On the Log Rank Conjecture

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Current Topics Seminar

UNIVERSITY *of* WASHINGTON

Communication complexity

Setting:

- Function $f : X \times Y \to \{0, 1\}$

Alice **Bob**

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$x \in X$	$y \in Y$

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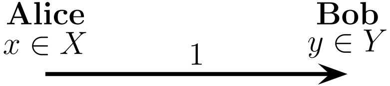

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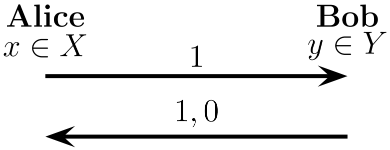

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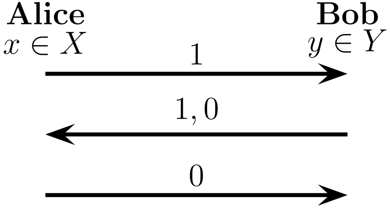

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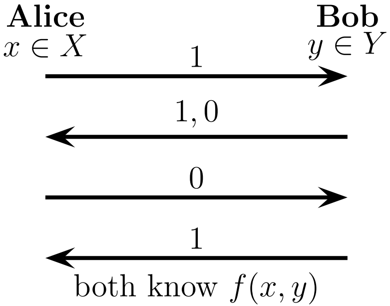

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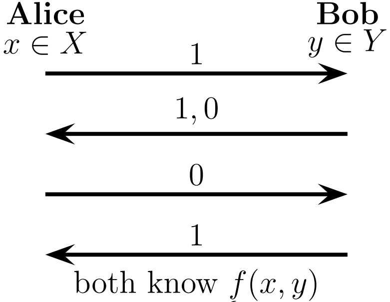

$$CC(f) = \min_{\text{protocoll}} \max_{x \in X, y \in Y} \{\text{bits to compute } f(x, y)\}$$

Communication complexity (2)

Example:

- Input for Alice: $x \in \{0,1\}^n$
- Input for Bob: $y \in \{0,1\}^n$

$$f(x,y) \quad = \quad x_1 + \ldots + x_n + y_1 + \ldots + y_n \qquad \text{mod } 2$$

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(1) Alice send $x_1 + \ldots + x_n$ mod 2 to Bob.

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	odd y				even y			
odd x	0	0	0	0	1	1	1	1
	0	0	0	0	1	1	1	1
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	0	0	0	0	1	1	1	1
even x	1	1	1	1	0	0	0	0
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- Function

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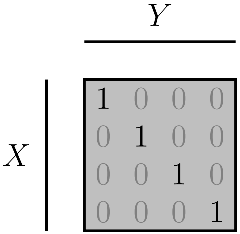
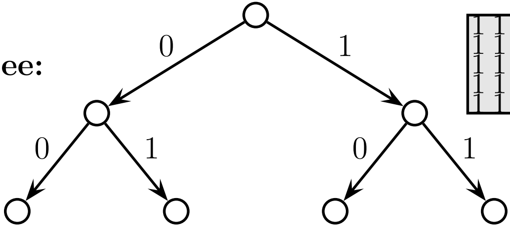


Communication complexity (3)

decision tree:



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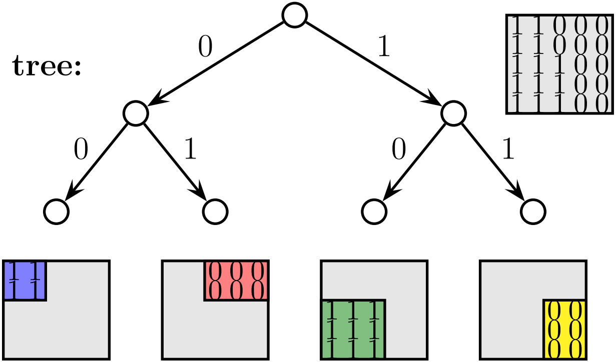

