0/1 Polytopes with Quadratic Chvátal Rank

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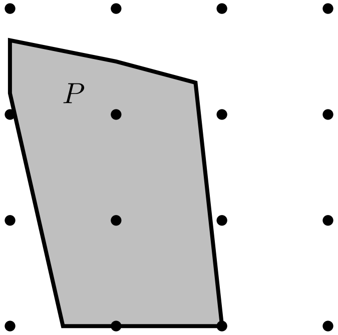
3rd Cargese Workshop on Combinatorial Optimization

Massachusetts Institute of Technology

Alexander von Humboldt Stiftung/Foundation

Gomory Chvátal Cuts

- **Given:** Polytope $P \subseteq \mathbb{R}^n$



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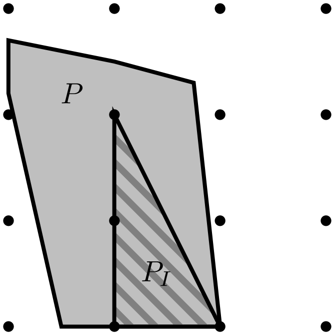

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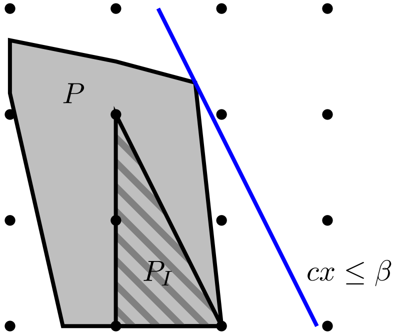

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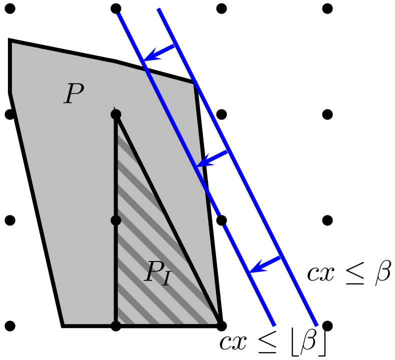

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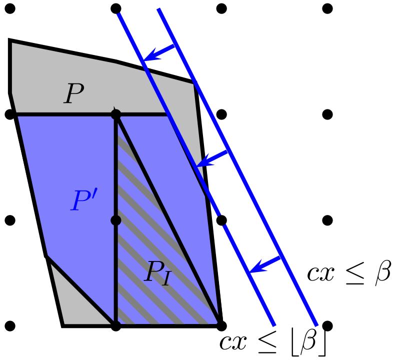

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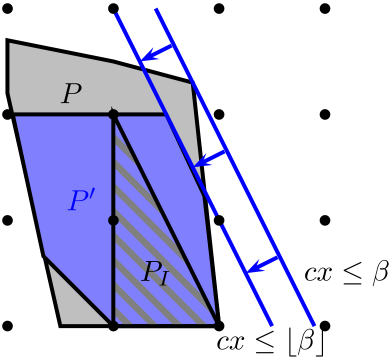

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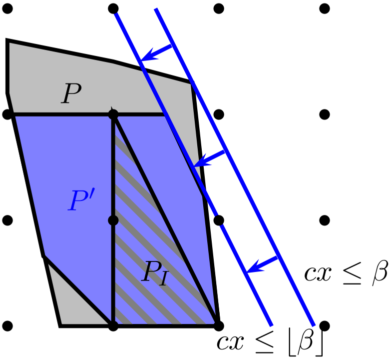

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- **Chvátal rank:** $P^{(\text{rk}(P))} = P_I$

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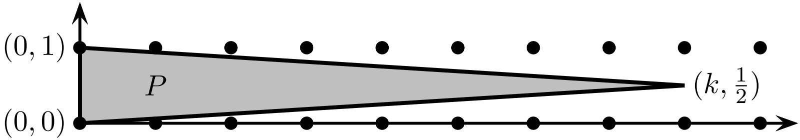

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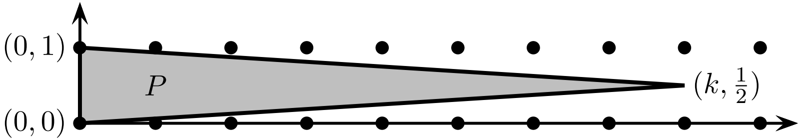

- For the rest of the talk assume $P \subseteq [0,1]^n$

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Theorem (Sanità, R. '12)

There exists a family of polytopes $P \subseteq [0,1]^n$ with Chvátal rank $\text{rk}(P) \geq \Omega(n^2)$.

The polytope

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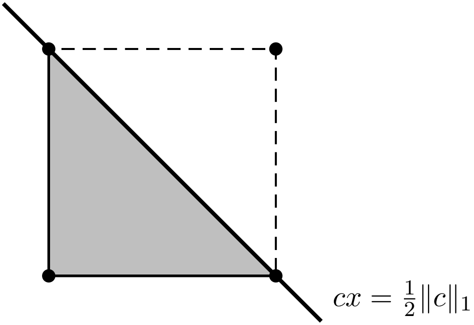
- Let $c \in \mathbb{Z}^n_{\geq 0}$ be a vector

$$P(c) := \text{conv} \underbrace{\left\{ x \in \{0,1\}^n : cx \leq \frac{\|c\|_1}{2} \right\}}_{\text{Knapsack solutions}}$$



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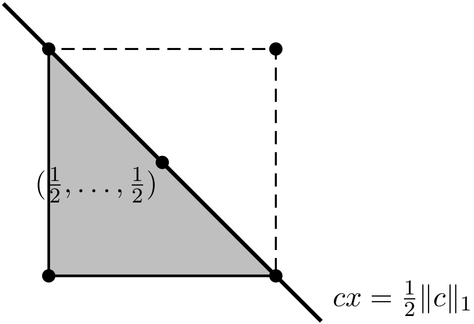

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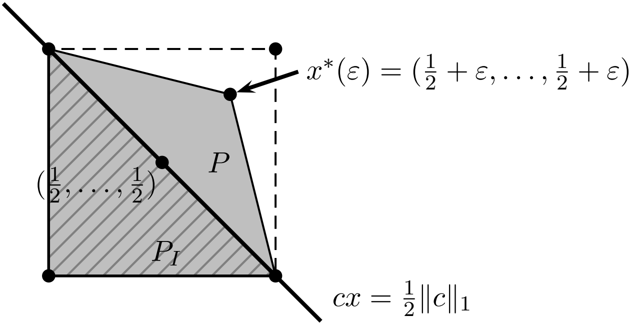
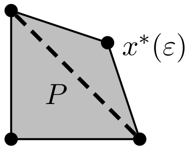


