A parameterized linear formulation of the integer hull

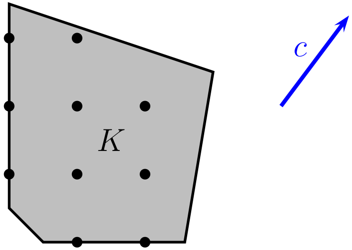
Thomas Rothvoss

Joint work with Fritz Eisenbrand

UNIVERSITY *of* WASHINGTON

Integer Programming

$$\begin{aligned} \max c^T x & & \\ Ax & \leq & b \\ x & \in & \mathbb{Z}^n \end{aligned}$$



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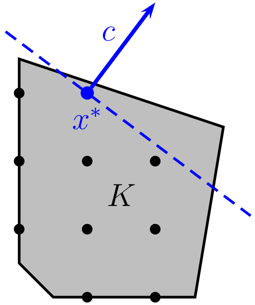

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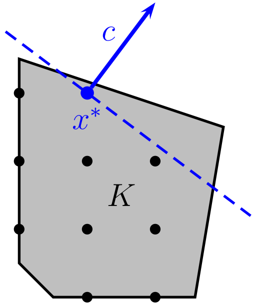

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- Powerful heuristics to solve in practice $\rightarrow$ **Branch & Bound**
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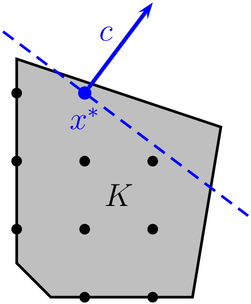

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Theoretical IP: What special cases can be solved efficiently?

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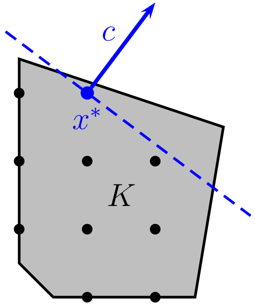

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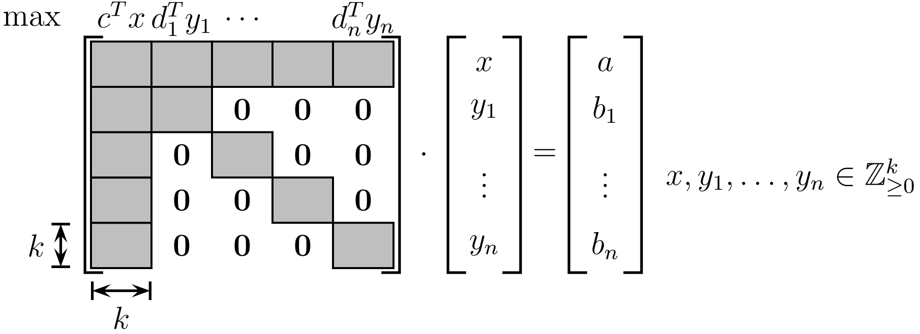
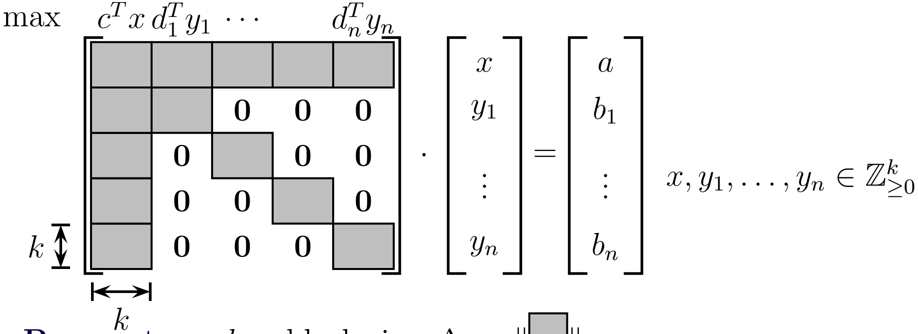
Examples:

- A is **totally unimodular** $\rightarrow$ IP solvable in polytime!
- All subdeterminants of A bounded by $O(1)$ $\rightarrow$ open!

n-fold IP


max $c^T x$ $d_1^T y_1$ $\cdots$ $d_n^T y_n$
0
k
k
x
y_1
$\vdots$
y_n
$=$
a
b_1
$\vdots$
b_n
$x, y_1, \ldots, y_n \in \mathbb{Z}^k_{\geq 0}$

n-fold IP



Parameters: k = block size; $\Delta := \|\square\|_\infty$

Results:

- General n-fold IP can be solved in time $n^{f(k,\Delta)}$ [HKW'13]
- **Open:** Does $f(k,\Delta) \cdot n^{O(1)}$ suffices?

