Iterative Randomized Rounding

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Joint work with Jarosław Byrka,
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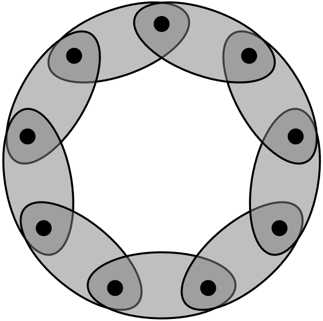
Massachusetts Institute of Technology

Alexander von Humboldt Stiftung/Foundation

What is Iterative Randomized Rounding?

Set Cover:

- Input: Sets $S_1, \dots, S_m$ over elements $1, \dots, n$; cost $c(S_i)$
- Goal: $\min_{I \subseteq [m]} \{ \sum_{i \in I} c_i \mid \bigcup_{i \in I} S_i = [n] \}$



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Standard LP:

$$\min \sum_{i=1}^{m} c(S_i) \cdot x_i$$

$$\sum_{i : j \in S_i} x_i \geq 1 \quad \forall j \in [n]$$

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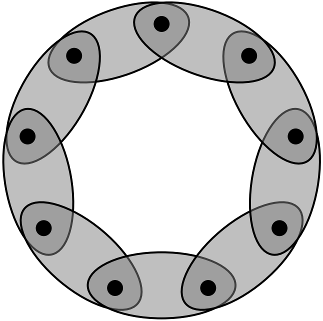

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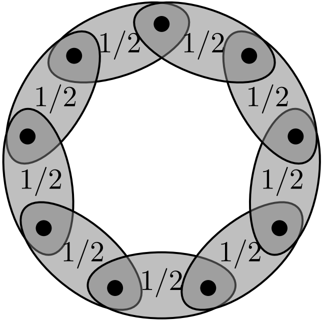

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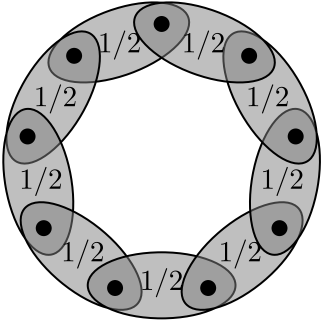

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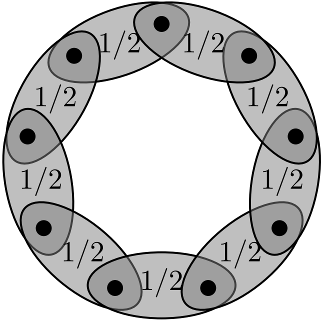

Known:

- Integrality gap is $\Theta(\ln n)$
- Suppose $|S_i| \leq k$. Then gap is $\Theta(\ln k)$.

Iterative randomized rounding algorithm:

(1) FOR $t = 1$ TO ∞

  (2) Solve LP $\rightarrow x$

  (3) FOR ALL i: Buy S_i with prob. x_i (remove covered el.)

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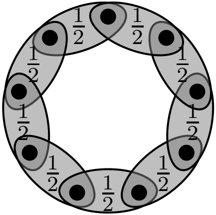
x			
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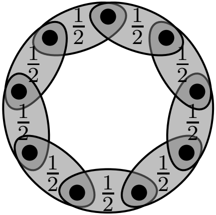
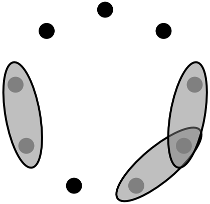
x			
bought sets			

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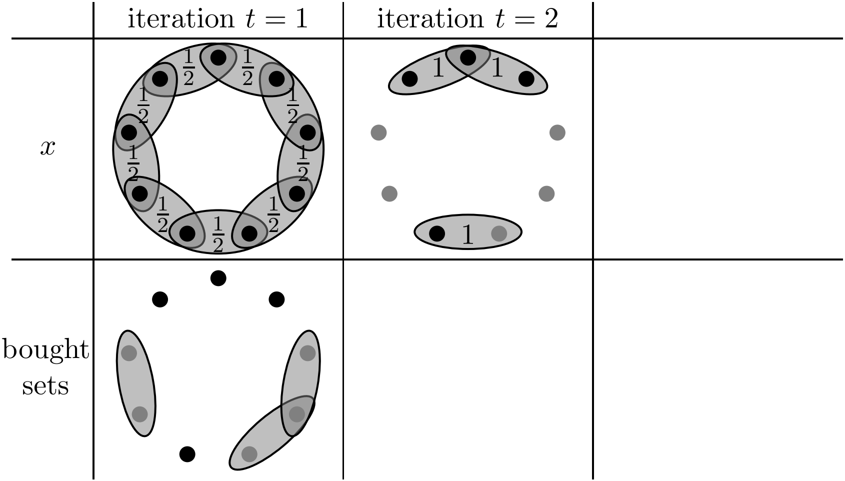

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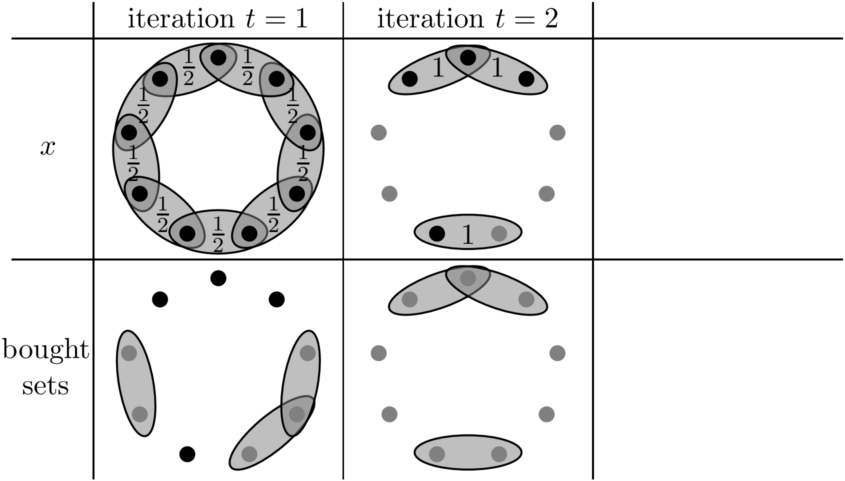

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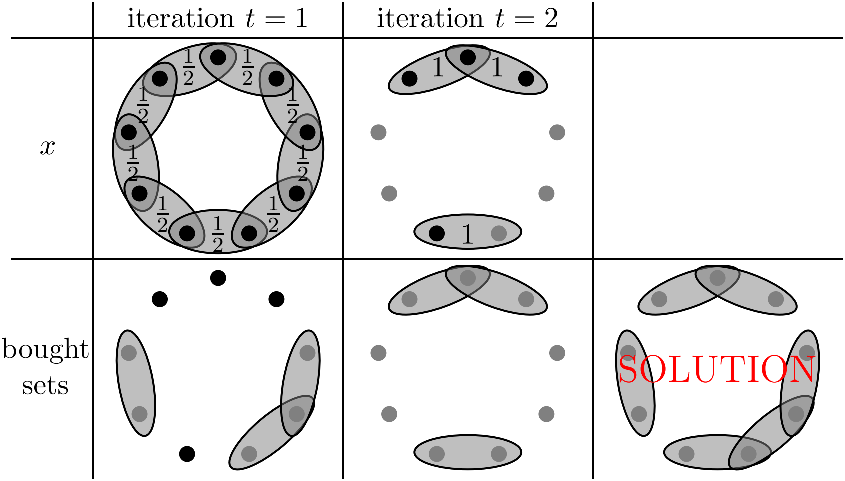

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- Let I^* be optimal Set Cover solution.