Critical vectors

- Call $\tilde{c}$ **critical** $\Leftrightarrow$ $\tilde{c}$ maximized at x^*



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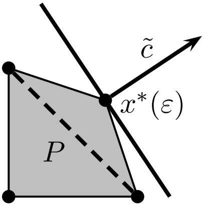

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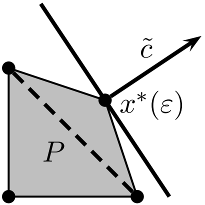

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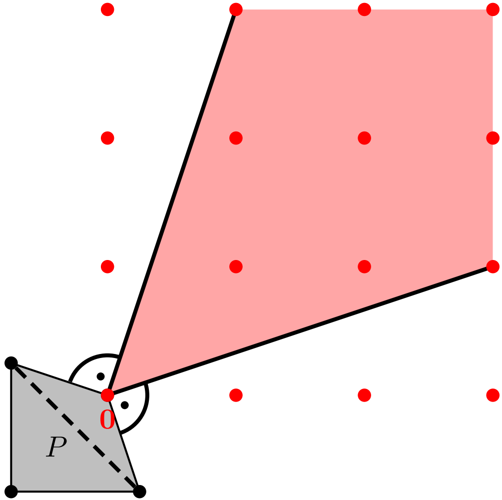

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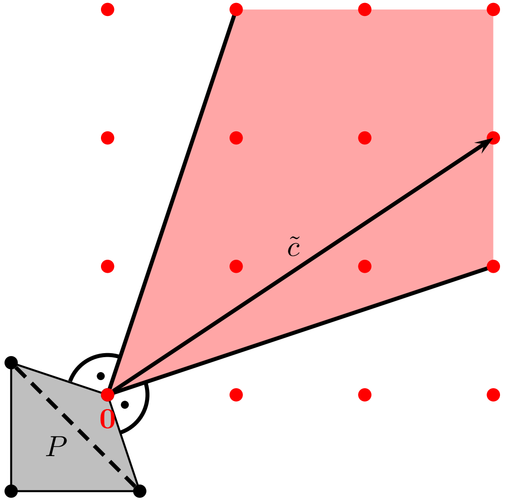

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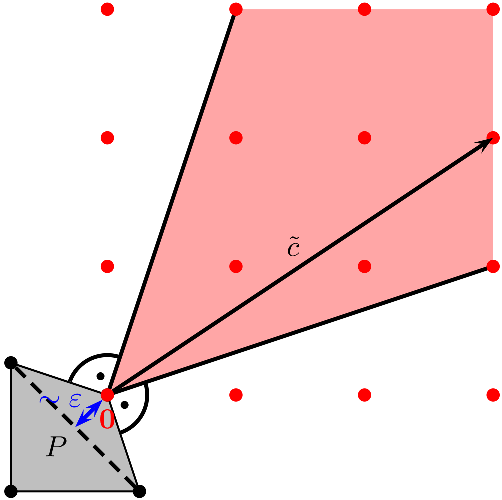

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random vector has no short, good SDA

↓

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A lower bound strategy

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Assume: $\forall \varepsilon \in [(\frac{1}{2})^{\Theta(n)}, \Theta(1)]: \quad \tilde{c}$ *critical* $\Longrightarrow \|\tilde{c}\|_1 \geq \Omega(\frac{n}{\varepsilon})$.
Then $rk(P) \geq \Omega(n^2)$.

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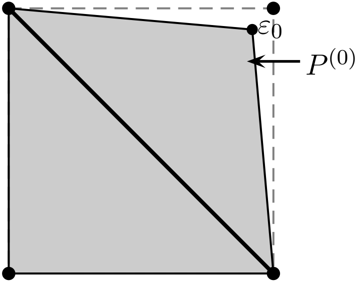

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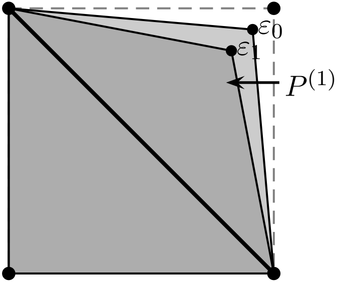

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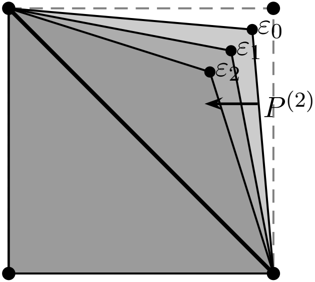

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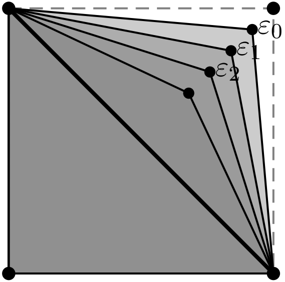

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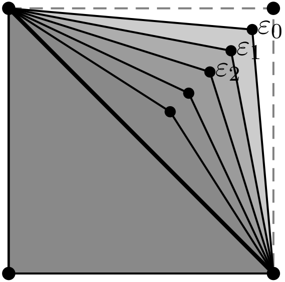

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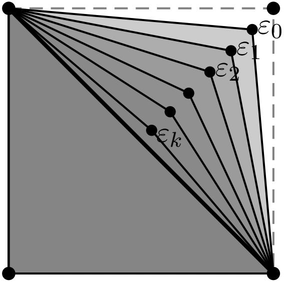

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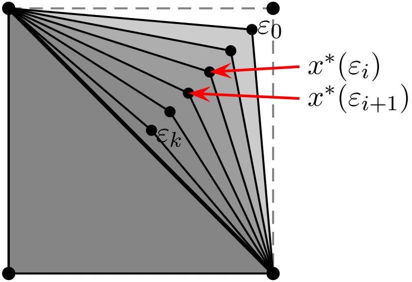

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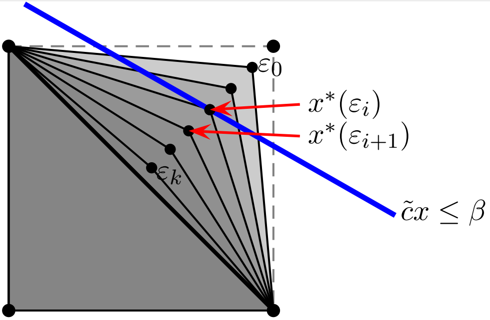

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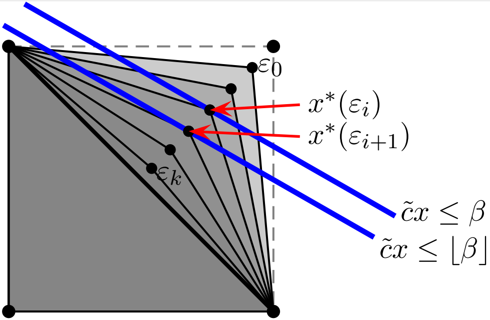

