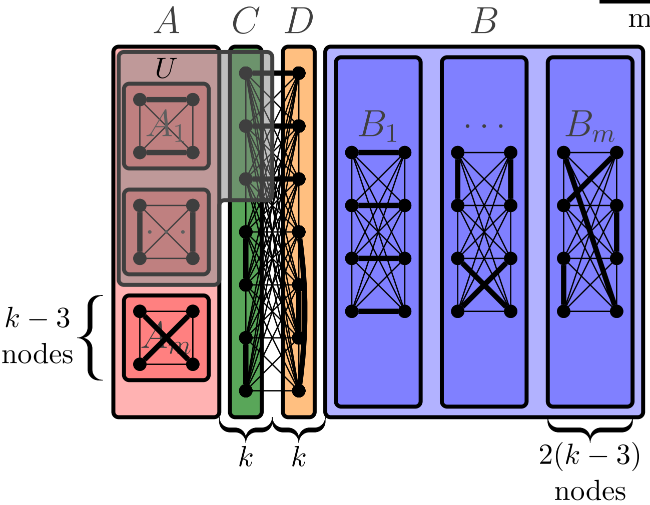
- Partition $T = (A, C, D, B)$
- Edges $E(T)$



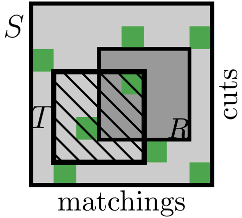
Partitions

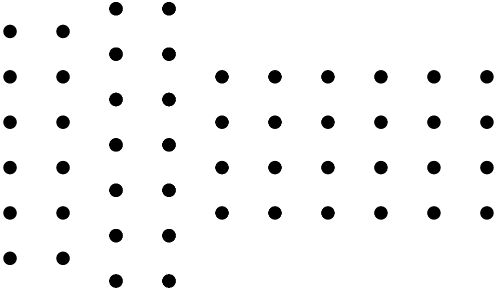


- Partition $T = (A, C, D, B)$
- Edges $E(T)$



Rewriting $\mu_3(R)$

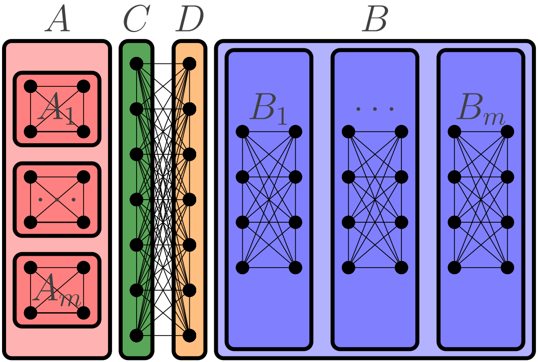


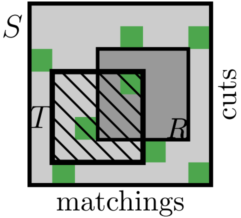


Randomly generate $(U, M) \sim Q_3$:

$$\mu_3(R) =$$

Rewriting $\mu_3(R)$



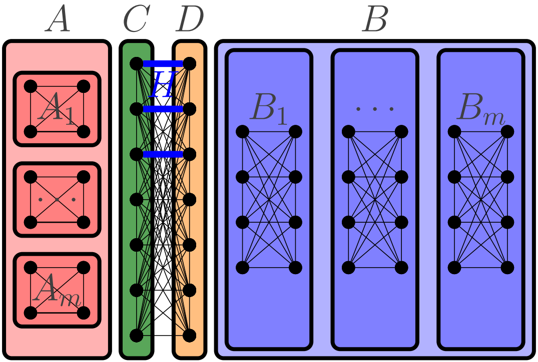


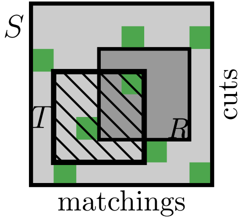
Randomly generate $(U, M) \sim Q_3$:

1. Choose T

$$\mu_3(R) = \mathop{\mathbb{E}}_{T}\Big[\qquad \Big]$$

Rewriting $\mu_3(R)$



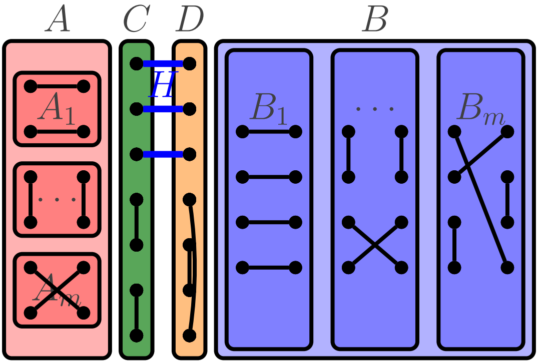


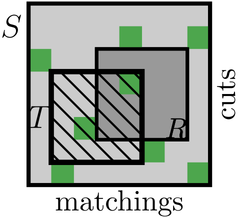
Randomly generate $(U, M) \sim Q_3$:

1. Choose T
2. Choose 3 edges $H \subseteq C \times D$

$$\mu_3(R) = \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[\qquad \right]\right]$$

Rewriting $\mu_3(R)$



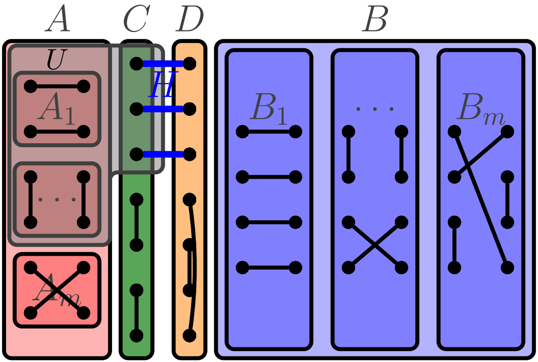


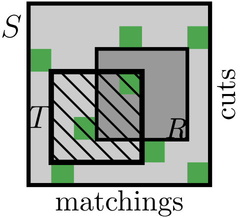
Randomly generate $(U, M) \sim Q_3$:

1. Choose T
2. Choose 3 edges $H \subseteq C \times D$
3. Choose $M \supseteq H$ (not cutting any other edge in $C \times D$)

$$\mu_3(R) = \mathop{\mathbb{E}}_{T}\Big[\mathop{\mathbb{E}}_{|H|=3}\Big[\qquad\qquad \Big]\Big]$$

Rewriting $\mu_3(R)$



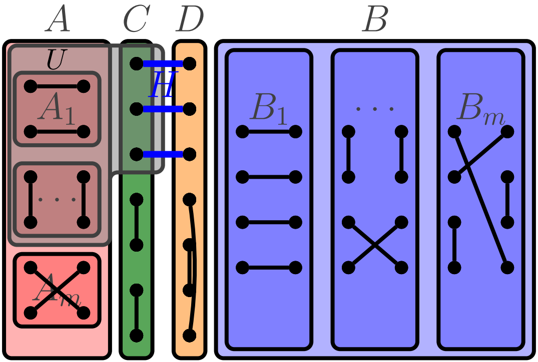


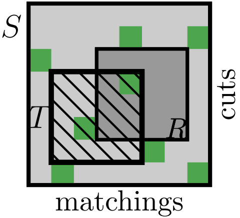
Randomly generate $(U, M) \sim Q_3$:

1. Choose T
2. Choose 3 edges $H \subseteq C \times D$
3. Choose $M \supseteq H$ (not cutting any other edge in $C \times D$)
4. Choose U cutting H (not cutting any A_i)

$$\mu_3(R) = \mathop{\mathbb{E}}_{T}\Big[\mathop{\mathbb{E}}_{|H|=3}\Big[\Pr[(U, M) \in R \mid T, H]\Big]\Big]$$

Rewriting $\mu_3(R)$



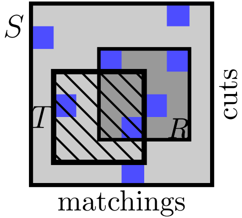


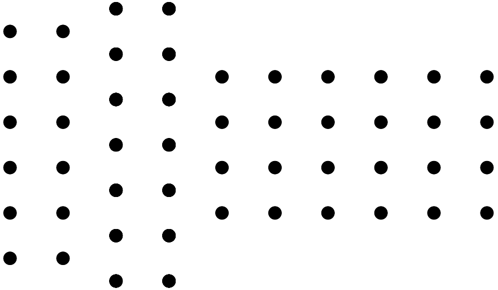
Randomly generate $(U, M) \sim Q_3$:

1. Choose T
2. Choose 3 edges $H \subseteq C \times D$
3. Choose $M \supseteq H$ (not cutting any other edge in $C \times D$)
4. Choose U cutting H (not cutting any A_i)

$$\mu_3(R) = \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[\Pr[U \in R \mid T, H] \cdot \Pr[M \in R \mid T, H]\right]\right]$$

Rewriting $\mu_k(R)$

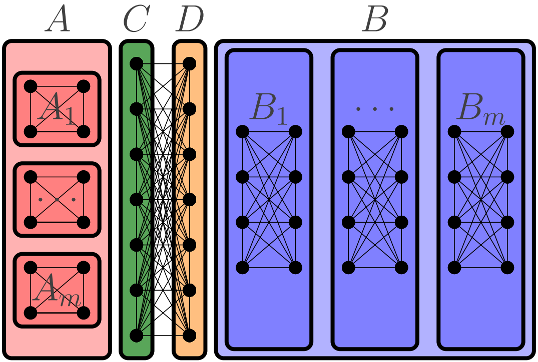


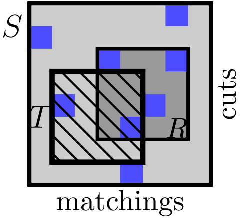


Randomly generate $(U, M) \sim Q_k$:

$\mu_k(\mathcal{R}) =$

Rewriting $\mu_k(R)$



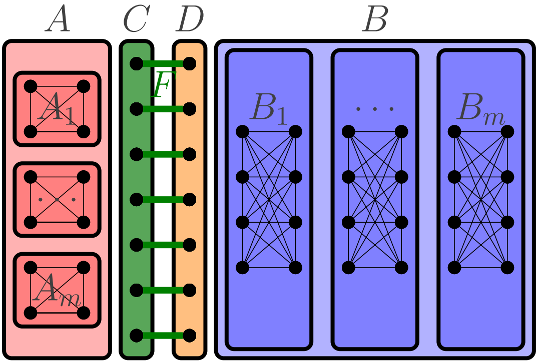


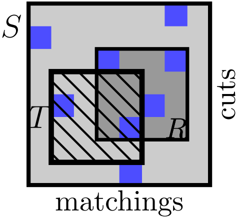
Randomly generate $(U, M) \sim Q_k$:

1. Choose T

$$\mu_k(\mathcal{R}) = \mathop{\mathbb{E}}_{T} \Big[\qquad \Big]$$

Rewriting $\mu_k(R)$



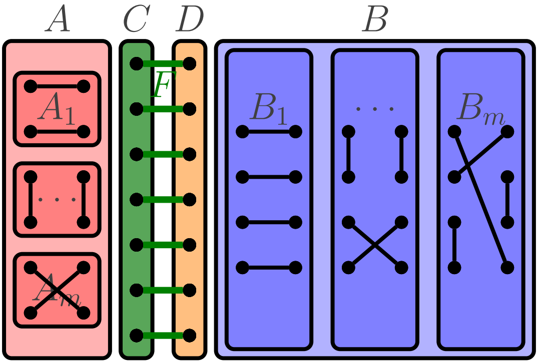


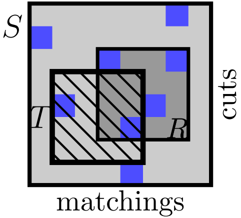
Randomly generate $(U, M) \sim Q_k$:

1. Choose T
2. Choose k edges $F \subseteq C \times D$

$$\mu_k(\mathcal{R}) = \mathop{\mathbb{E}}_{T}\Big[\mathop{\mathbb{E}}_{|F|=k} \Big[\qquad \Big]\Big]$$

Rewriting $\mu_k(R)$



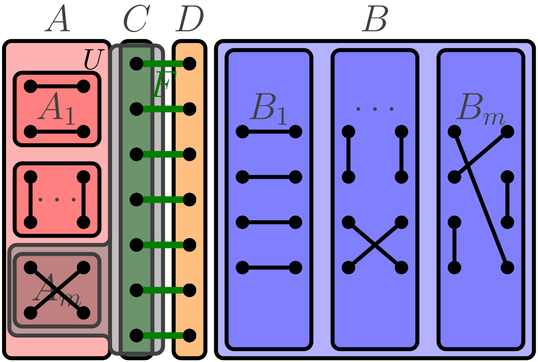


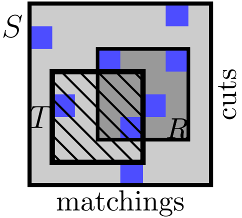
Randomly generate $(U, M) \sim Q_k$:

1. Choose T
2. Choose k edges $F \subseteq C \times D$
3. Choose $M \supseteq F$

$$\mu_k(\mathcal{R}) = \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|F|=k} \left[\Pr[M \in \mathcal{R} \mid T, H] \right]\right]$$

Rewriting $\mu_k(R)$





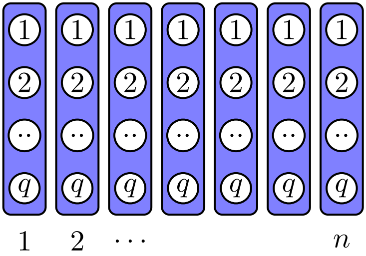
Randomly generate $(U, M) \sim Q_k$:

1. Choose T
2. Choose k edges $F \subseteq C \times D$
3. Choose $M \supseteq F$
4. Choose $U \supseteq C$ (not cutting any A_i)

$$\mu_k(\mathcal{R}) = \mathop{\mathbb{E}}_{T}\Big[\mathop{\mathbb{E}}_{|F|=k}\Big[\Pr[M \in \mathcal{R} \mid T, H] \cdot \Pr[U \in R \mid T, H]\Big]\Big]$$

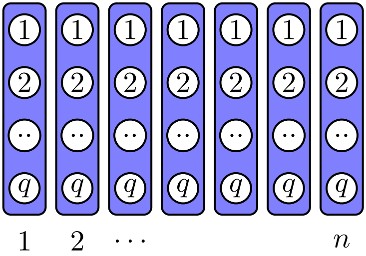
Pseudorandom-behaviour of large sets

- Consider vectors $X \subseteq [q]^n$.