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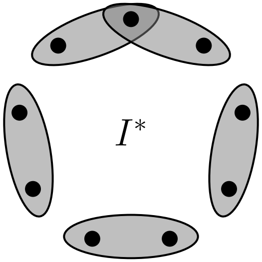

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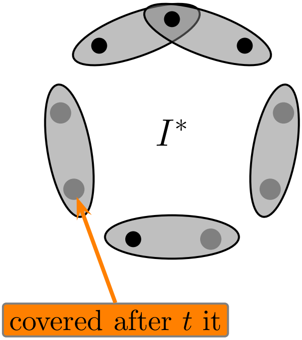

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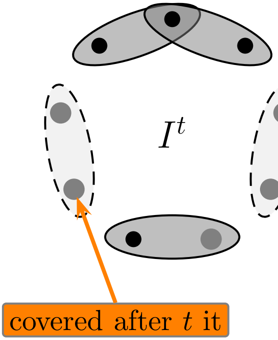

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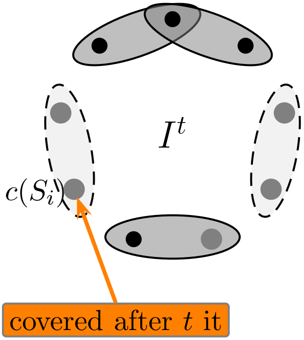

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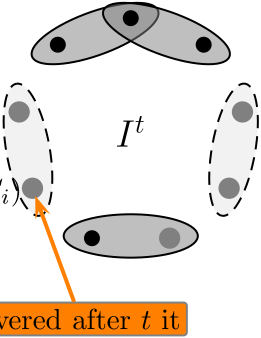

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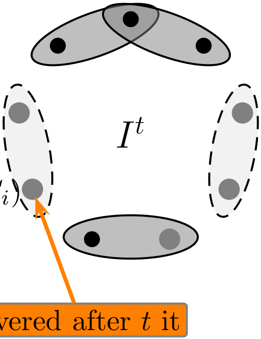


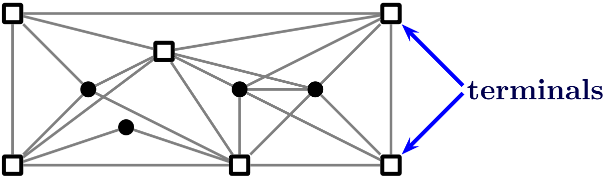
Steiner Tree

Given:

- undirected graph $G = (V, E)$
- cost $c : E \to \mathbb{Q}_+$
- terminals $R \subseteq V$

Find: Min-cost Steiner tree, spanning R.

$$OPT := \min\{c(S) \mid S \text{ spans } R\}$$



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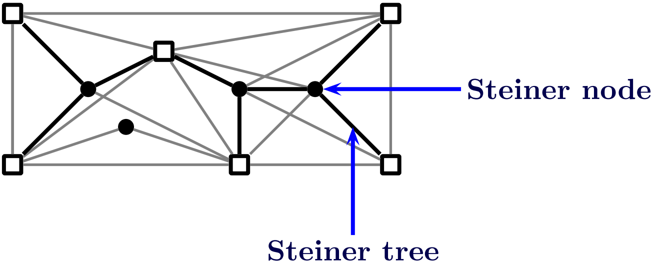

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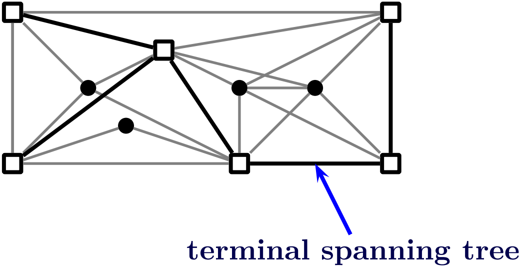
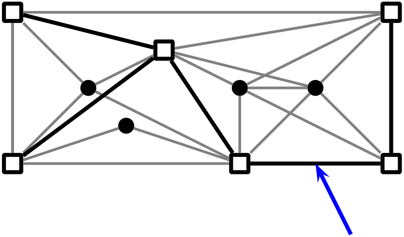


Min-cost terminal spanning tree (MST):

Spanning tree



Min-cost terminal spanning tree (MST):

- Can be computed in poly-time.

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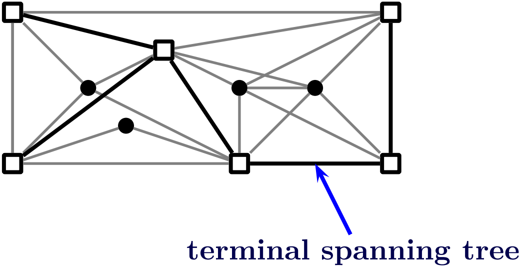

Min-cost terminal spanning tree (MST):

- Can be computed in poly-time.
- Costs $\leq 2 \cdot OPT$.

Known results for Steiner tree:

Approximations:

- 2-apx (*minimum spanning tree heuristic*)
- 1.83-apx [Zelikovsky '93]
- 1.667-apx [Prömel & Steger '97]
- 1.644-apx [Karpinski & Zelikovsky '97]
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Hardness:

- **NP**-hard even if edge costs $\in \{1, 2\}$ [Bern & Plassmann '89]
- no $< \frac{96}{95}$-apx unless $\mathbf{NP} = \mathbf{P}$ [Chlebik & Chlebikova '02]

Our results:

Theorem

There is a polynomial time 1.39-approximation.

- LP-based! *(Directed-Component Cut Relaxation)*
- Algorithmic framework: *Iterative Randomized Rounding*

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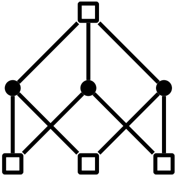
Theorem

The Directed-Component Cut Relaxation has an integrality gap of at most 1.55.

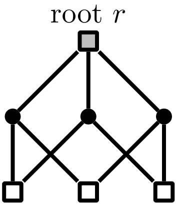
- First < 2 bound for *any* LP-relaxation.