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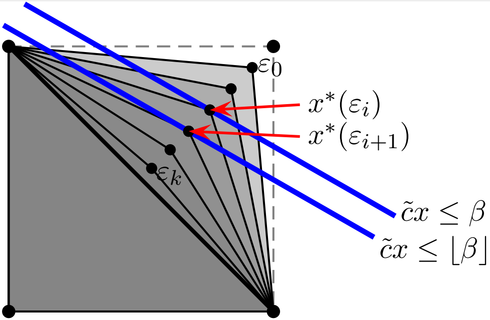

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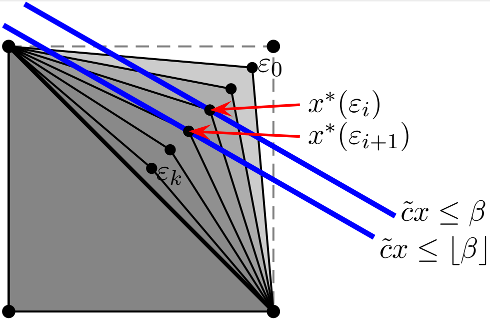

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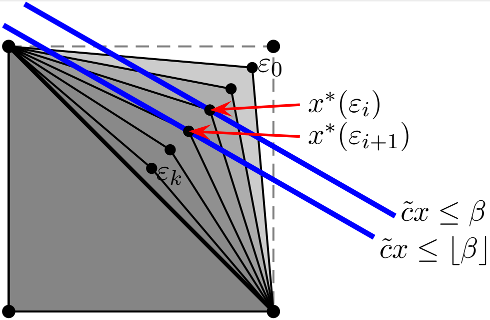

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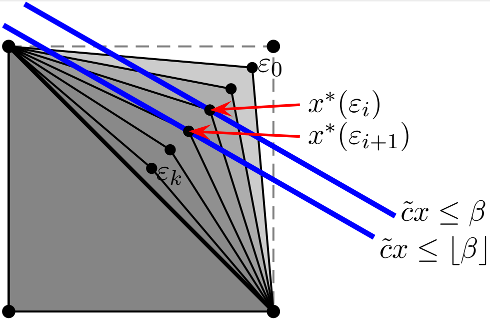

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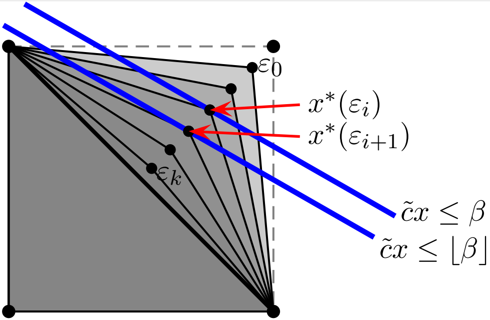

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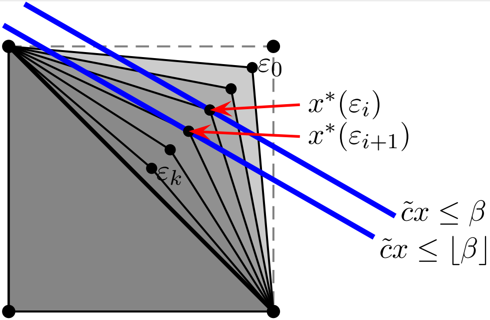

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- Then $\frac{\varepsilon_{i+1}}{\varepsilon_i} \geq 1 - \frac{\Theta(1)}{n} \implies k \geq \Omega(n^2) \quad \square$

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critical vectors are good
Simultaneous Diophantine Approximations to c

critical vectors are long $\Longrightarrow \Omega(n^2)$ rank

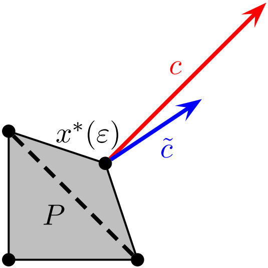
Simultaneous Diophantine Approximation

Lemma

Under magical assumptions

$$\tilde{c} \text{ critical} \implies \|\lambda\tilde{c} - c\|_1 \leq O(\varepsilon) \cdot \|c\|_1 \quad (\text{for some } \lambda > 0)$$

- **Intuition:** $\tilde{c}$ critical $\implies$ $\tilde{c}$ well-approximates c



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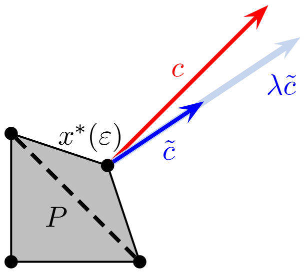

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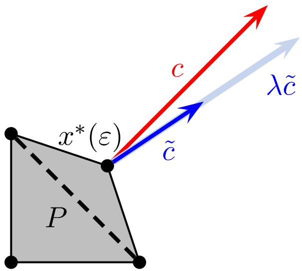

- **Lemma follows from:**

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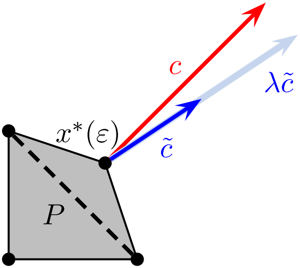

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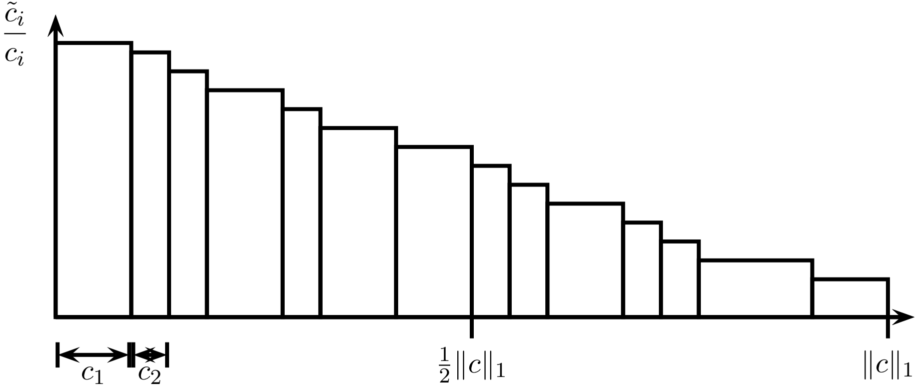