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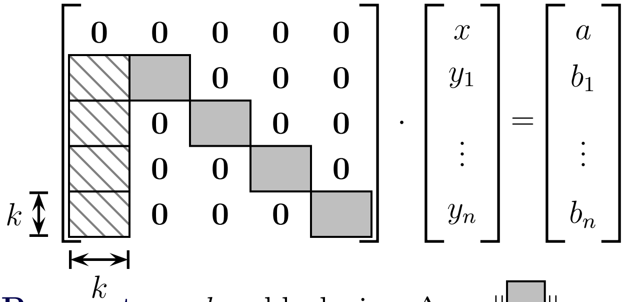

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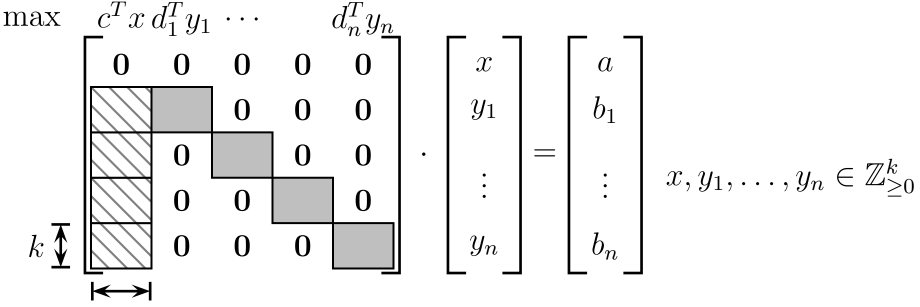

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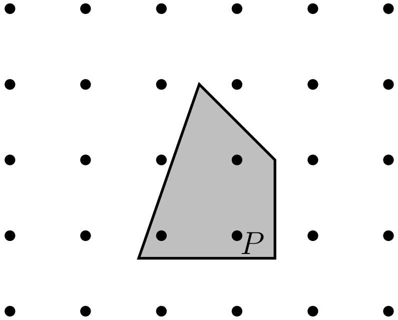
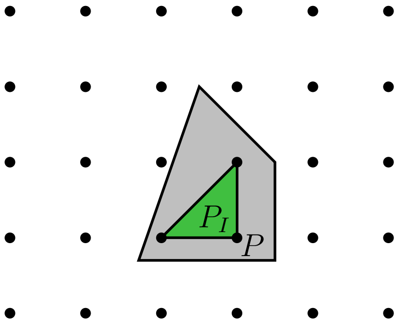
- Solve 2SSIP **with objective** in time $f(k,\Delta) \cdot n^{O(1)}$ [here]

Cutting planes



- For a polytope P

Cutting planes



- For a polytope P, $P_I := \mathrm{conv}(P \cap \mathbb{Z}^n)$ is the **integer hull**.

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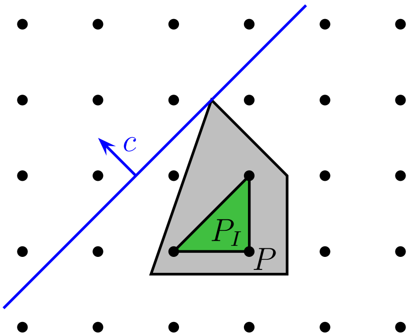

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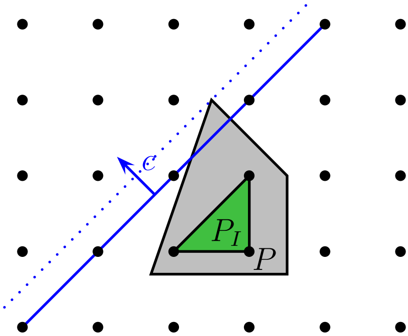
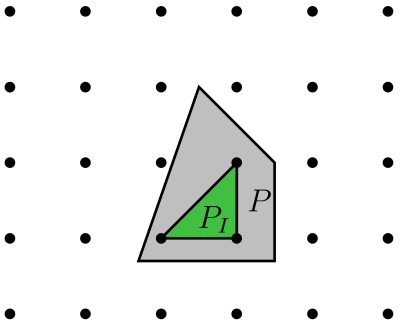