Bi-directed cut relaxation



Bi-directed cut relaxation

- Pick a **root** $r \in R$



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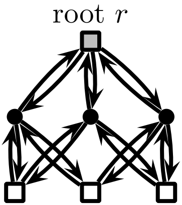

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$$\sum_{e \in \delta^+(U)} z_e \geq 1 \qquad \forall U \subseteq V \setminus \{r\} : U \cap R \neq \emptyset$$

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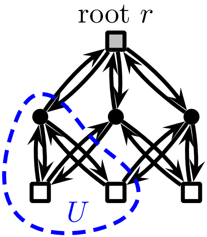

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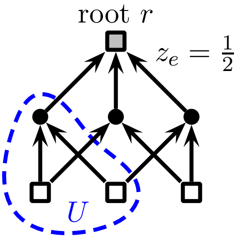

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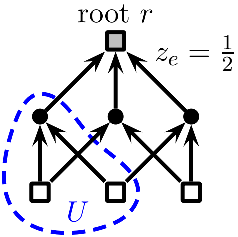
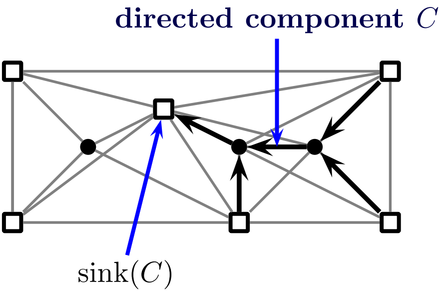


Theorem (Edmonds '67)

$R = V \Rightarrow BCR$ integral

- Integrality gap $\leq 4/3$ for quasi-bipartite graphs [Chakrabarty, Devanur, Vazirani '08]
- Integrality gap $\in [1.16, 2]$

Components



- $\mathbf{C}$ = set of directed components

Directed component cut relaxation

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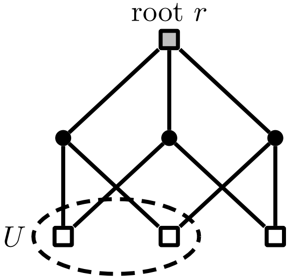

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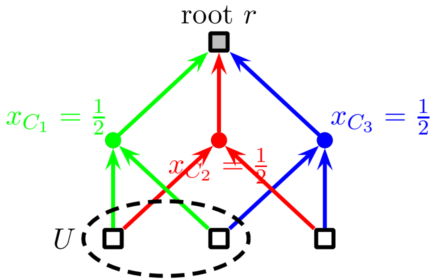

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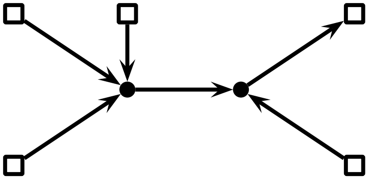
Properties:

- Number of variables: exponential
- Number of constraints: exponential
- Approximable within $1 + \varepsilon$ (we ignore the ε here).

Solvability of the LP

Lemma

For any $\varepsilon > 0$, a solution x of cost $\leq (1+\varepsilon)OPT_f$ can be computed in polynomial time.



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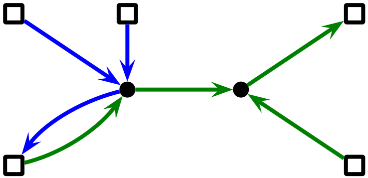

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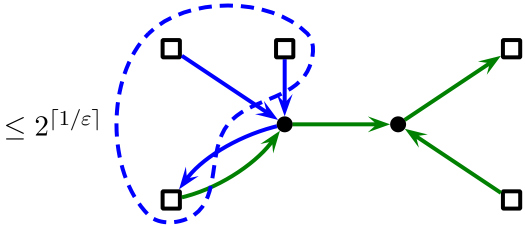

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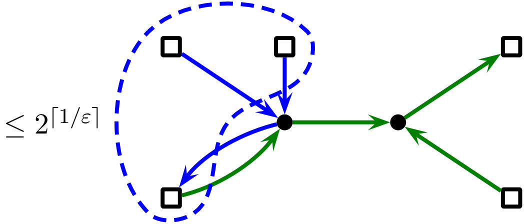

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  $\rightarrow$ # variables polynomial
- Compact flow formulation $\rightarrow$ # constraints polynomial
  (or solve with ellipsoid method). □

An iterative randomized rounding algo

(1) FOR $t = 1, \ldots, \infty$ DO

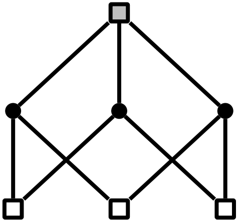
(2) Compute opt. LP solution x

(3) Sample a component:

$$\Pr[\text{sample } C] = \frac{x_C}{\mathbf{1}^T x}$$

and contract it.

(4) IF all terminals connected THEN output sampled components



An iterative randomized rounding algo

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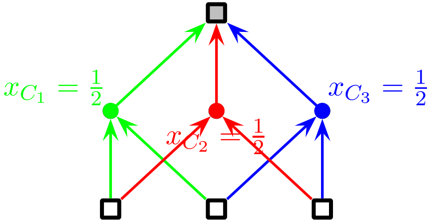

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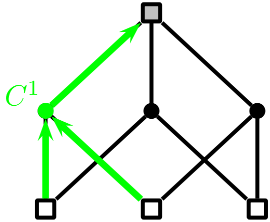

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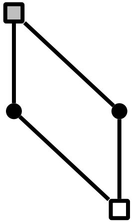

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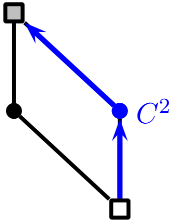

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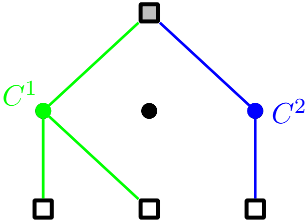

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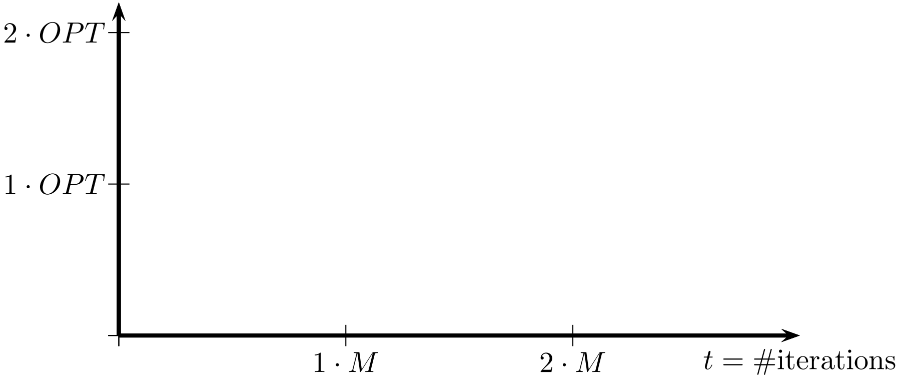
- W.l.o.g. $M := \mathbf{1}^T x$ invariant