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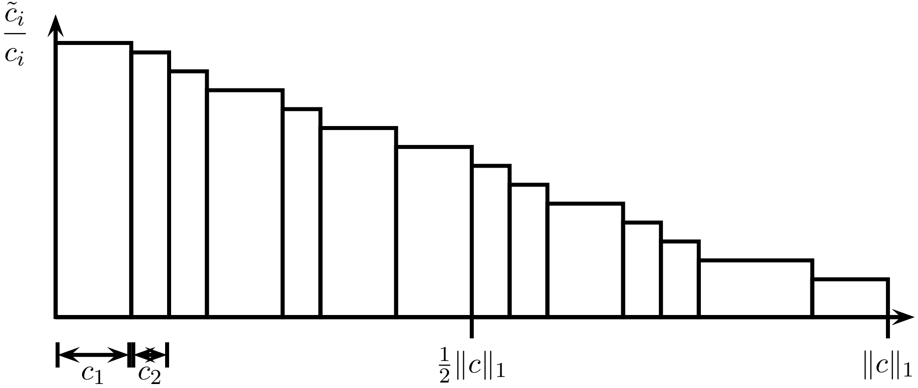

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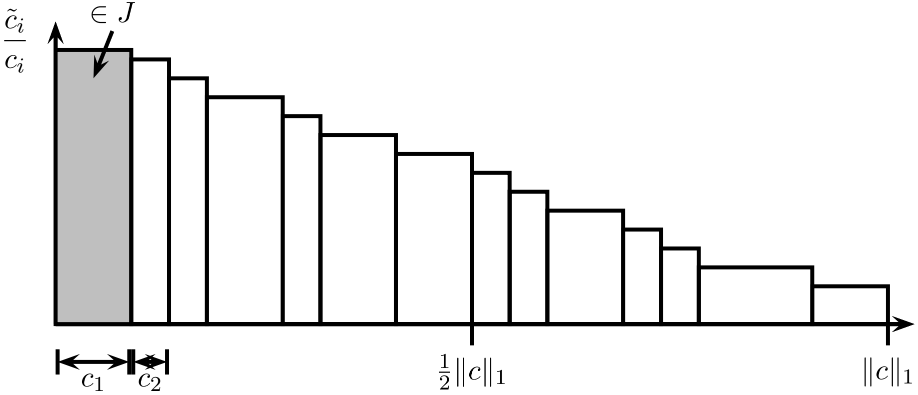

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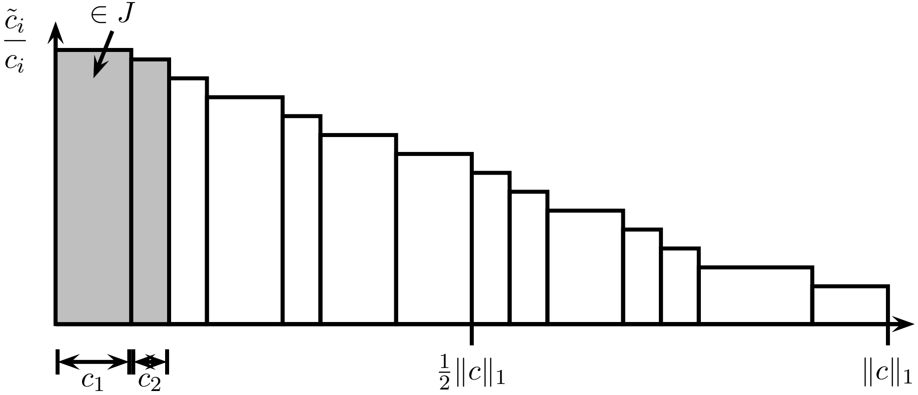

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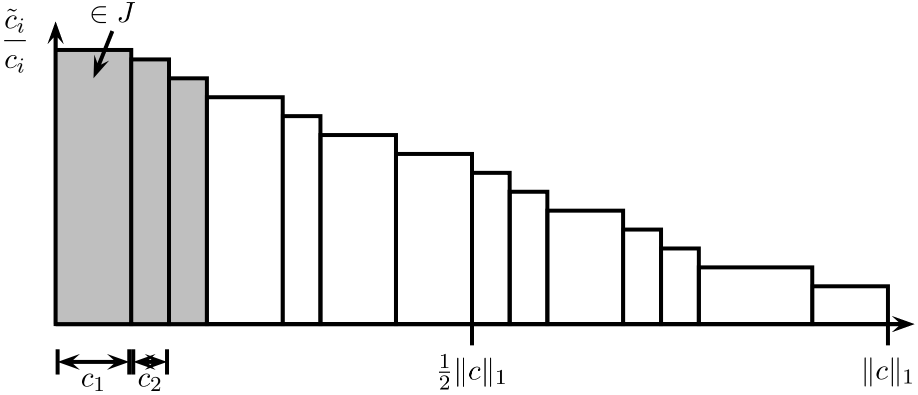

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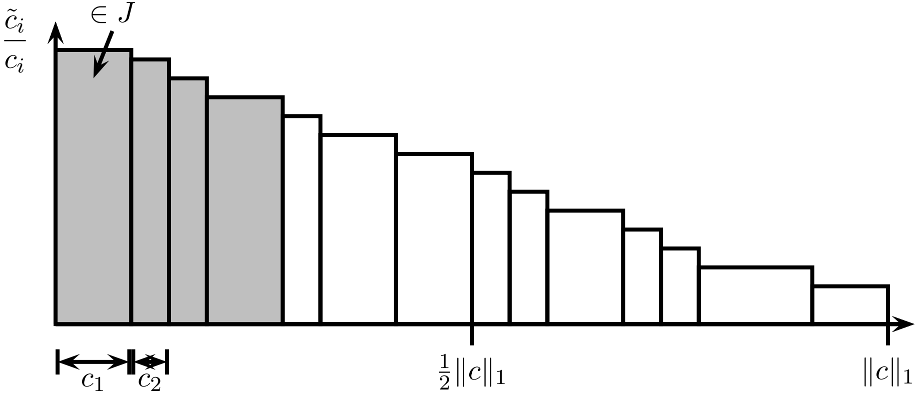

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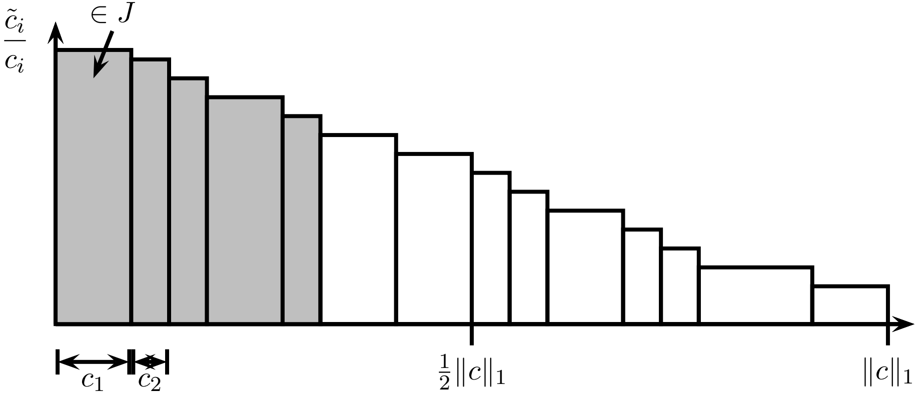