- For a polytope P, $P_I := \text{conv}(P \cap \mathbb{Z}^n)$ is the **integer hull**.
- If $c^T x \leq \delta$ is feasible for P with $c \in \mathbb{Z}^n$, then $c^T x \leq \lfloor \delta \rfloor$ is feasible for P_I → **Chvátal Gomory cutting plane**

Cutting planes (2)



- Adding **all** possible Chvátal Gomory cutting planes simultaneously gives the **elementary closure**

$$P' = \bigcap_{\substack{(c^T x \leq \delta) \supseteq P \\ c \in \mathbb{Z}^n}} \left(c^T x \leq \lfloor \delta \rfloor\right).$$

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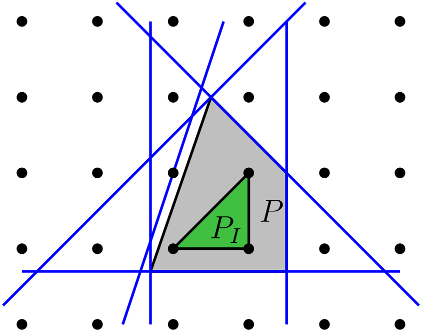

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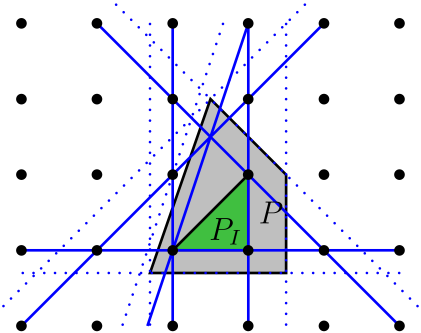

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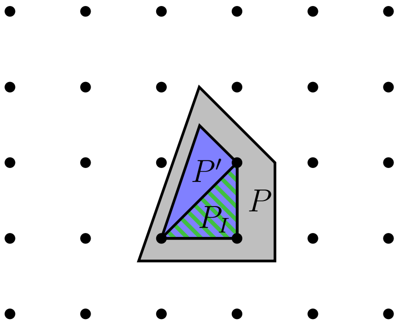

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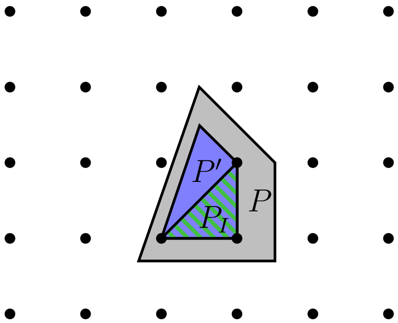

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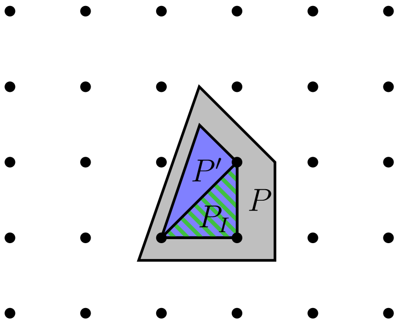
- Iterating $P^{(i)} := ((P')')\ldots'$ gives i**th closure**

Cutting planes (3)



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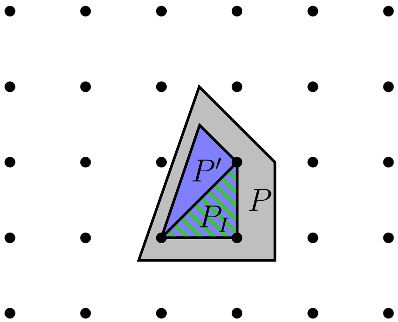
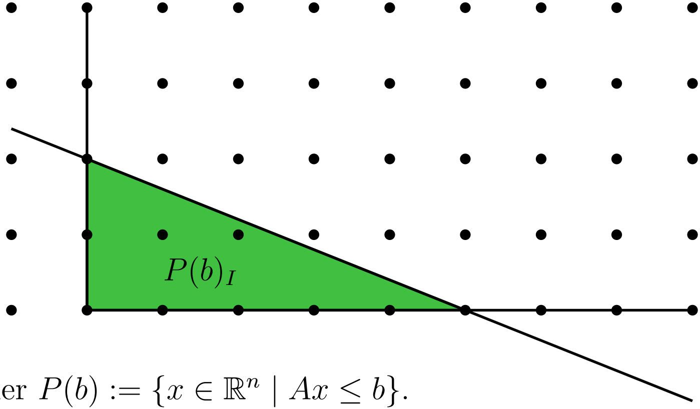


- Smallest i with $P^{(i)} = P_I$ is **Chvátal rank** of P.
- Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ with integer A, b.
  Then Chvátal rank $\leq (n\|A\|_\infty)^{O(n^2)}$
  [Cook, Gerards, Schrijver, Tardos '86]

Integer hull



- Consider $P(b) := \{x \in \mathbb{R}^n \mid Ax \leq b\}$.
- In general $P(b)_I$ **not** changing **linear** in b!