Roadmap

- In one iteration t:

  $$E[c(\text{comp. sampled in it. } t)] = \sum_C \frac{x_C}{M} \cdot c(C) \leq \frac{1}{M} \cdot OPT \text{ in it } t$$



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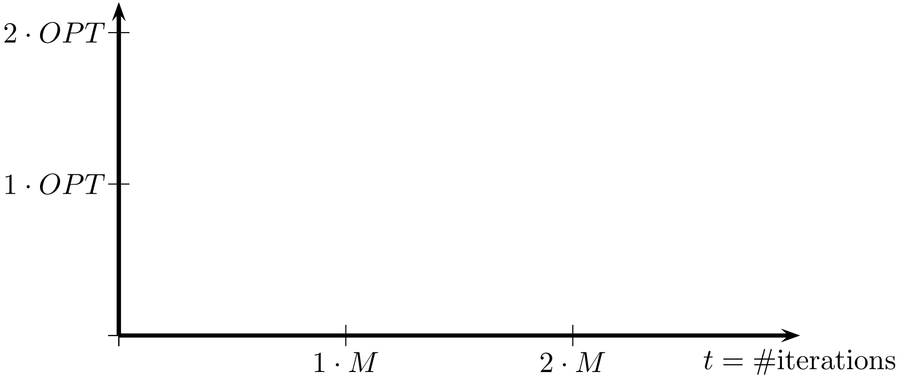

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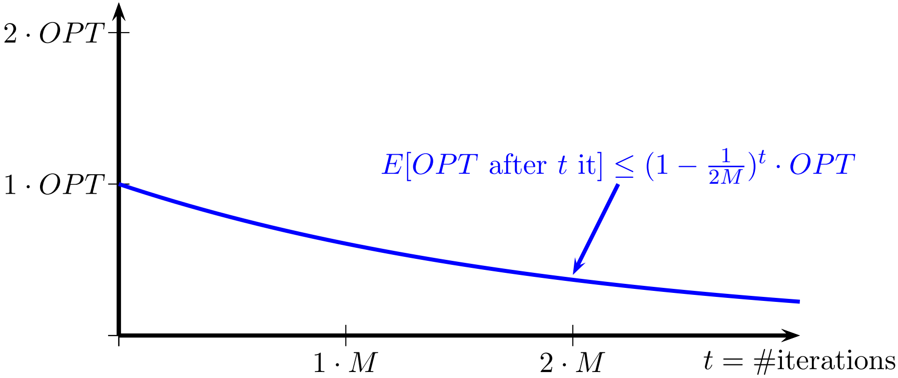

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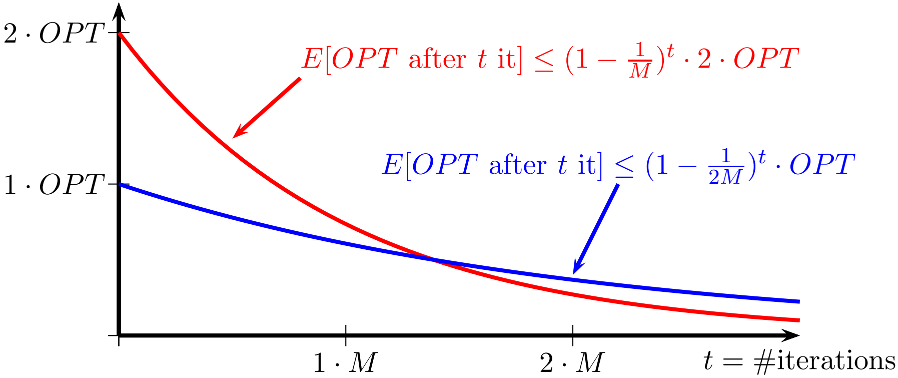
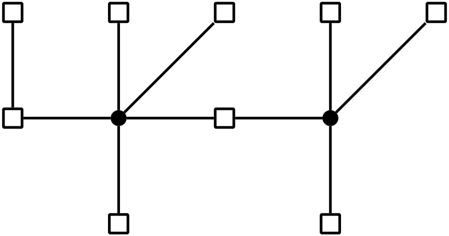


Bridges

- Let S be Steiner tree



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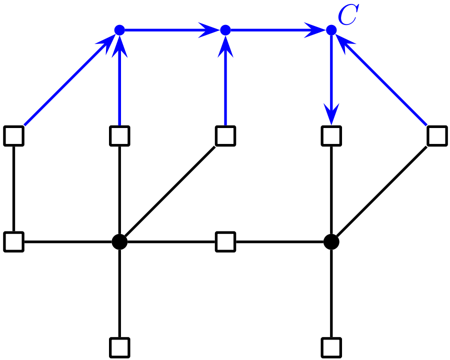

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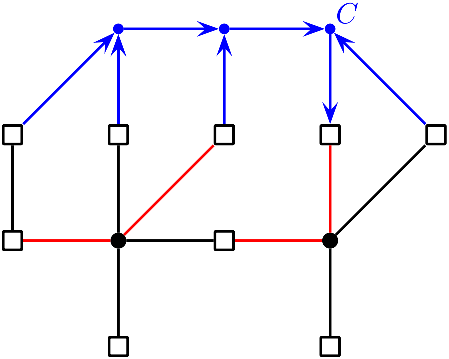

- **Bridges:**

$$Br_S(C) = \text{argmax}\{c(B) \mid B \subseteq S,\ S \backslash B \cup C \text{ is connected}\}$$

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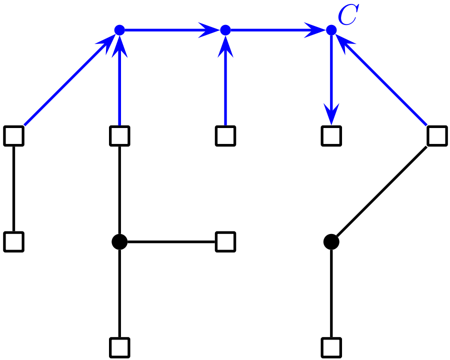

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The saving function

Definition

For a Steiner tree S, the **saving function** $w : E \to \mathbb{Q}_+$ is defined as

$$w(u,v) := \max\{c(e) \mid e \text{ on } u-v \text{ path in } S\}.$$

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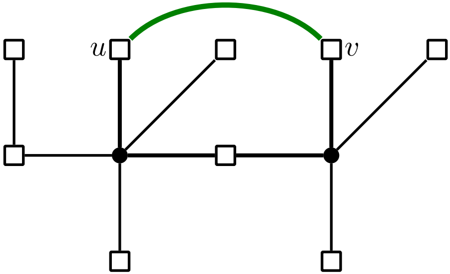
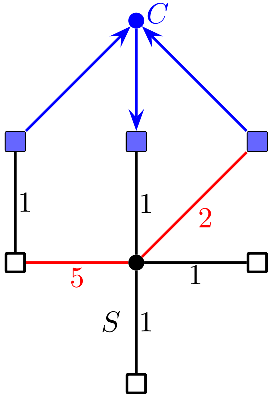