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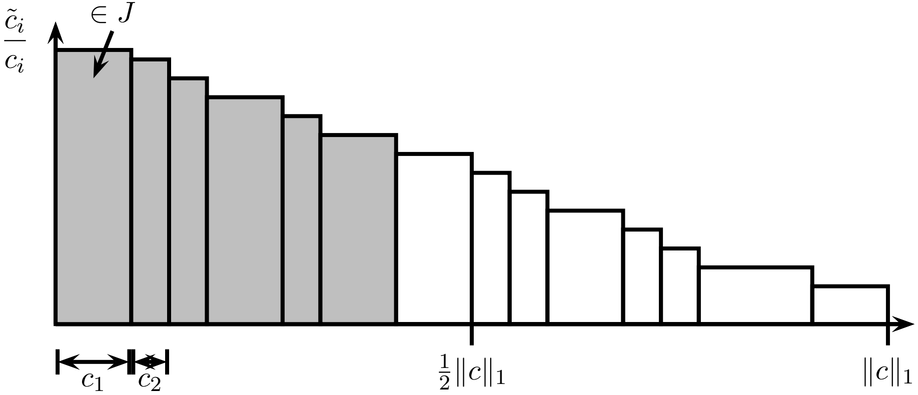

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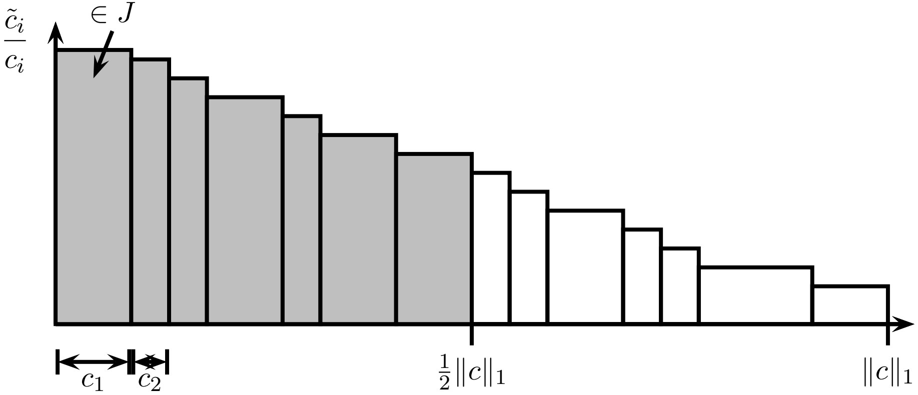

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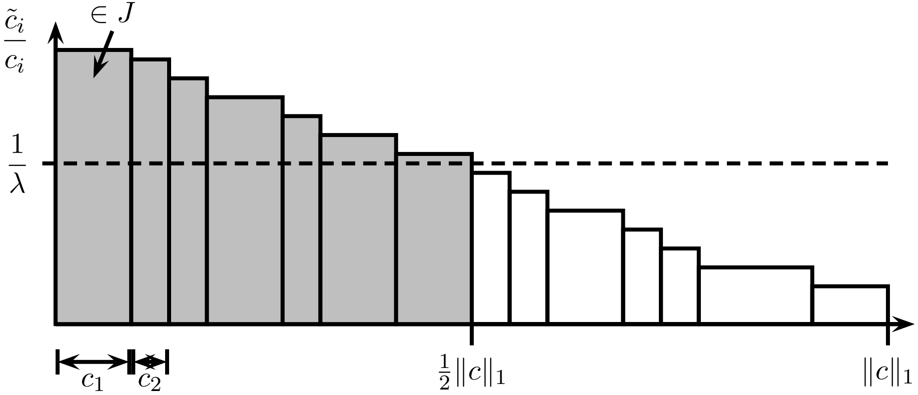

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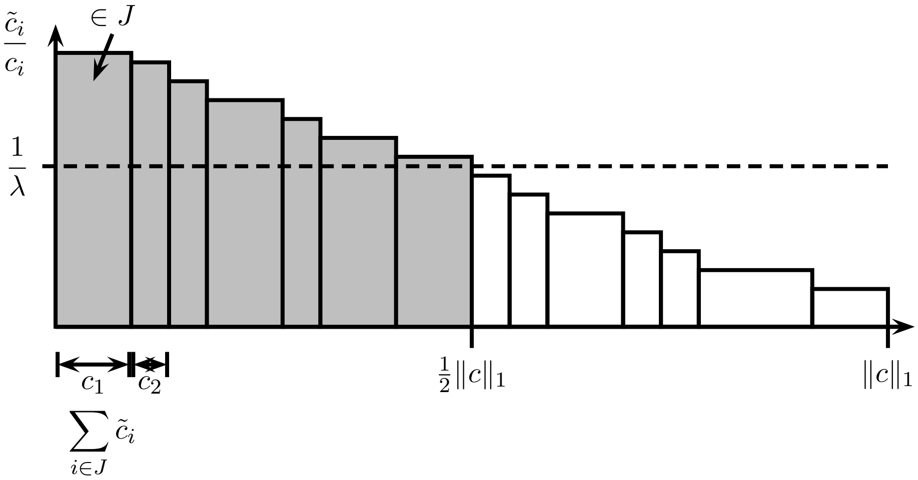

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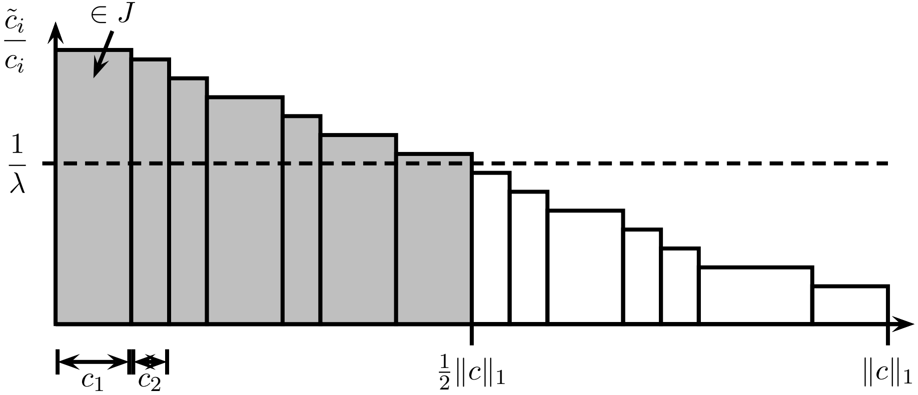