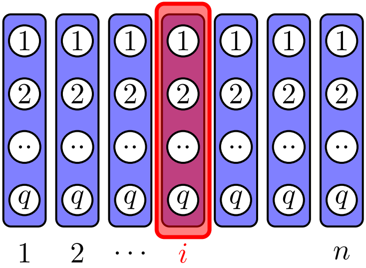
Pseudorandom-behaviour of large sets

- Consider vectors $X \subseteq [q]^n$.
- Draw $x \sim X$.



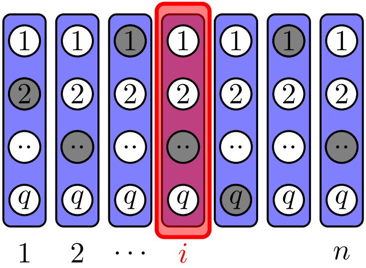
Pseudorandom-behaviour of large sets

- Consider vectors $X \subseteq [q]^n$.
- Draw $x \sim X$.



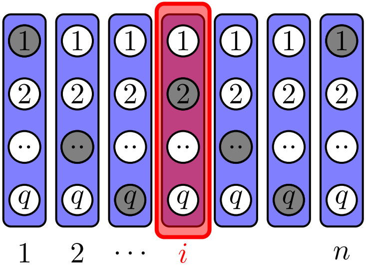
Pseudorandom-behaviour of large sets

- Consider vectors $X \subseteq [q]^n$.
- Draw $x \sim X$.



Pseudorandom-behaviour of large sets

- Consider vectors $X \subseteq [q]^n$.
- Draw $x \sim X$.

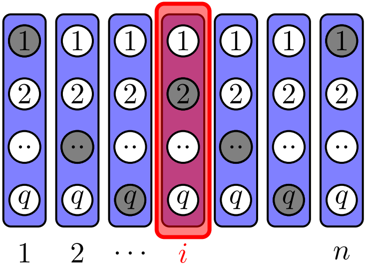


Pseudorandom-behaviour of large sets

- Consider vectors $X \subseteq [q]^n$.
- Draw $x \sim X$.

Lemma

$|X|$ **large** $\Rightarrow$ for most indices x_i is **approx. uniform**

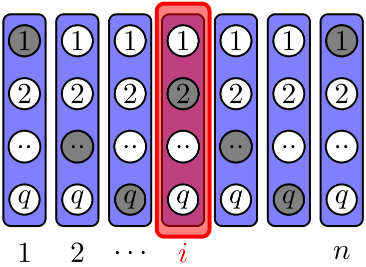


Pseudorandom-behaviour of large sets

- Consider vectors $X \subseteq [q]^n$.
- Draw $x \sim X$.

Lemma

εn **biased** indices $\Rightarrow \frac{|X|}{q^n} \leq 2^{-\Omega(n)}$.



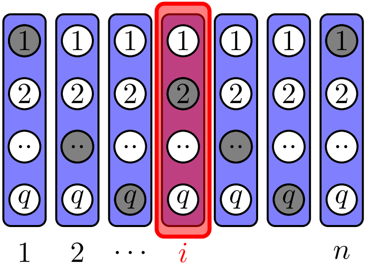
Pseudorandom-behaviour of large sets

- Consider vectors $X \subseteq [q]^n$.
- Draw $x \sim X$.

Lemma

εn **biased** indices $\Rightarrow \frac{|X|}{q^n} \leq 2^{-\Omega(n)}$.

$$\log_2(|X|) = H(x)$$



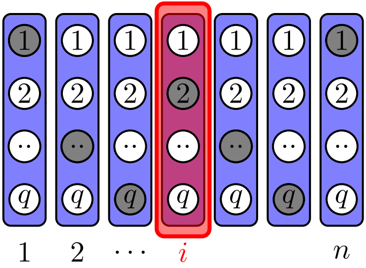
Pseudorandom-behaviour of large sets

- Consider vectors $X \subseteq [q]^n$.
- Draw $x \sim X$.

Lemma

εn **biased** indices $\Rightarrow \frac{|X|}{q^n} \leq 2^{-\Omega(n)}$.

$$\log_2(|X|) = H(x) \leq \sum_{i=1}^{n} H(x_i)$$



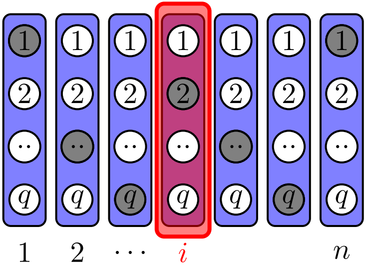
Pseudorandom-behaviour of large sets

- Consider vectors $X \subseteq [q]^n$.
- Draw $x \sim X$.

Lemma

εn **biased** indices $\Rightarrow \frac{|X|}{q^n} \leq 2^{-\Omega(n)}$.

$$\log_2(|X|) = H(x) \leq \sum_{i \text{ biased}} H(x_i) + \sum_{i \text{ unbiased}} H(x_i)$$



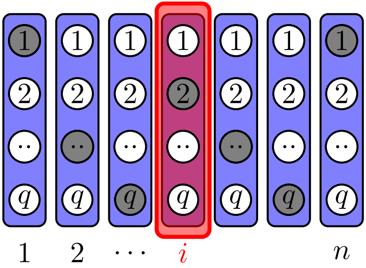
Pseudorandom-behaviour of large sets

- Consider vectors $X \subseteq [q]^n$.
- Draw $x \sim X$.

Lemma

εn **biased** indices $\Rightarrow \frac{|X|}{q^n} \leq 2^{-\Omega(n)}$.

$$\log_2(|X|) = H(x) \leq \sum_{i \text{ biased}} \underbrace{H(x_i)}_{\leq \log_2(q) - \Theta(1)} + \sum_{i \text{ unbiased}} \underbrace{H(x_i)}_{\leq \log_2(q)}$$



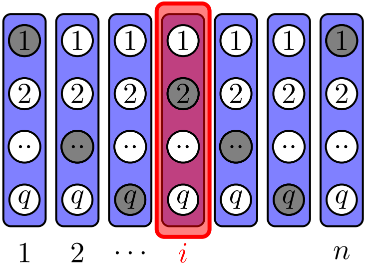
Pseudorandom-behaviour of large sets

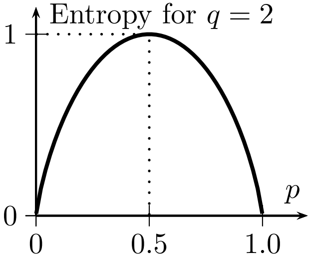
- Consider vectors $X \subseteq [q]^n$.
- Draw $x \sim X$.