A saving lemma

Lemma

For any component C, $\exists$ saving tree spanning the terminals of C with

$$c(Br_S(C)) = w(\textit{saving tree})$$



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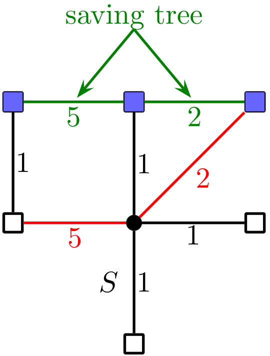
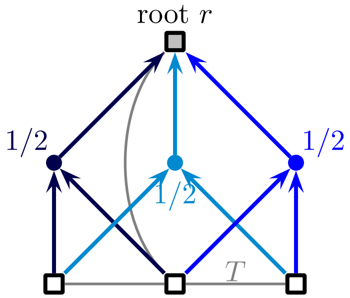


The Bridge Lemma (1)

Lemma (Bridge Lemma)

For T terminal spanning tree, x LP solution:

$$E\left[c\left(\begin{array}{c}\textit{terminal spanning tree}\\ \textit{after } 1 \textit{ sampling step}\end{array}\right)\right] \leq \left(1-\frac{1}{M}\right)\cdot c(T)$$



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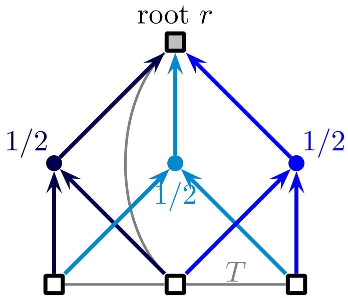

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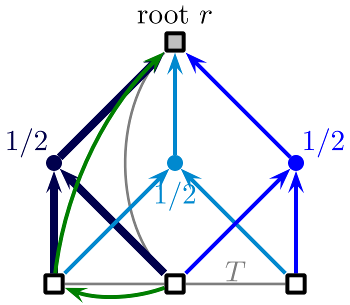

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- Transfer capacity from component to its saving tree
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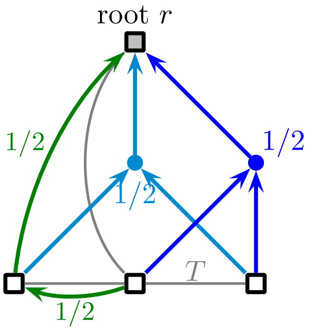

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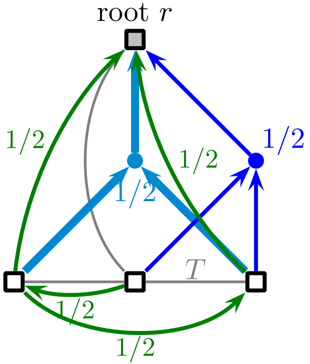

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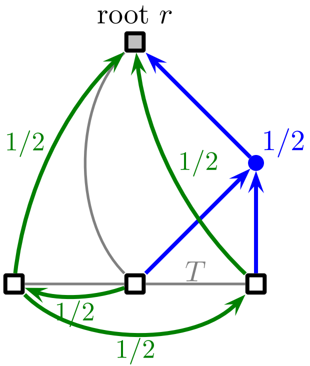

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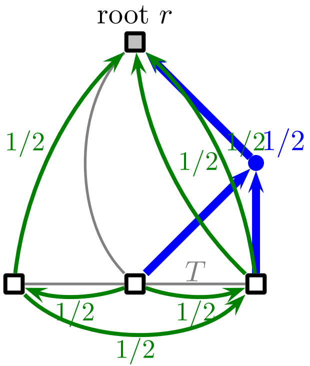

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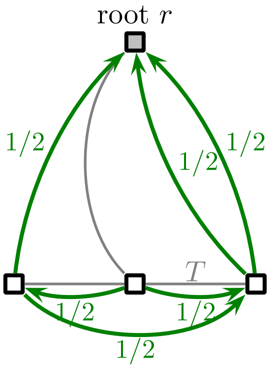

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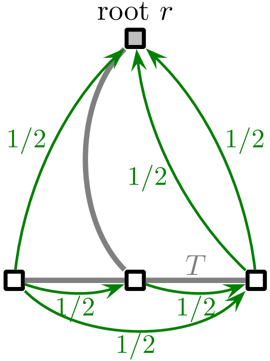
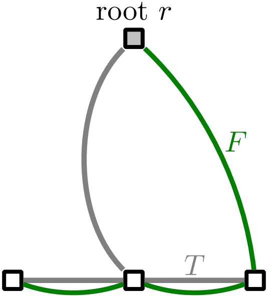


The Bridge Lemma (2)

Edmonds Thm

$$\sum_{C \in \mathbf{C}} x_C \cdot c(Br_T(C)) = w(y) \geq w(F)$$



The Bridge Lemma (2)

Cycle rule

Edmonds Thm

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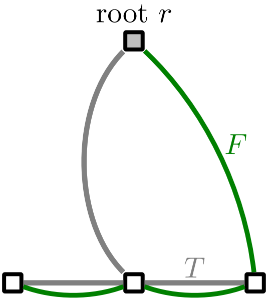

A 1st bound on OPT

Lemma

$E[OPT \ after \ it. \ t] \leq \left(1 - \frac{1}{M}\right)^t \cdot 2 \cdot OPT.$

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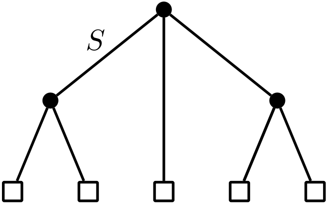
A 2nd bound on OPT

Theorem

In any iteration

$$E[new\ OPT] \leq \left(1 - \frac{1}{2M}\right) \cdot old\ OPT$$

- Let S be opt. Steiner tree



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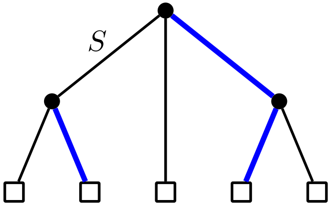

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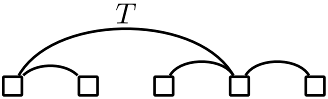

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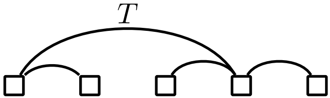