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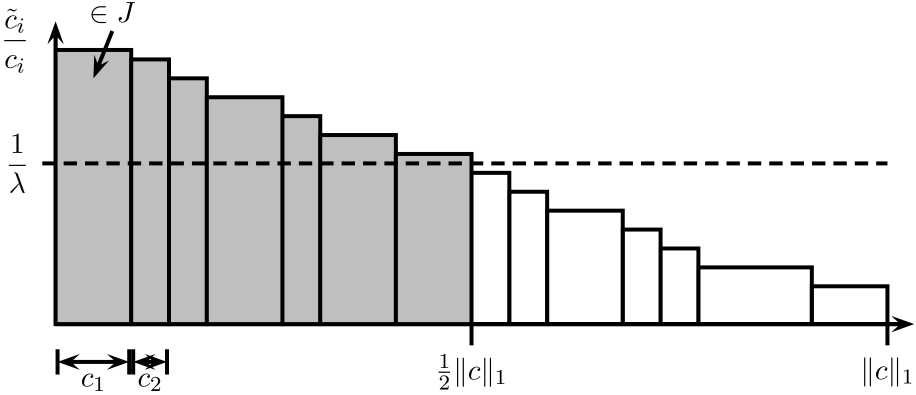

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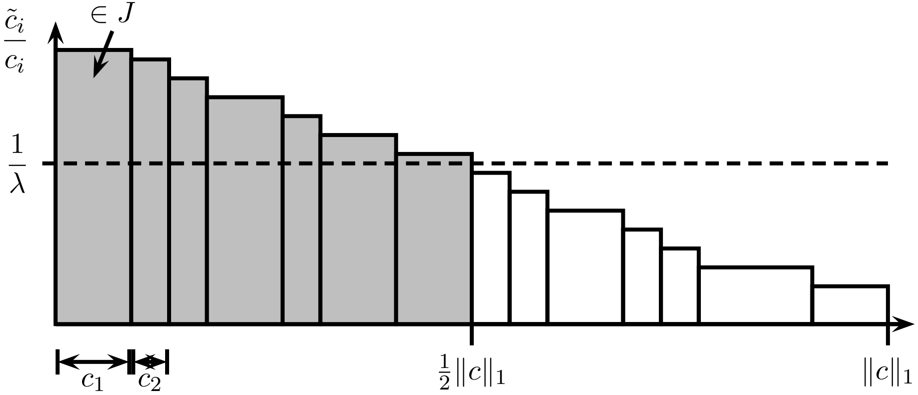

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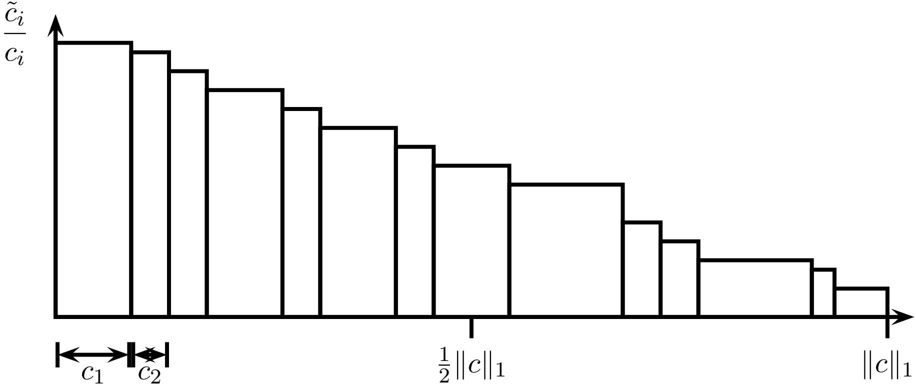

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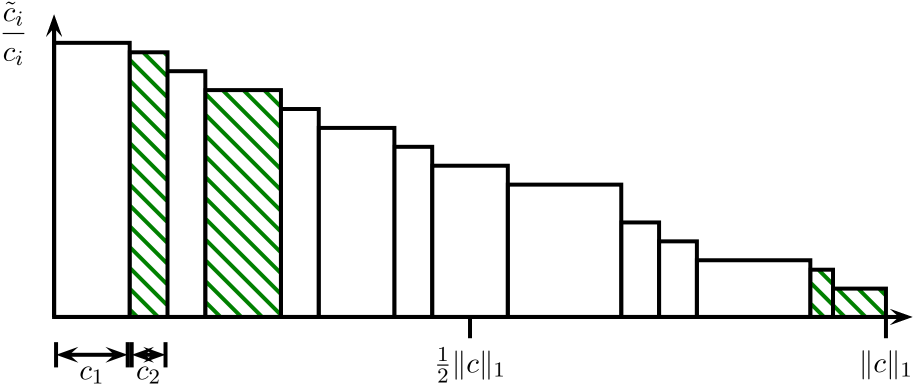

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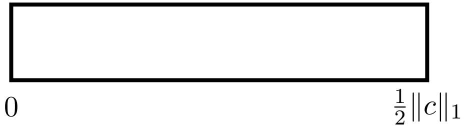

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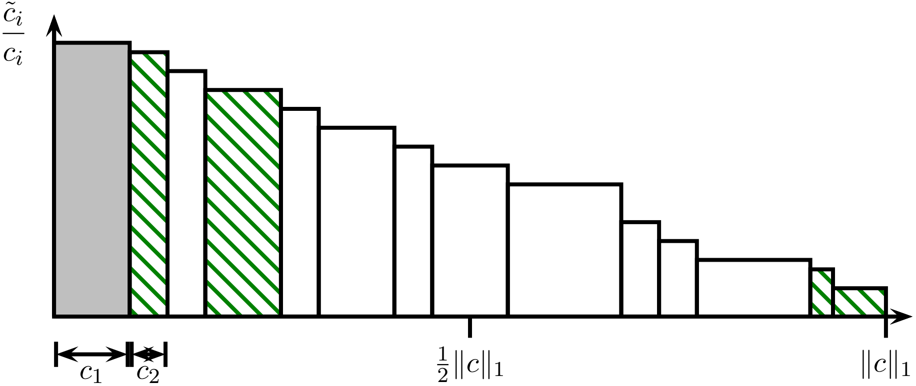