Lemma

εn **biased** indices $\Rightarrow \frac{|X|}{q^n} \leq 2^{-\Omega(n)}$.

$$\log_2(|X|) = H(x) \leq \sum_{i \text{ biased}} \underbrace{H(x_i)}_{\leq \log_2(q) - \Theta(1)} + \sum_{i \text{ unbiased}} \underbrace{H(x_i)}_{\leq \log_2(q)}$$





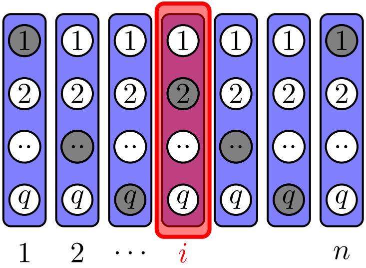
Pseudorandom-behaviour of large sets

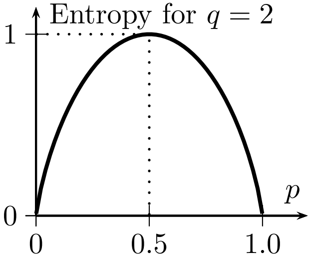
- Consider vectors $X \subseteq [q]^n$.
- Draw $x \sim X$.

Lemma

εn **biased** indices $\Rightarrow \frac{|X|}{q^n} \leq 2^{-\Omega(n)}$.

$$\log_2(|X|) = H(x) \leq \sum_{i \text{ biased}} \underbrace{H(x_i)}_{\leq \log_2(q) - \Theta(1)} + \sum_{i \text{ unbiased}} \underbrace{H(x_i)}_{\leq \log_2(q)} \leq n \log_2(q) - \Omega(n)$$





Pseudorandom-behaviour of large sets

- Consider vectors $X \subseteq [q]^n$.
- Draw $x \sim X$.

Lemma

εn **biased** indices $\Rightarrow \frac{|X|}{q^n} \leq 2^{-\Omega(n)}$.

$$\log_2(|X|) = H(x) \leq \sum_{i \text{ biased}} \underbrace{H(x_i)}_{\leq \log_2(q) - \Theta(1)} + \sum_{i \text{ unbiased}} \underbrace{H(x_i)}_{\leq \log_2(q)} \leq n \log_2(q) - \Omega(n)$$

Corollary

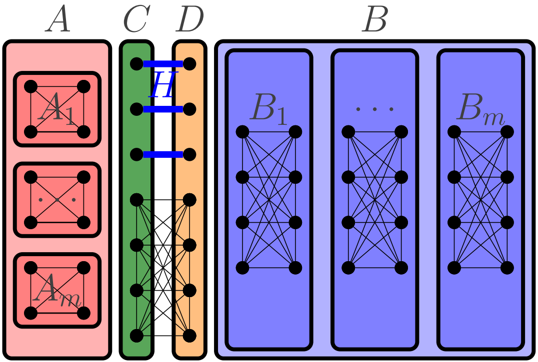
If X large, then for most i

$$\Pr_{x \sim [q]^n}[x \in X] \approx \Pr_{x \sim [q]^n}[x \in X \mid x_i = j]$$

M-good

Definition

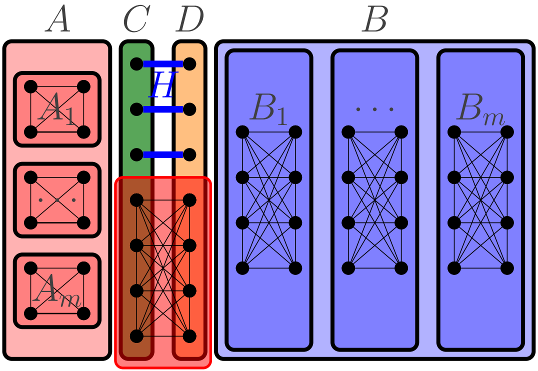
(T, H) ***M*-good** if $M \sim \{M \in R \mid H \subseteq M \subseteq E(T)\}$ is ε-uniform on $(C \cup D) \backslash V(H)$.



M-good

Definition

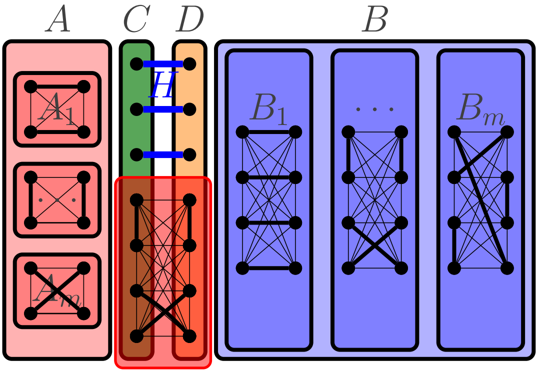
(T, H) M-**good** if $M \sim \{M \in R \mid H \subseteq M \subseteq E(T)\}$ is ε-uniform on $(C \cup D) \setminus V(H)$.



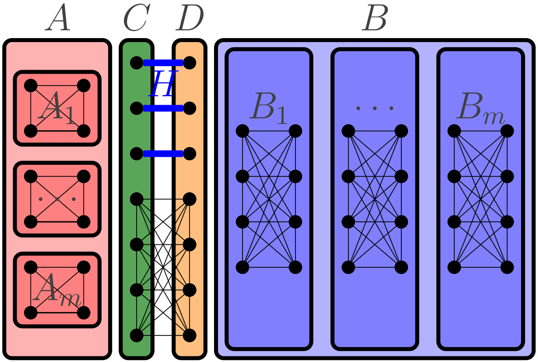
M-good

Definition

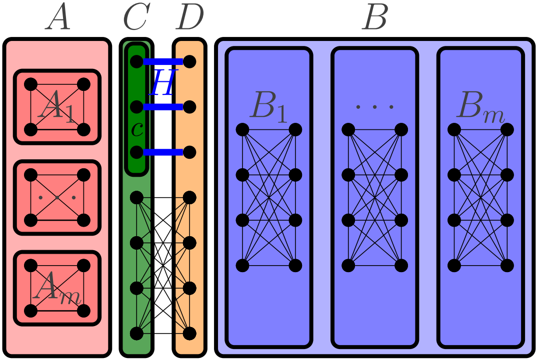
(T, H) **M-good** if $M \sim \{M \in R \mid H \subseteq M \subseteq E(T)\}$ is ε-uniform on $(C \cup D) \backslash V(H)$.