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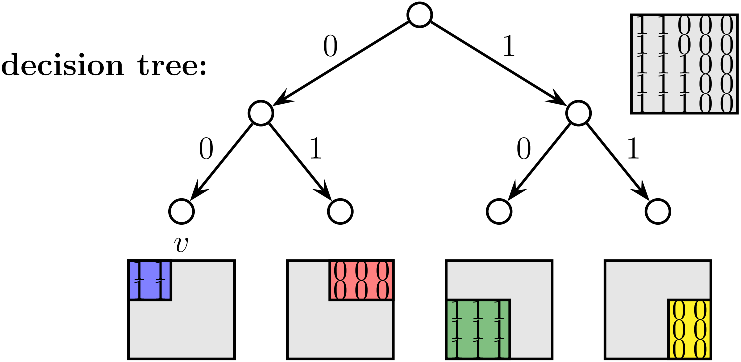

Observations:

- For a leave v of tree, $R_v := \{(x, y) : \text{protocoll ends in } v\}$ is a **monochromatic rectangle**

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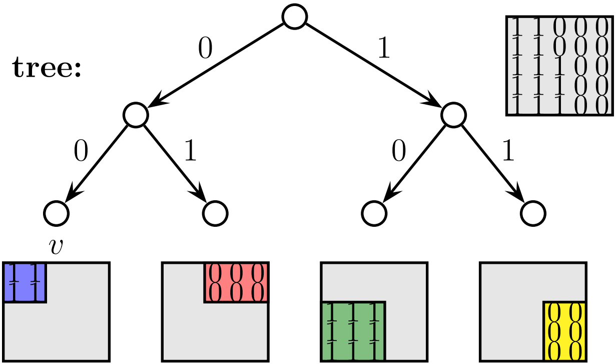

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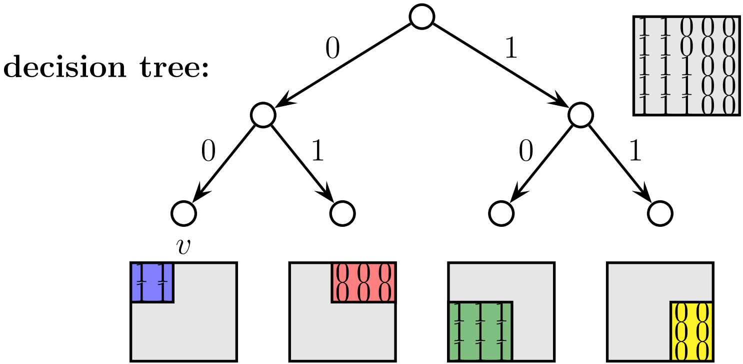

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- $CC(f) \geq \log \text{rank}(f)$

Relation to rank

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- Here: A much shorter and direct proof by me.

The technical main result

- It suffices to show

Lemma

Any 0/1 matrix A

$$\begin{array}{|cccccccc|}\hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline \end{array} \; A$$

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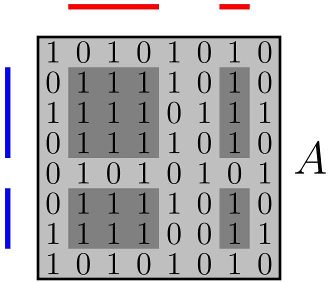
Any 0/1 matrix A has a monochromatic submatrix R of size

$$|R| \geq 2^{-\tilde{O}(\sqrt{\mathrm{rank}(A)})} \cdot |A|$$



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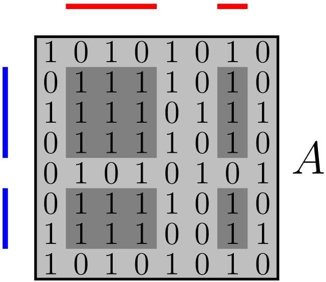
Lemma

Any 0/1 matrix A has a **almost** monochromatic submatrix R of size

$$|R| \geq 2^{-\tilde{O}(\sqrt{\text{rank}(A)})} \cdot |A|$$

- Almost means $\frac{\#\text{zeroes}}{\#\text{ones}} \leq \frac{1}{8 \cdot \text{rank}(A)}$