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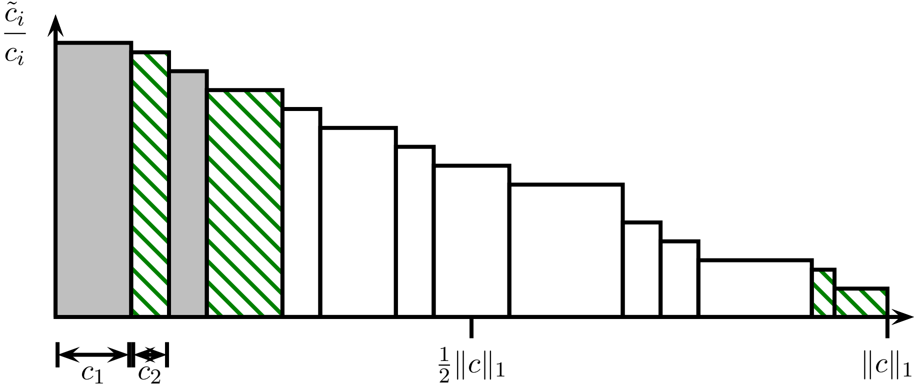

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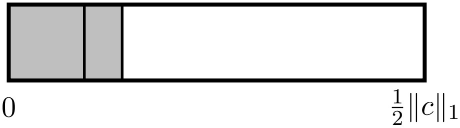

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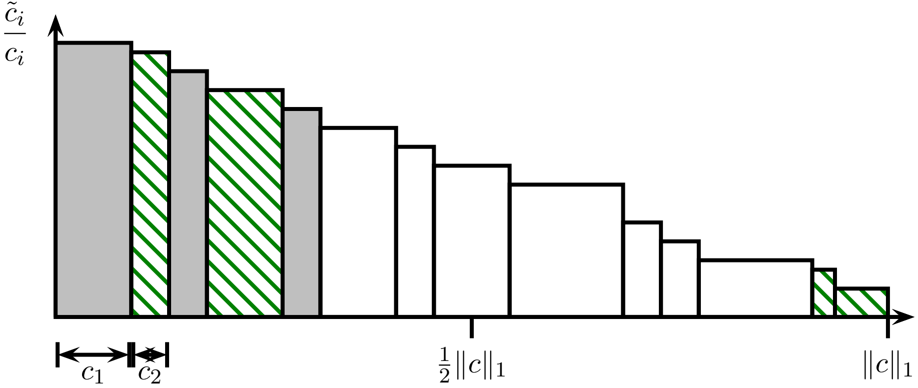

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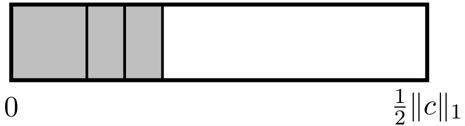

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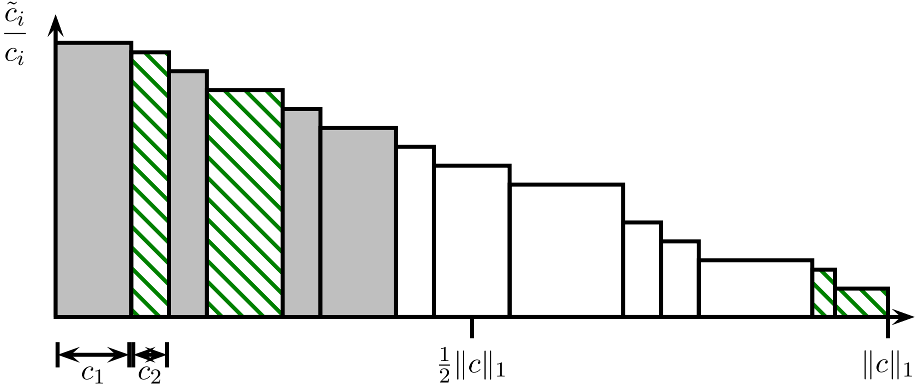

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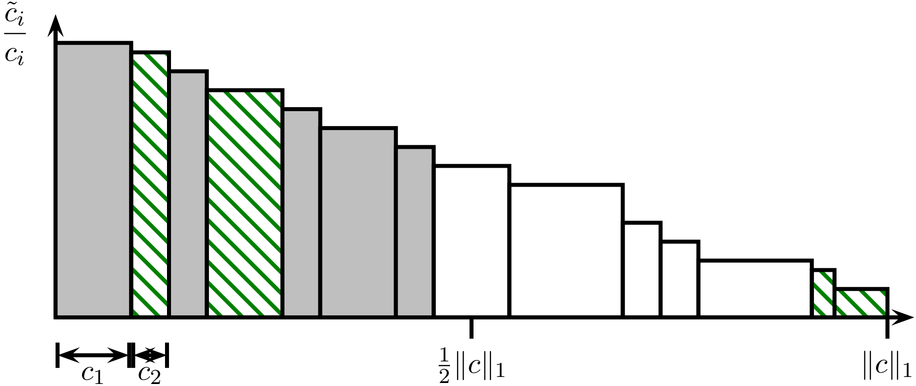

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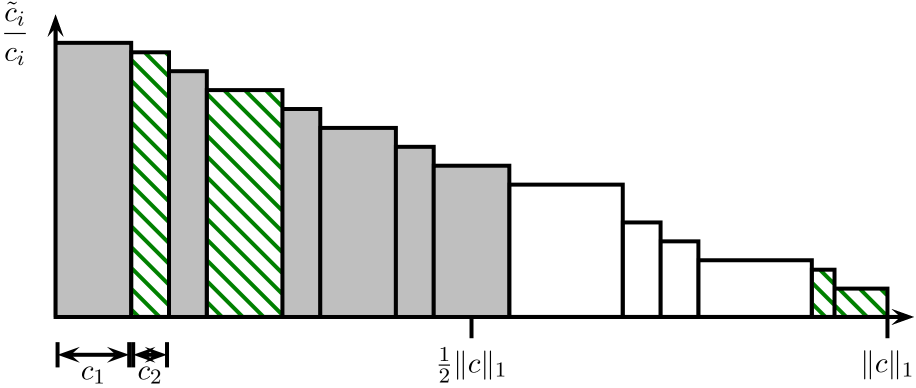

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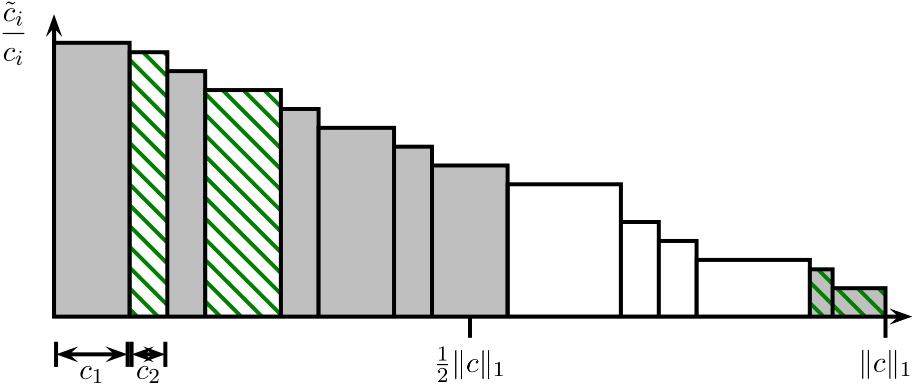