U-good



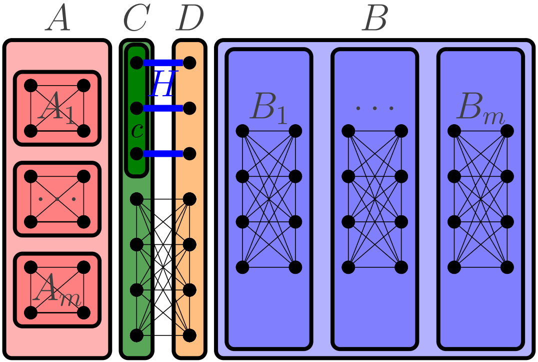
U-good



U-good

Definition

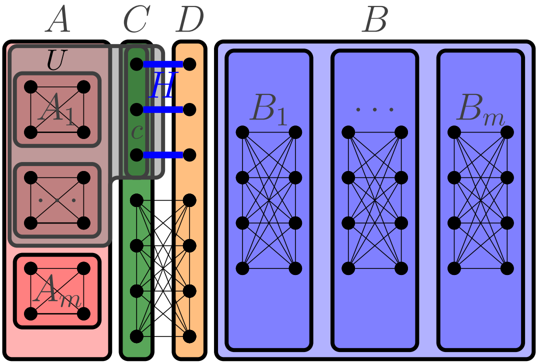
(T, H) **U-good** if $U \sim \{U \in R \mid c \subseteq U; \text{doesn't cut any } A_i\}$ has $\Pr[U \cap C = c] \approx \frac{1}{2} \approx \Pr[U \cap C = C]$.



U-good

Definition

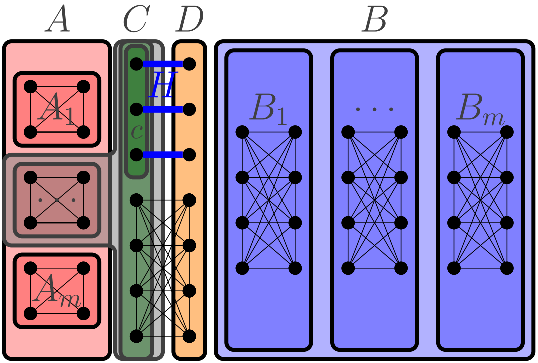
(T, H) ***U*-good** if $U \sim \{U \in R \mid c \subseteq U; \text{doesn't cut any } A_i\}$ has $\Pr[U \cap C = c] \approx \frac{1}{2} \approx \Pr[U \cap C = C]$.



U-good

Definition

(T, H) **U-good** if $U \sim \{U \in R \mid c \subseteq U; \text{doesn't cut any } A_i\}$ has $\Pr[U \cap C = c] \approx \frac{1}{2} \approx \Pr[U \cap C = C]$.



Splitting $\mu_3(R)$

$\mu_3(R)$

Splitting $\mu_3(R)$

$$\mu_3(R) \quad = \quad \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[\Pr[(U,M) \in R \mid T,H]\right]\right]$$

Splitting $\mu_3(R)$

$$\begin{aligned}
\mu_3(R) \quad &= \quad \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[\Pr[(U,M)\in R \mid T,H]\right]\right] \\
&\leq \quad \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[\mathtt{GOOD}(T,H)\cdot\Pr[(U,M)\in R \mid T,H]\right]\right] \\
&\quad + \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[M{-}\mathtt{BAD}(T,H)\cdot\Pr[(U,M)\in R \mid T,H]\right]\right] \\
&\quad + \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[U{-}\mathtt{BAD}(T,H)\cdot\Pr[(U,M)\in R \mid T,H]\right]\right]
\end{aligned}$$

Splitting $\mu_3(R)$

$$
\begin{aligned}
\mu_3(R) \quad &= \quad \mathop{\mathbb{E}}_{T}\Big[\mathop{\mathbb{E}}_{|H|=3}\big[\Pr[(U,M)\in R \mid T,H]\big]\Big] \\
&\leq \quad \mathop{\mathbb{E}}_{T}\Big[\mathop{\mathbb{E}}_{|H|=3}\Big[\texttt{GOOD}(T,H)\cdot\Pr[(U,M)\in R \mid T,H]\Big]\Big] \\
&\quad + \mathop{\mathbb{E}}_{T}\Big[\mathop{\mathbb{E}}_{|H|=3}\Big[M\text{−}\texttt{BAD}(T,H)\cdot\Pr[(U,M)\in R \mid T,H]\Big]\Big] \\
&\quad + \mathop{\mathbb{E}}_{T}\Big[\mathop{\mathbb{E}}_{|H|=3}\Big[U\text{−}\texttt{BAD}(T,H)\cdot\Pr[(U,M)\in R \mid T,H]\Big]\Big]
\end{aligned}
$$

$\leq O(\frac{1}{k^2})\mu_k(R)$

Splitting $\mu_3(R)$

$$
\begin{aligned}
\mu_3(R) &= \mathop{\mathbb{E}}_{T}\Big[\mathop{\mathbb{E}}_{|H|=3}\big[\Pr[(U,M)\in R \mid T,H]\big]\Big] \\
&\le \mathop{\mathbb{E}}_{T}\Big[\mathop{\mathbb{E}}_{|H|=3}\Big[\mathtt{GOOD}(T,H)\cdot \Pr[(U,M)\in R \mid T,H]\Big]\Big] \\
&\quad + \mathop{\mathbb{E}}_{T}\Big[\mathop{\mathbb{E}}_{|H|=3}\Big[M{-}\mathtt{BAD}(T,H)\cdot \Pr[(U,M)\in R \mid T,H]\Big]\Big] \\
&\quad + \mathop{\mathbb{E}}_{T}\Big[\mathop{\mathbb{E}}_{|H|=3}\Big[U{-}\mathtt{BAD}(T,H)\cdot \Pr[(U,M)\in R \mid T,H]\Big]\Big]
\end{aligned}
$$

$\le O(\frac{1}{k^2})\mu_k(R)$

$\le \varepsilon\cdot\mu_3(R) + 2^{-\Omega(n)}$

Splitting $\mu_3(R)$

$$\begin{aligned}
\mu_3(R) \quad &= \quad \mathop{\mathbb{E}}_{T}\Big[\mathop{\mathbb{E}}_{|H|=3}\big[\Pr[(U,M)\in R \mid T,H]\big]\Big] \\
&\leq \quad \mathop{\mathbb{E}}_{T}\Big[\mathop{\mathbb{E}}_{|H|=3}\Big[\mathtt{GOOD}(T,H)\cdot\Pr[(U,M)\in R \mid T,H]\Big]\Big] \\
&\quad + \mathop{\mathbb{E}}_{T}\Big[\mathop{\mathbb{E}}_{|H|=3}\Big[M\text{-}\mathrm{BAD}(T,H)\cdot\Pr[(U,M)\in R \mid T,H]\Big]\Big] \\
&\quad + \mathop{\mathbb{E}}_{T}\Big[\mathop{\mathbb{E}}_{|H|=3}\Big[U\text{-}\mathrm{BAD}(T,H)\cdot\Pr[(U,M)\in R \mid T,H]\Big]\Big]
\end{aligned}$$