Thoughts about rank

Let A be a 0/1 matrix of rank r. By definition,

$$A_{ij} = \langle u_i, v_j \rangle \quad \text{with} \quad u_i, v_j \in \mathbb{R}^r$$

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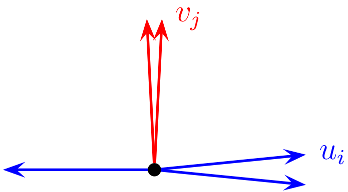

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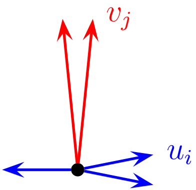

- For any regular matrix T: $u_i' := Tu_i$ & $v_j' := (T^{-1})^T v_j$
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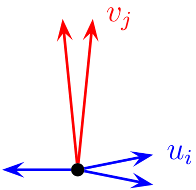

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Lemma

Vectors can be chosen so that $\|u_i\|_2, \|v_j\|_2 \leq r^{1/4} \; \forall i, j$

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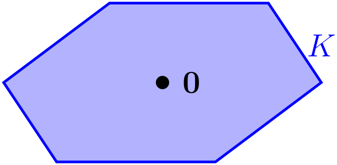
For any symmetric convex body $K \subseteq \mathbb{R}^n$, there is an ellipsoid E so that $E \subseteq K \subseteq \sqrt{n} \cdot E$.



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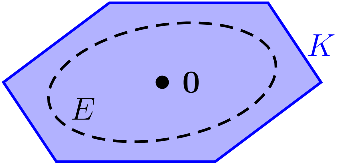

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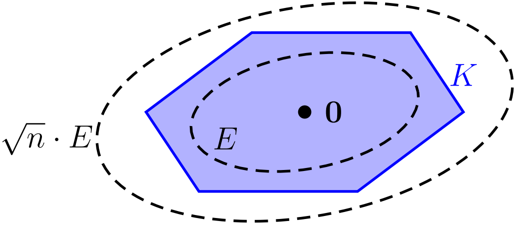
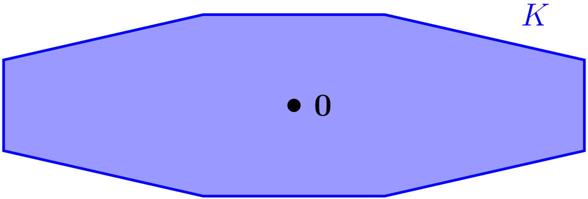


- $T : \mathbb{R}^n \to \mathbb{R}^n$ linear map $\Rightarrow T(\text{ball})$ is an **ellipsoid**

John's Theorem (2)

Proof:



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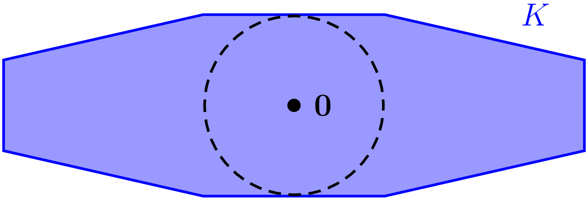

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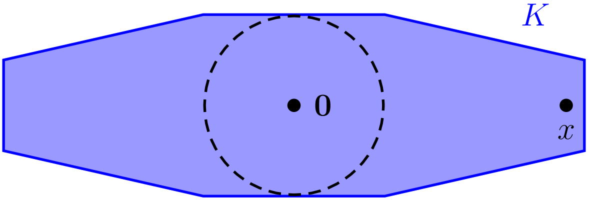

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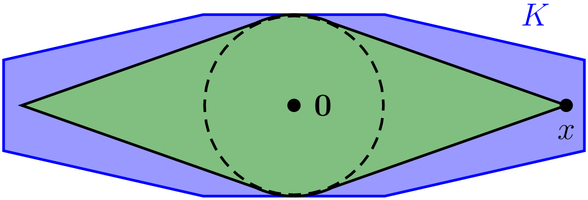