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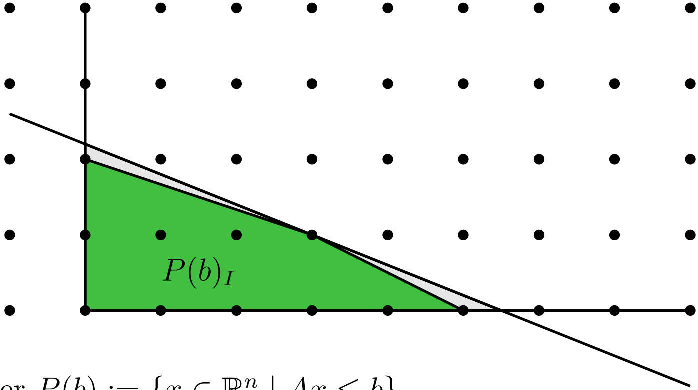

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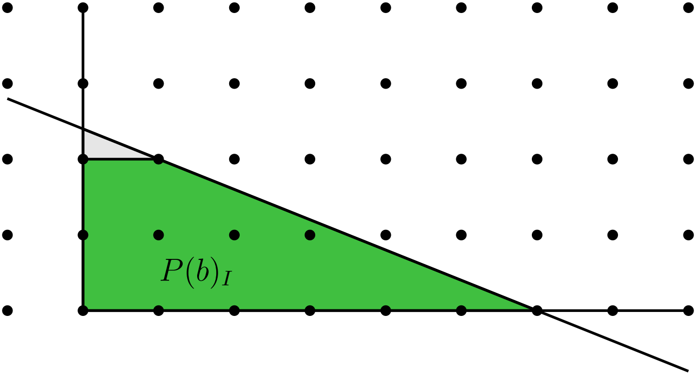

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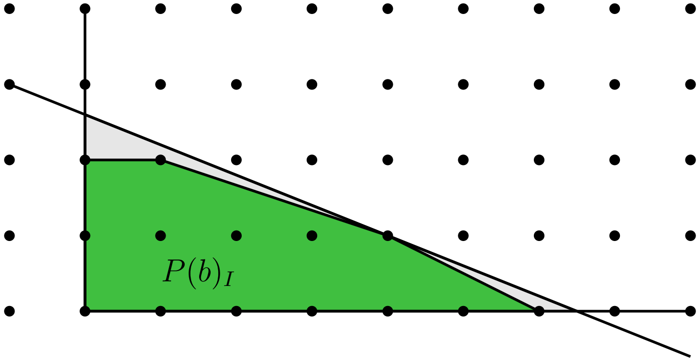

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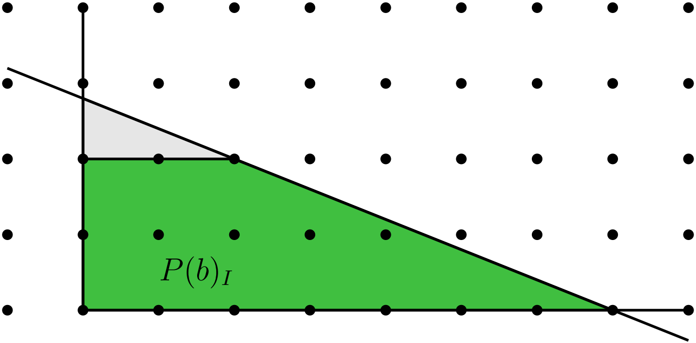

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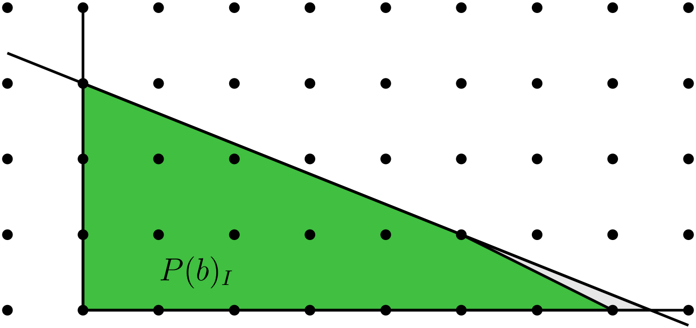