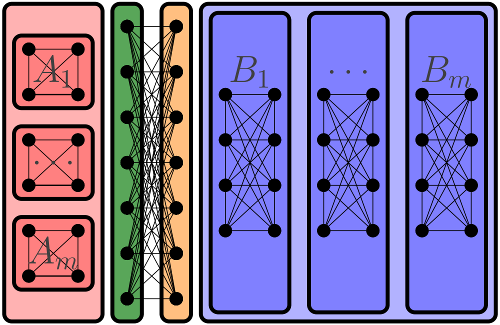
$\leq O(\frac{1}{k^2})\mu_k(R)$

$\leq \varepsilon \cdot \mu_3(R) + 2^{-\Omega(n)}$

$\leq \varepsilon \cdot \mu_3(R) + 2^{-\Omega(n)}$

Contribution of good partitions

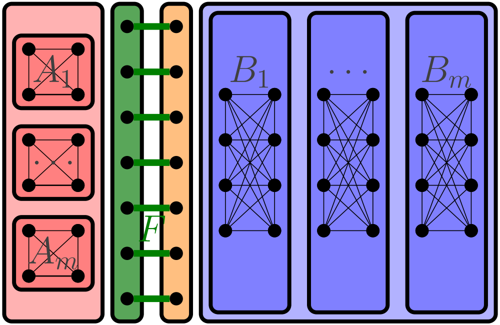
For T



Contribution of good partitions

For T and $F \subseteq C \times D$ with $|F| = k$ compare:

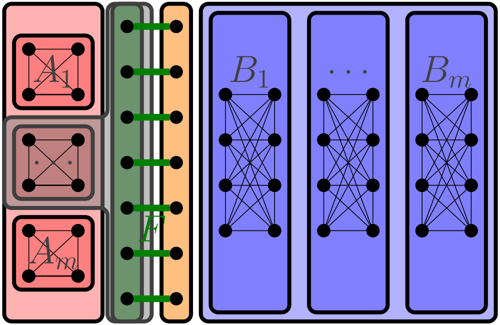
- Contribution to $\mu_k(R)$:



Contribution of good partitions

For T and $F \subseteq C \times D$ with $|F| = k$ compare:

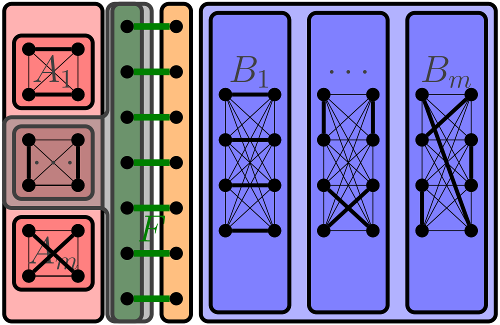
- Contribution to $\mu_k(R)$:



Contribution of good partitions

For T and $F \subseteq C \times D$ with $|F| = k$ compare:

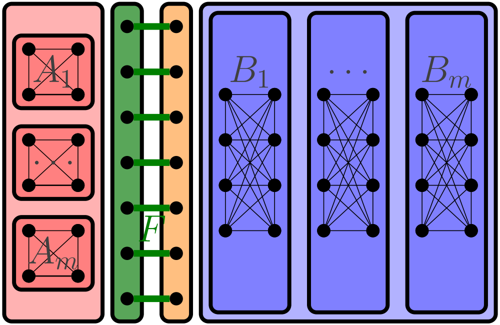
- Contribution to $\mu_k(R)$: $\Pr[(U, M) \in R \mid T, F]$



Contribution of good partitions

For T and $F \subseteq C \times D$ with $|F| = k$ compare:

- Contribution to $\mu_k(R)$: $\Pr[(U, M) \in R \mid T, F]$
- Contribution to $\mu_3(R)$:

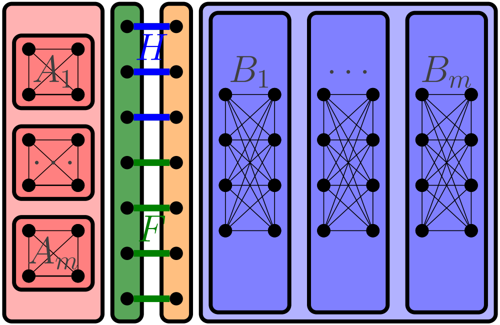


Contribution of good partitions

For T and $F \subseteq C \times D$ with $|F| = k$ compare:

- Contribution to $\mu_k(R)$: $\Pr[(U, M) \in R \mid T, F]$
- Contribution to $\mu_3(R)$:

$$\mathop{\mathbb{E}}_{H \sim \binom{F}{3}} [\texttt{GOOD}(T, H) \cdot \Pr[(U, M) \in R \mid T, H]]$$

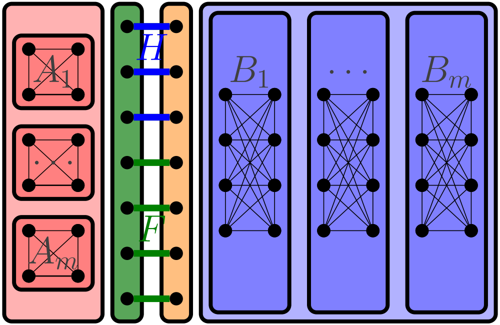


Contribution of good partitions

For T and $F \subseteq C \times D$ with $|F| = k$ compare:

- Contribution to $\mu_k(R)$: $\Pr[(U,M) \in R \mid T, F]$
- Contribution to $\mu_3(R)$:

$$\mathop{\mathbb{E}}_{H\sim\binom{F}{3}}[\mathtt{GOOD}(T,H)\cdot \Pr[(U,M)\in R \mid T,H]] \lesssim \Pr[\ldots \mid T,F]\cdot \mathop{\Pr}_{H\sim\binom{F}{3}}[\mathtt{GOOD}(T,H)]$$

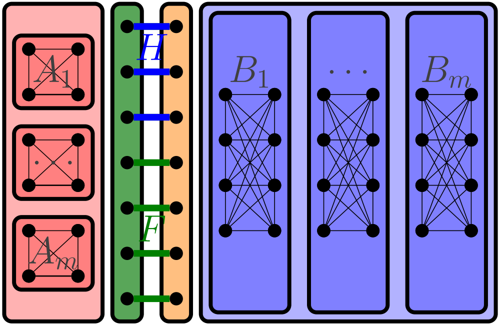


Contribution of good partitions

For T and $F \subseteq C \times D$ with $|F| = k$ compare:

- Contribution to $\mu_k(R)$: $\Pr[(U, M) \in R \mid T, F]$
- Contribution to $\mu_3(R)$:

$$\mathop{\mathbb{E}}_{H \sim \binom{F}{3}} [\mathtt{GOOD}(T, H) \cdot \Pr[(U, M) \in R \mid T, H]] \lesssim \Pr[\ldots \mid T, F] \cdot \underbrace{\Pr_{H \sim \binom{F}{3}} [\mathtt{GOOD}(T, H)]}_{= O(1/k^2)}$$