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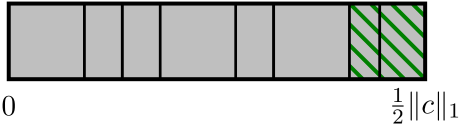

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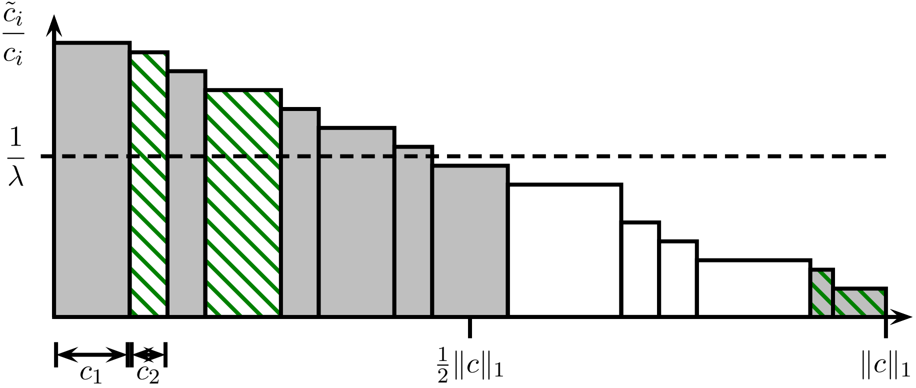

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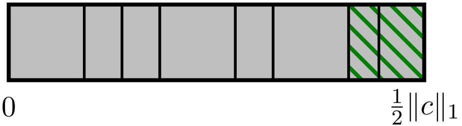

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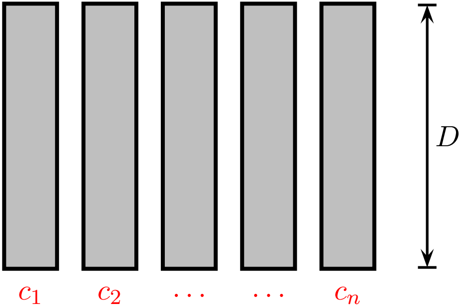

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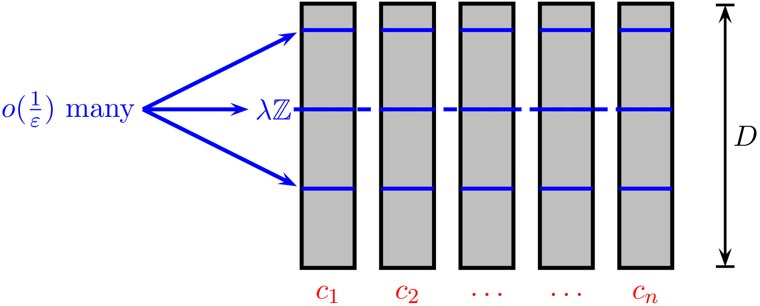

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- **Reason:** Number of λ's and ε's is $2^{O(n)}$; there must be a $\Omega(n)$-size subset of indices satisfying $(*)$

Lemma

For $D := 2^{n/8}$, *pick* $c_i \in \{1, \ldots, D\}$ **at random**. *W.h.p.*

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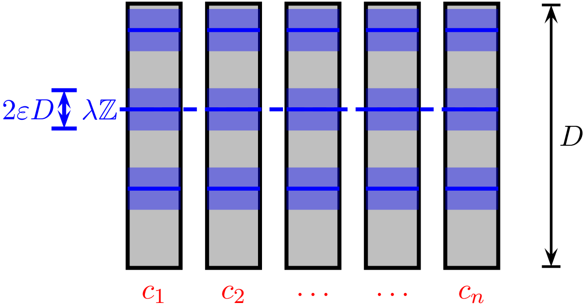

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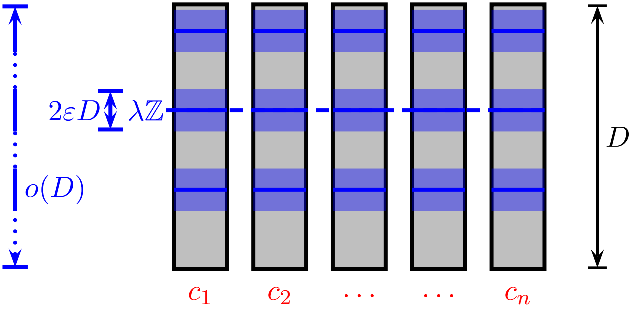

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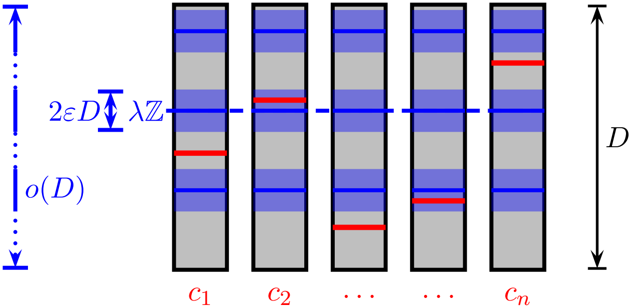

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Overview

random vector has no short, good SDA

↓

critical vectors are good
Simultaneous Diophantine Approximations to c

↓

critical vectors are long $\Longrightarrow \Omega(n^2)$ rank

The end

Thanks for your attention

Where is the bottleneck for $\omega(n^2)$ bound?

- **Problem 1:** *Our proof technique does not extent!*
  $\frac{n}{2}$ random numbers in $\{1, \ldots, D\}$ + $\frac{n}{2}$ "fill numbers" cannot work for $D \gg 2^n$
- **Problem 2:** *Set of normal vectors with* $c_i \geq 2^{\Omega(n \log n)}$ *is extremely sparse!*
  ($2^{O(n^2)}$ potential normal vectors, but $2^{\Omega(n^2 \log n)}$ vectors with $n \log n$ bits per coefficient)
- **Problem 3:** *For coefficients* $> 2^{\omega(n)}$*, better SDAs exist!*
  For $c \in [0,1]^n$ and $N \in \mathbb{N}$. Find $Q \in \{1, \ldots, N\}$ s.t. minimize $\|c - \frac{\mathbb{Z}^n}{Q}\|_\infty$.
  - For $Q := N$, error $\leq \frac{1}{N}$
  - Dirichlet's Theorem: error $\leq \frac{1}{Q \cdot N^{1/n}}$