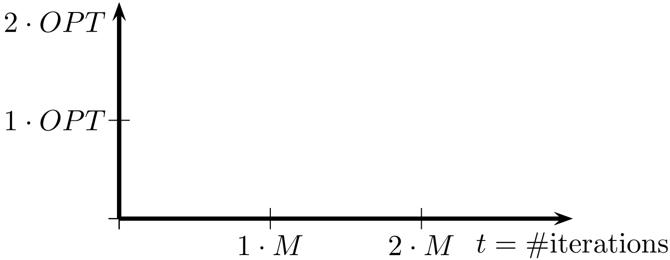
$$E[\text{save on } S] \geq E[\text{save on } T] \overset{\text{Bridge Lem}}{\geq} \frac{1}{M} \cdot \underbrace{c(T)}_{\geq \frac{1}{2}c(S)} \geq \frac{1}{2M} \cdot c(S)$$

$\square$

The approximation guarantee

Theorem

$E[APX] \leq (1.5{+}\varepsilon) \cdot OPT.$



- Cost of sampled components:

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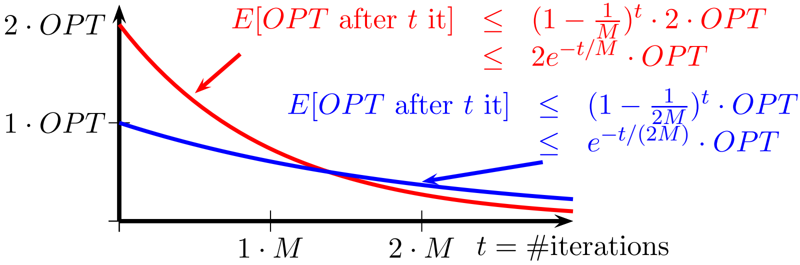

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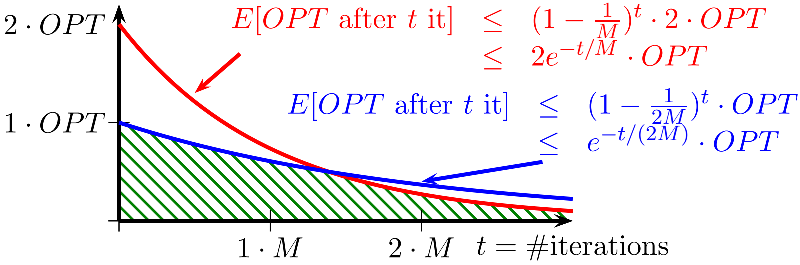

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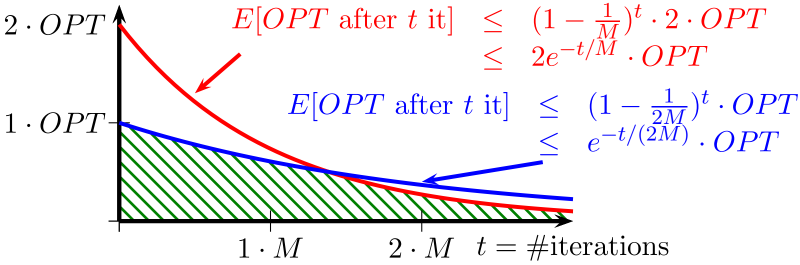

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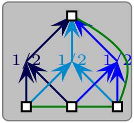
A generalized bridge lemma



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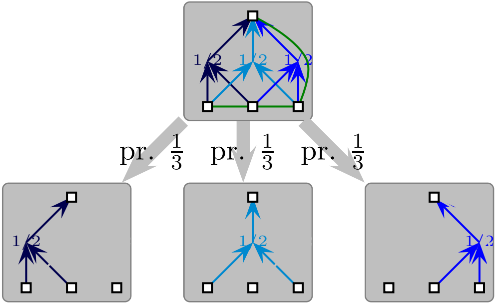
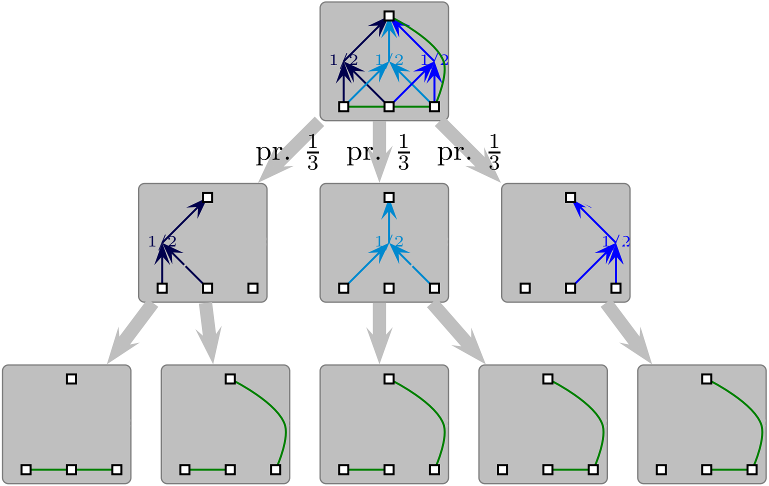


A generalized bridge lemma



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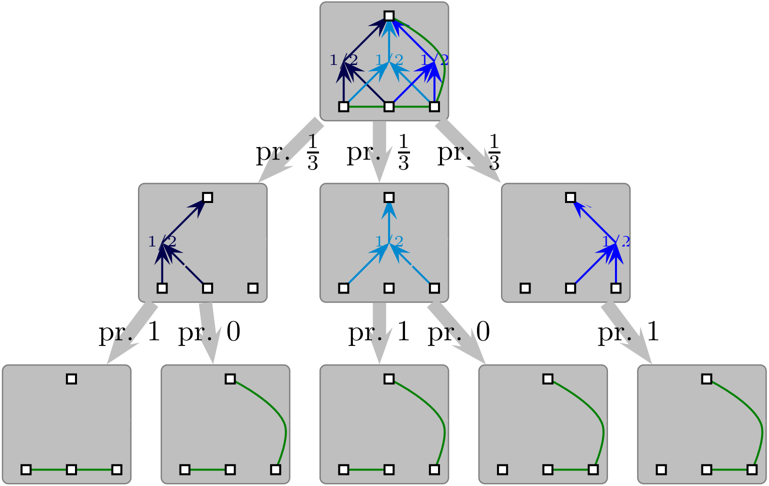
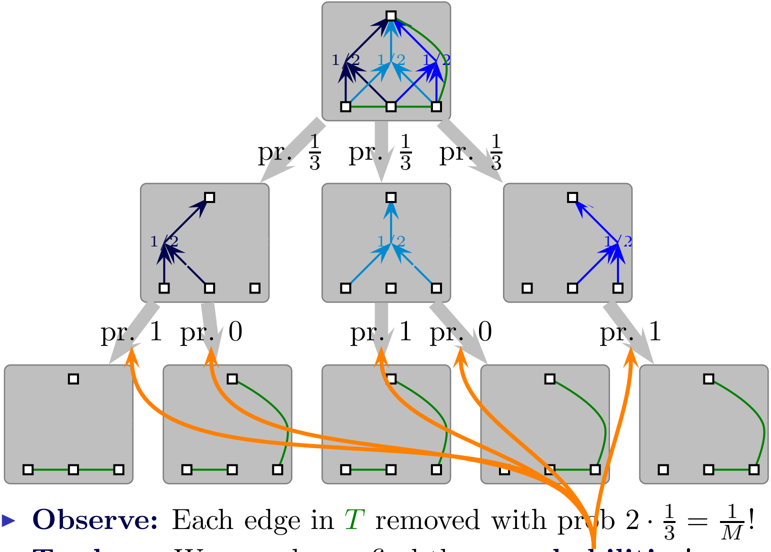


- **Observe:** Each edge in T removed with prob $2 \cdot \frac{1}{3} = \frac{1}{M}$!