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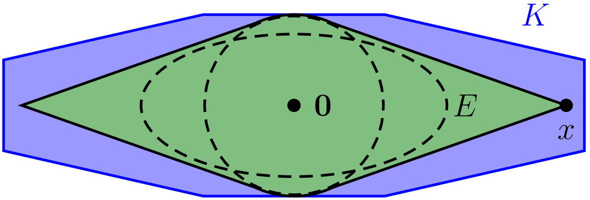

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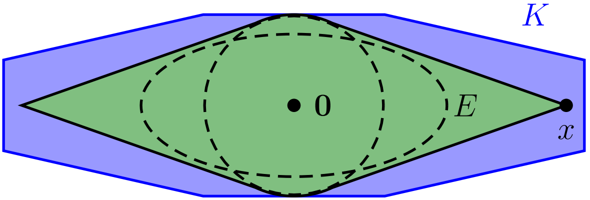

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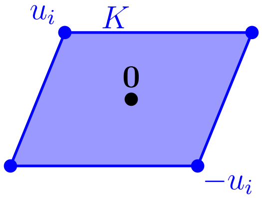

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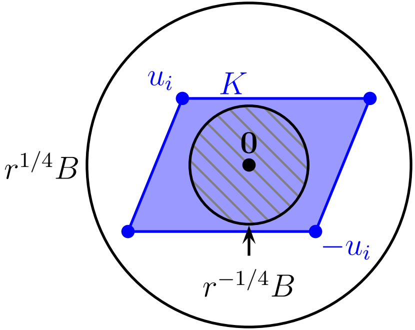

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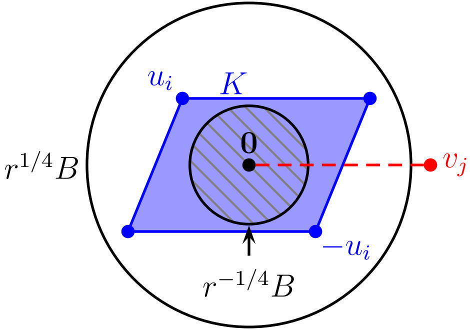

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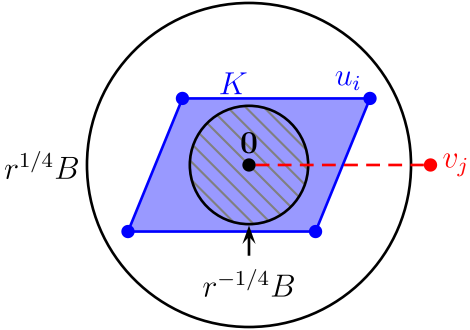

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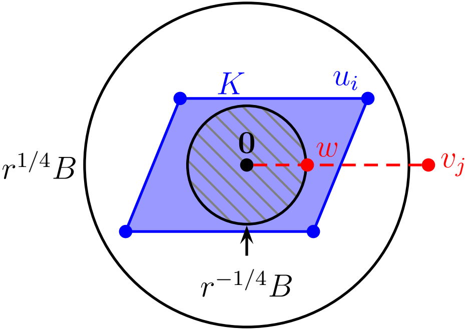

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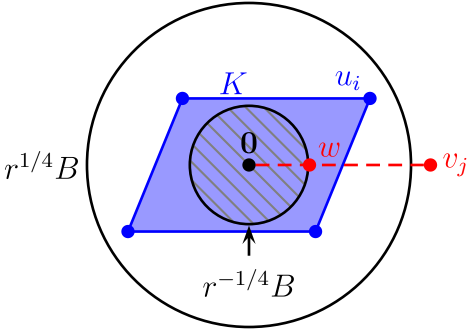