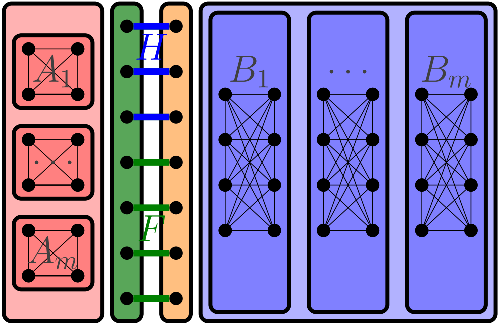
Contribution of good partitions

For T and $F \subseteq C \times D$ with $|F| = k$ compare:

- Contribution to $\mu_k(R)$: $\Pr[(U, M) \in R \mid T, F]$
- Contribution to $\mu_3(R)$:

$$\mathop{\mathbb{E}}_{H \sim \binom{F}{3}} [\mathtt{GOOD}(T, H) \cdot \Pr[(U, M) \in R \mid T, H]] \lesssim \Pr[\ldots \mid T, F] \cdot \underbrace{\Pr_{H \sim \binom{F}{3}} [\mathtt{GOOD}(T, H)]}_{= O(1/k^2)}$$

- Suffices to show: $H, H^* \subseteq F$ good $\Rightarrow |H \cap H^*| \geq 2$



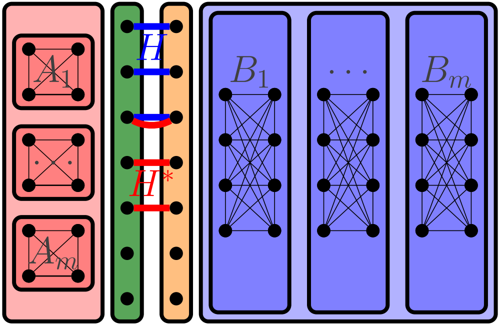
Contribution of good partitions

For T and $F \subseteq C \times D$ with $|F| = k$ compare:

- Contribution to $\mu_k(R)$: $\Pr[(U, M) \in R \mid T, F]$
- Contribution to $\mu_3(R)$:

$$\mathop{\mathbb{E}}_{H\sim\binom{F}{3}}[\mathtt{GOOD}(T,H)\cdot\Pr[(U,M)\in R \mid T,H]] \lesssim \Pr[\ldots \mid T,F]\cdot \underbrace{\Pr_{H\sim\binom{F}{3}}[\mathtt{GOOD}(T,H)]}_{=O(1/k^2)}$$

- Suffices to show: $H, H^* \subseteq F$ good $\Rightarrow |H \cap H^*| \geq 2$
- Suppose $|H \cap H^*| \leq 1$



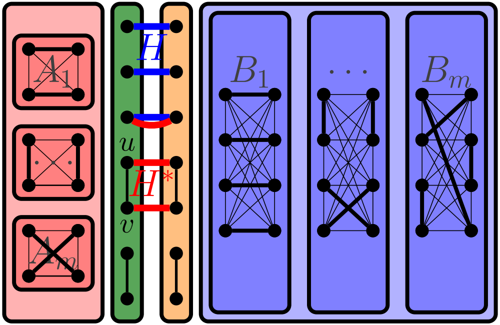
Contribution of good partitions

For T and $F \subseteq C \times D$ with $|F| = k$ compare:

- Contribution to $\mu_k(R)$: $\Pr[(U,M) \in R \mid T, F]$
- Contribution to $\mu_3(R)$:

$$\mathop{\mathbb{E}}_{H \sim \binom{F}{3}} [\mathtt{GOOD}(T,H) \cdot \Pr[(U,M) \in R \mid T, H]] \lesssim \Pr[\ldots \mid T, F] \cdot \underbrace{\Pr_{H \sim \binom{F}{3}} [\mathtt{GOOD}(T,H)]}_{=O(1/k^2)}$$

- Suffices to show: $H, H^* \subseteq F$ good $\Rightarrow |H \cap H^*| \geq 2$
- Suppose $|H \cap H^*| \leq 1$
- (T, H) good $\Rightarrow \exists M : \{u, v\} \in M$



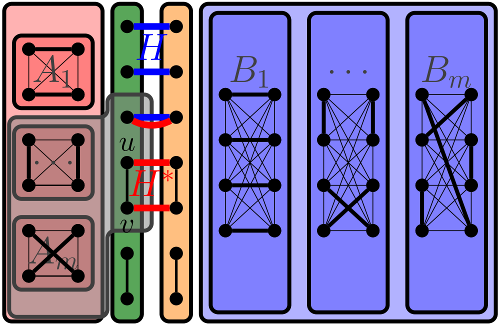
Contribution of good partitions

For T and $F \subseteq C \times D$ with $|F| = k$ compare:

- Contribution to $\mu_k(R)$: $\Pr[(U,M) \in R \mid T, F]$
- Contribution to $\mu_3(R)$:

$$\mathop{\mathbb{E}}_{H \sim \binom{F}{3}} [\mathtt{GOOD}(T,H) \cdot \Pr[(U,M) \in R \mid T,H]] \lesssim \Pr[\ldots \mid T,F] \cdot \underbrace{\Pr_{H \sim \binom{F}{3}} [\mathtt{GOOD}(T,H)]}_{=O(1/k^2)}$$

- Suffices to show: $H, H^* \subseteq F$ good $\Rightarrow |H \cap H^*| \geq 2$
- Suppose $|H \cap H^*| \leq 1$
- (T,H) good $\Rightarrow \exists M : \{u,v\} \in M$
- (T,H^*) good $\Rightarrow \exists U : u,v \in U$



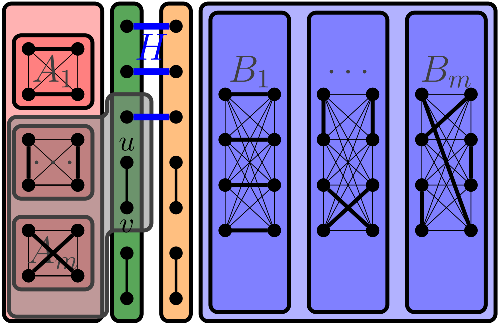
Contribution of good partitions

For T and $F \subseteq C \times D$ with $|F| = k$ compare:

- Contribution to $\mu_k(R)$: $\Pr[(U,M) \in R \mid T, F]$
- Contribution to $\mu_3(R)$:

$$\mathop{\mathbb{E}}_{H \sim \binom{F}{3}} [\texttt{GOOD}(T,H) \cdot \Pr[(U,M) \in R \mid T, H]] \lesssim \Pr[\ldots \mid T, F] \cdot \underbrace{\Pr_{H \sim \binom{F}{3}} [\texttt{GOOD}(T,H)]}_{=O(1/k^2)}$$

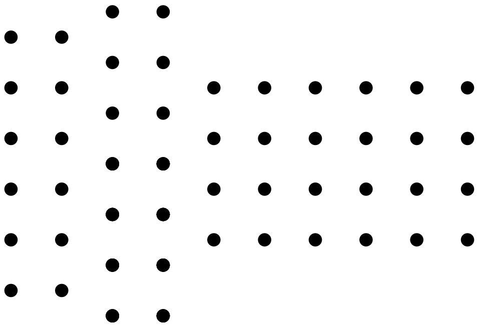
- Suffices to show: $H, H^* \subseteq F$ good $\Rightarrow |H \cap H^*| \geq 2$
- Suppose $|H \cap H^*| \leq 1$
- (T, H) good $\Rightarrow \exists M : \{u, v\} \in M$
- (T, H^*) good $\Rightarrow \exists U : u, v \in U$
- $|\delta(U) \cap M| = 1$ Contradiction!