A generalized bridge lemma



- **Observe:** Each edge in T removed with prob $2 \cdot \frac{1}{3} = \frac{1}{M}$!
- **To show:** We can always find these **probabilities**!

A generalized bridge lemma (2)

Lemma

Let T be terminal spanning tree. Sample C from LP solution x. There are $B \subseteq T$ (dep. on C) s.t.

- $(T \setminus B) \cup C$ *spans all terminals*
- $\Pr[e \in B] \geq \frac{1}{M} \; \forall e \in T$.

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$$\begin{aligned} \sum_{B:(T\setminus B)\cup C \text{ conn.}} \Pr[\text{rem. } B \mid C] &= 1 \ \forall C \\ \sum_{B \ni e, C} \Pr[C] \cdot \Pr[\text{rem. } B \mid C] &\geq \frac{1}{M} \ \forall e \end{aligned}$$

- Suppose system (1) has no non-negative solution.

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$$\begin{aligned} y_C &\geq x_C \cdot c(B) \ \forall B : T \setminus B \cup C \text{ conn.} \\ \textstyle\sum_C y_C &< c(T) \end{aligned}$$

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- **Farkas Lemma:** System (2) has solution $(y, c) \geq \mathbf{0}$

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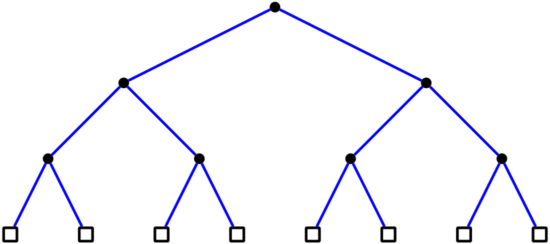
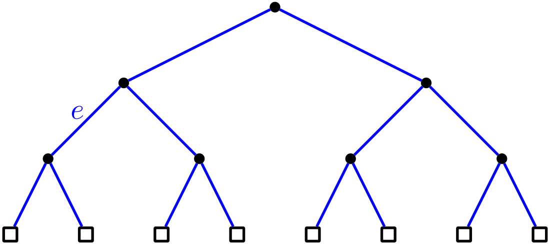
- Suppose system (1) has no non-negative solution.
- **Farkas Lemma:** System (2) has solution $(y, c) \geq \mathbf{0}$
- Contradiction to Bridge Lemma! □

The 1.39 bound



- Let S^* optimum Steiner tree.

The 1.39 bound



- Let S^* optimum Steiner tree.
- **Goal:** Define Steiner tree $S^t \subseteq S^*$ after t iterations with $E[t : e \in S^t] \leq 1.39 \cdot M$.

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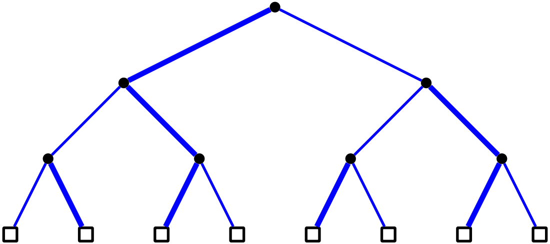

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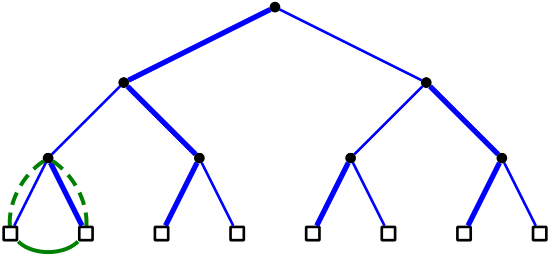

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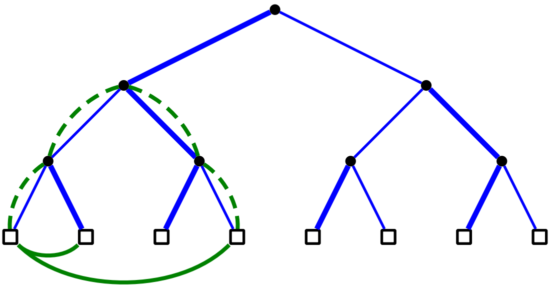

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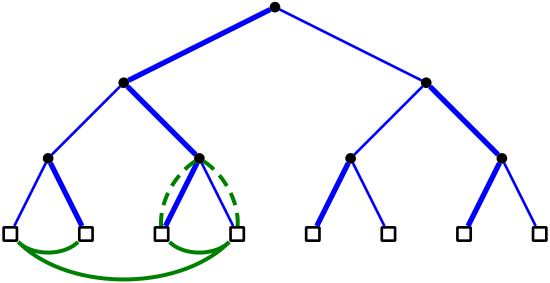

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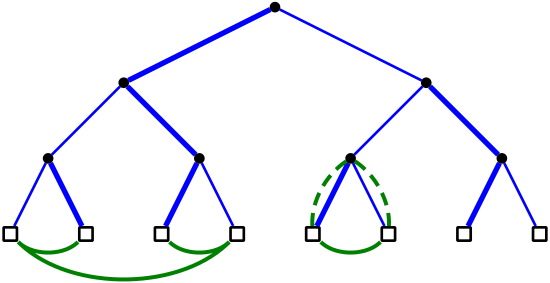

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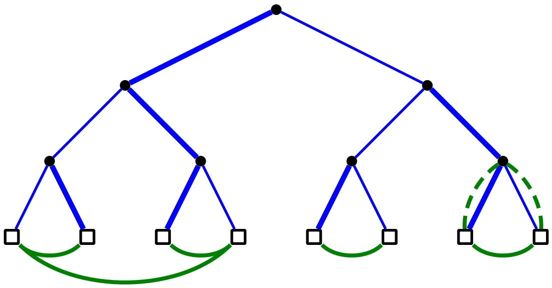