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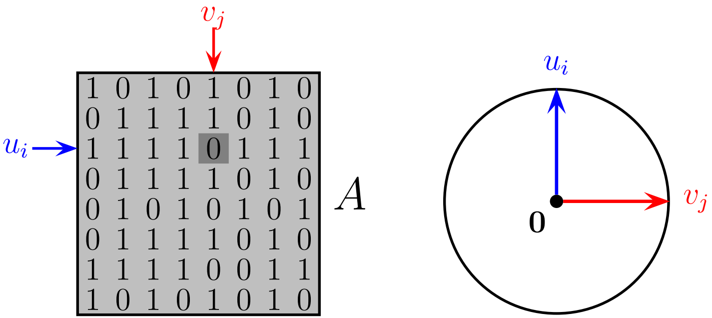

- There are unit vectors u_i, v_j so that

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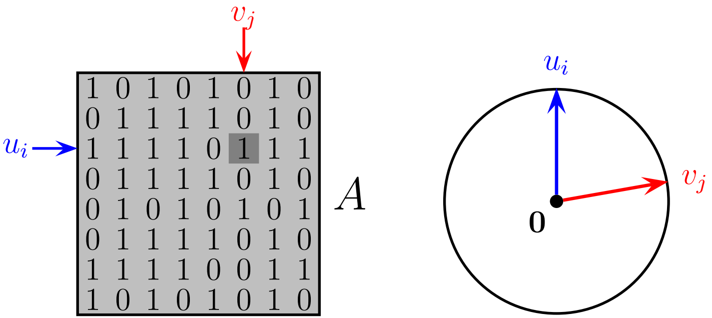

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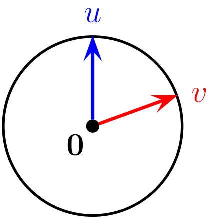
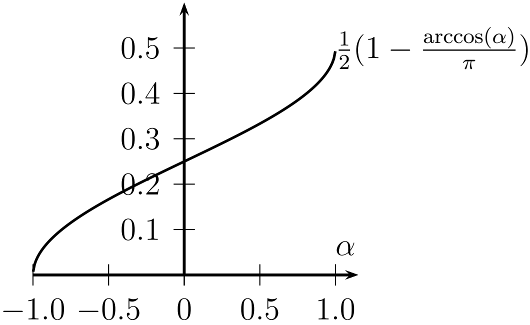
For unit vectors $u, v \in \mathbb{R}^2$, take a random direction g. Then

$$\Pr[\langle g, u\rangle \geq 0 \text{ and } \langle g, v\rangle \geq 0] = \frac{1}{2}\left(1 - \frac{\arccos(\langle u, v\rangle)}{\pi}\right)$$





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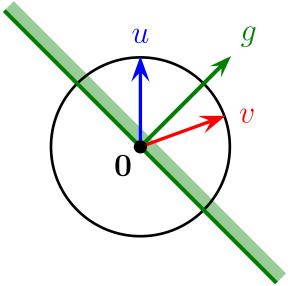

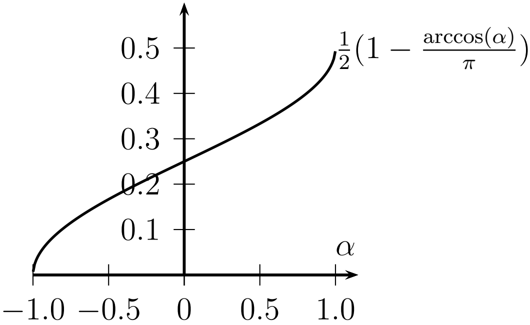

Sheppard's formula

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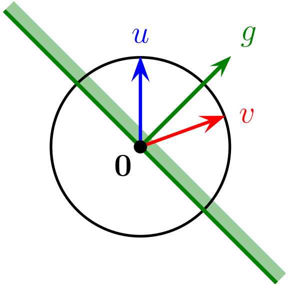
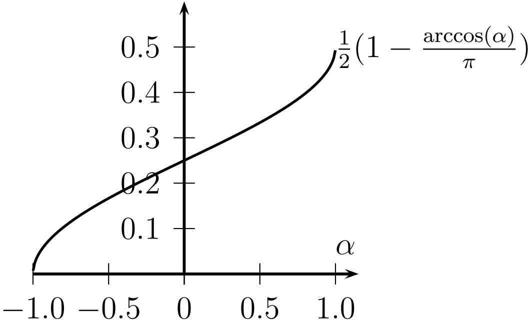
For unit vectors $u, v \in \mathbb{R}^2$, take a random direction g. Then

$$\Pr[\langle g, u\rangle \geq 0 \text{ and } \langle g, v\rangle \geq 0] \approx \frac{1}{4} + const \cdot \langle u, v\rangle$$