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- Right hand side of CG cut is **affine linear** in b!!!
- Coefficients bounded by $\|\lambda^T A\|_\infty \leq n\|A\|_\infty$

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- Iterating this $f(n, \Delta)$ times gives same for $P(b)_I$:

Theorem (Eisenbrand, R. 2024)

Let $P(b) := \{x \in \mathbb{R}^n \mid Ax \leq b\}$ where $A \in \mathbb{Z}^{m \times n}$ and $\|A\|_\infty \leq \Delta$. Then there exist $D \in \mathbb{N}$, $B \in \mathbb{Z}^{m' \times n}$, $C \in \mathbb{Z}^{m' \times m}$ so that for all $r \in \{0, \ldots, D-1\}^m$ there is an $f_r \in \mathbb{Z}^{m'}$ with

$$P(b)_I = \left\{x \in \mathbb{R}^n \mid Bx \leq f_r + Cb\right\} \quad \forall b \in \mathbb{Z}^m \text{ with } b \equiv_D r$$

Moreover one can choose $D \leq n^{n^{(n\Delta)^{O(n^2)}}}$.

Back to 2-SSIP

Solving 2-stage stochastic IPs:

$$
\begin{aligned}
\max c^T x + \sum_{i=1}^{n} d_i^T y_i & & & \\
A_i x + B_i y_i &\leq & b_i & \quad \forall i \in [n] \\
x &\in & \mathbb{Z}^k & \\
y_1, \ldots, y_n &\in & \mathbb{Z}^k &
\end{aligned}
$$

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Solving 2-stage stochastic IPs:

$$\begin{aligned} \max c^T x + \sum_{i=1}^{n} d_i^T y_i & & \\ y_i &\in P_i(b_i - A_i x) \quad \forall i \in [n] \\ x &\in \mathbb{Z}^k \\ y_1, \ldots, y_n &\in \mathbb{Z}^k \end{aligned}$$

- Let $P_i(b) := \{y_i \in \mathbb{R}^k \mid B_i y_i \leq b\}$

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x &\equiv_D r
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$$

- Let $P_i(b) := \{y_i \in \mathbb{R}^k \mid B_i y_i \leq b\}$
- Guess remainder $r := (x \mod D)$ in optimum solution

Back to 2-SSIP

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$$
\begin{array}{rcll}
\multicolumn{4}{l}{\max c^T x + \sum_{i=1}^{n} d_i^T y_i} \\
y_i & \in & (P_i(\underbrace{b_i - A_i x}_{\text{fixed } \equiv_D}))_I & \forall i \in [n] \\
x & \in & \mathbb{Z}^k & \\
y_1, \ldots, y_n & \in & \mathbb{R}^k & \\
x & \equiv_D & r &
\end{array}
$$

- Let $P_i(b) := \{y_i \in \mathbb{R}^k \mid B_i y_i \leq b\}$
- Guess remainder $r := (x \mod D)$ in optimum solution

Back to 2-SSIP

Solving 2-stage stochastic IPs:

$$
\begin{aligned}
\max c^T x + \sum_{i=1}^{n} d_i^T y_i \\
y_i &\in (P_i(\underbrace{b_i - A_i x}_{\text{fixed } \equiv_D}))_I \quad \forall i \in [n] \\
x &\in \mathbb{Z}^k \\
y_1, \ldots, y_n &\in \mathbb{R}^k \\
x &\equiv_D r
\end{aligned}
$$

- Let $P_i(b) := \{y_i \in \mathbb{R}^k \mid B_i y_i \leq b\}$
- Guess remainder $r := (x \mod D)$ in optimum solution
- Solve remaining system with $O(k)$ integer x-variables (+poly many fractional y-variables)

Open problems

- Our bound gives $D \leq 2^{2^{\Delta^{\mathrm{poly}(n)}}}$ (**triple-exponential**).

Open problem 1

Is $D \leq 2^{\Delta^{\mathrm{poly}(n)}}$ (**double-exponential**) sufficient?

Open problems

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Is $D \leq 2^{\Delta^{\mathrm{poly}(n)}}$ (**double-exponential**) sufficient?

Open problem 2

Can one solve general n-fold IP in time $f(k, \Delta) \cdot n^{O(1)}$?

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Open problem 3

Can one solve every n-**variable ILP** in time $2^{O(n)}$?

- Best known is $(\log n)^{O(n)}$ [Reis, R. 2023].

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Thanks for your attention!