Most partitions are good

Lemma

$\Pr[(T, H) \text{ is } M\text{-bad}] \leq \varepsilon$

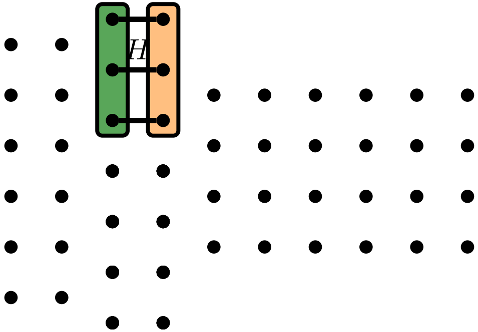


Most partitions are good

Lemma

$\Pr[(T,H) \text{ is } M\text{-bad}] \leq \varepsilon$

- Pick H

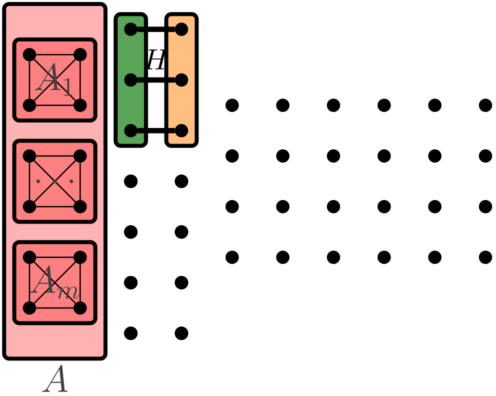


Most partitions are good

Lemma

$\Pr[(T, H) \text{ is } M\text{-bad}] \leq \varepsilon$

- Pick H, A

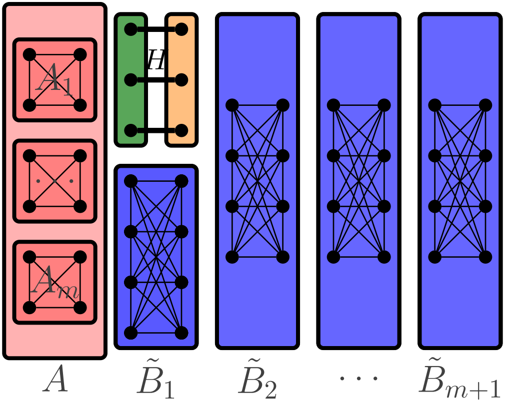


Most partitions are good

Lemma

$\Pr[(T, H) \text{ is } M\text{-bad}] \leq \varepsilon$

- Pick H, A, $\tilde{B}_1, \ldots, \tilde{B}_{m+1}$.

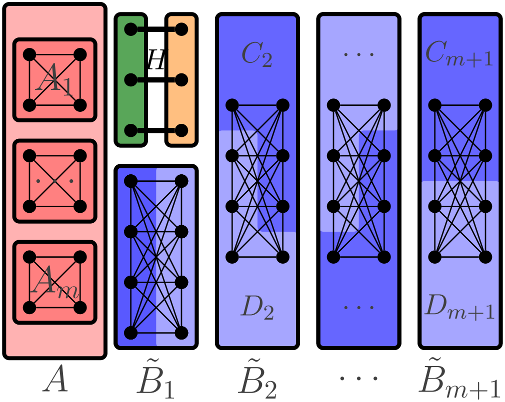


Most partitions are good

Lemma

$\Pr[(T,H) \text{ is } M\text{-bad}] \leq \varepsilon$

- Pick H, A, $\tilde{B}_1, \ldots, \tilde{B}_{m+1}$. Split $\tilde{B}_i = C_i \dot{\cup} D_i$.

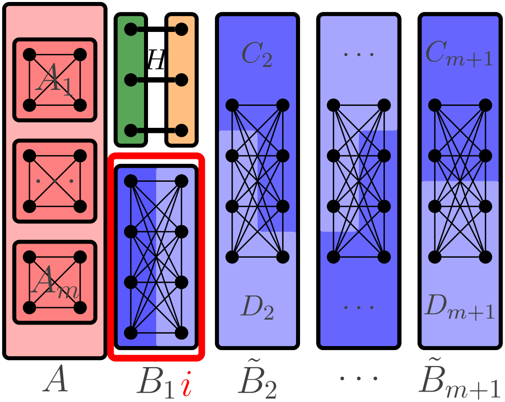


Most partitions are good

Lemma

$\Pr[(T, H) \text{ is } M\text{-bad}] \leq \varepsilon$

- Pick H, A, $\tilde{B}_1, \ldots, \tilde{B}_{m+1}$. Split $\tilde{B}_i = C_i \dot{\cup} D_i$.
- Pick randomly $i \in \{1, \ldots, m\}$

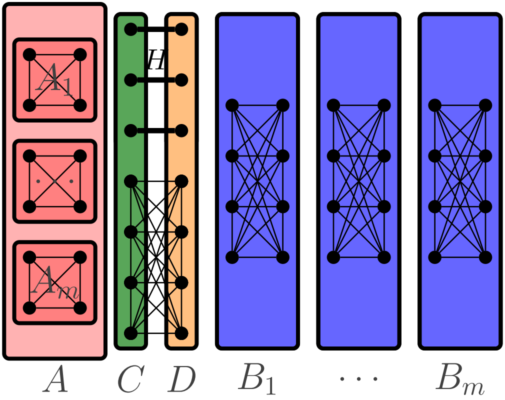


Most partitions are good

Lemma

$\Pr[(T, H) \text{ is } M\text{-bad}] \leq \varepsilon$

- Pick H, A, $\tilde{B}_1, \ldots, \tilde{B}_{m+1}$. Split $\tilde{B}_i = C_i \dot{\cup} D_i$.
- Pick randomly $i \in \{1, \ldots, m\}$ and let $C := C_i$, $D := D_i$



Open problems

Open problem

Show that there is no small **SDP** representing the Correlation/TSP/matching polytope!

Open problems

Open problem

Show that there is no small **SDP** representing the Correlation/TSP/matching polytope!

Thanks for your attention