Finding an almost monochr. submatrix

- Suppose A is 0/1 matrix with #ones$(A) \geq$ #zeroes(A)

1	0	1	0	1	0	1	0
0	1	1	1	1	0	1	0
1	1	1	1	0	1	1	1
0	1	1	1	1	0	1	0
0	1	0	1	0	1	0	1
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A

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- Suppose A is 0/1 matrix with $\#\text{ones}(A) \geq \#\text{zeroes}(A)$
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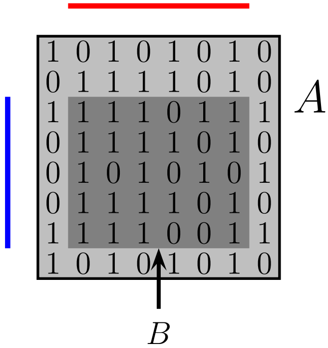

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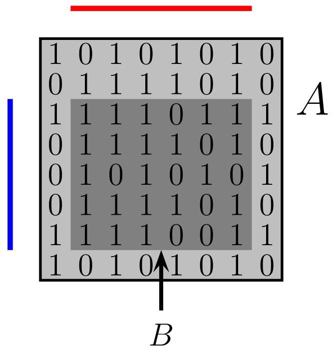

- We know

$$\mathbb{E}\left[\frac{\#\text{zeroes in } B}{\#\text{ones in } B}\right] \approx \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{\sqrt{r}}} = 1 - \Theta\left(\frac{1}{\sqrt{r}}\right)$$

$$\mathbb{E}[\text{fract of entries in } B] \approx \frac{1}{4}$$

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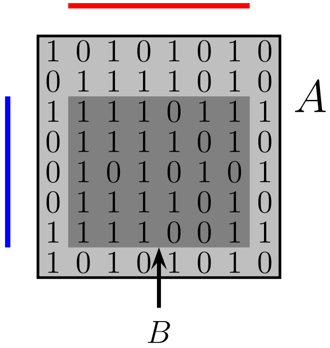

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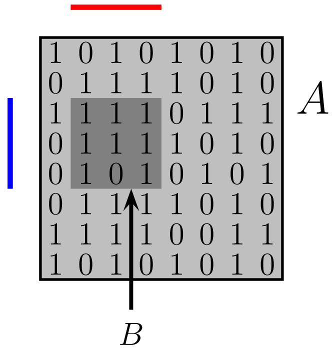

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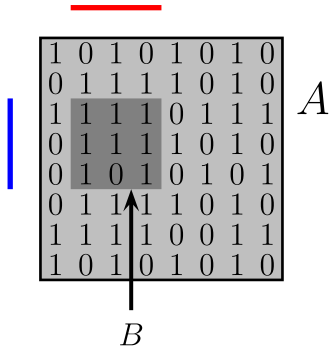

- We know

$$\mathbb{E}\left[\frac{\#\text{zeroes in } B}{\#\text{ones in } B}\right] \approx \left(\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{\sqrt{r}}}\right)^T = \frac{1}{8r}$$

$$\mathbb{E}[\text{fract of entries in } B] \approx \left(\frac{1}{4}\right)^T = 2^{-\Theta(\sqrt{r}\log r)}$$

The end

- The following is equivalent to log-rank conjecture:

Conjecture

Any rank-r 0/1-matrix A has a submatrix with

- $|B| \geq 2^{-(\log(r))^{O(1)}}$
- B is monochromatic (except of a $\frac{1}{8r}$-fraction of entries).

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Thanks for your attention