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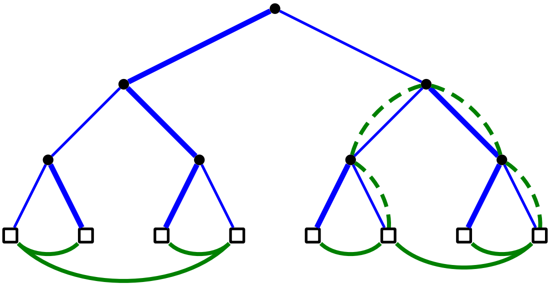

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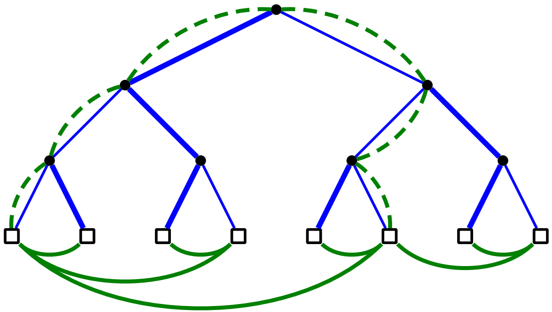

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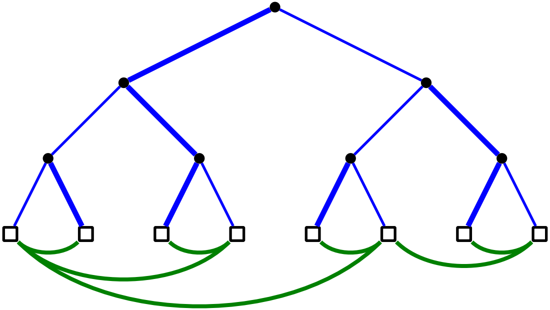

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- Such edges $\{u, v\}$ induce terminal spanning tree T

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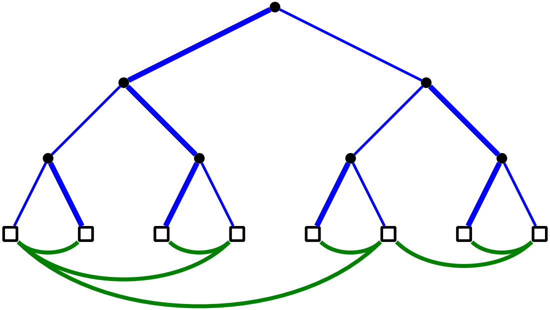

- Def S^t: $e \in T$ not deleted $\Rightarrow$ keep edges in corr. cycle in S^*

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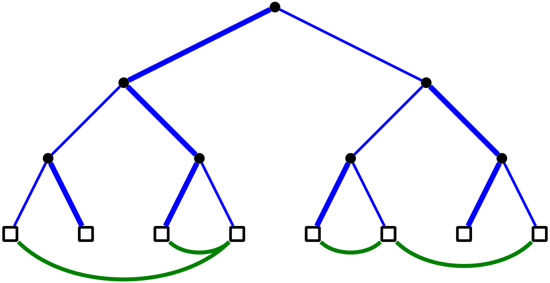

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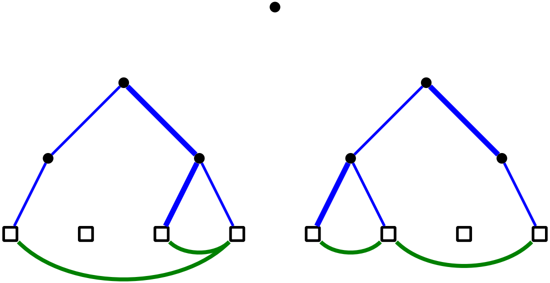

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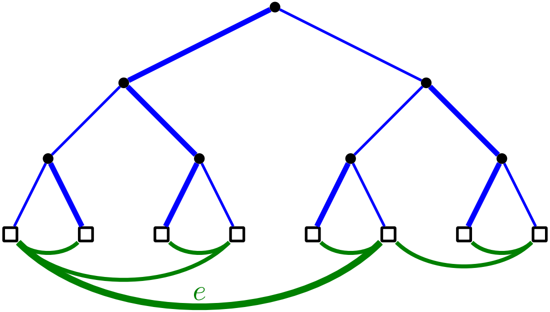

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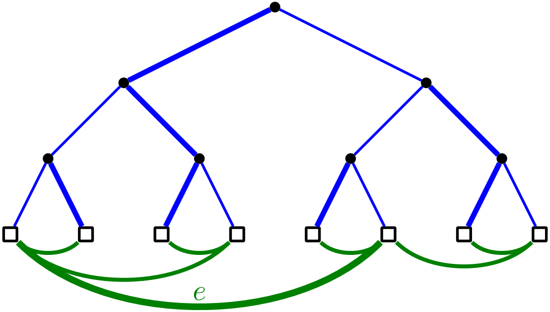

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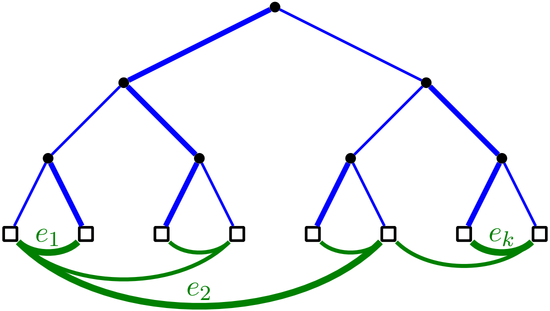

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  $\rightarrow$ **Coupon Collector Theorem**

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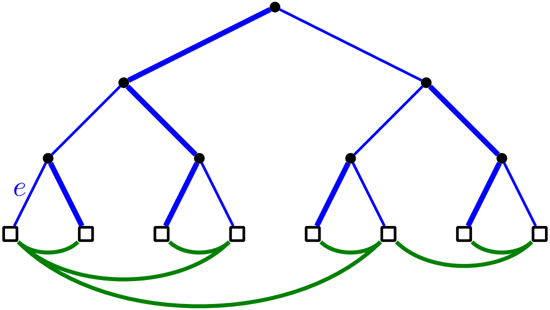

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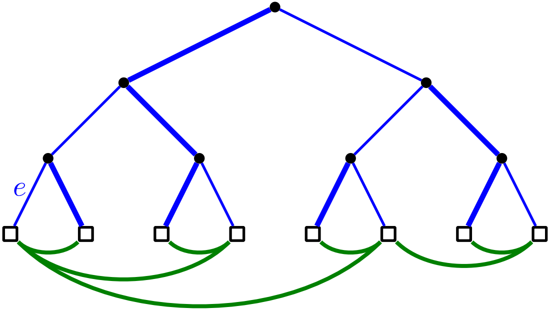

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- $\Pr[e \text{ in } k \text{ cycles}] = (\frac{1}{2})^k$
  $E[t : e \text{ deleted}] \leq \sum_{k \geq 1} \left(\frac{1}{2}\right)^k \cdot H(k) \cdot M = \ln(4) \cdot M \approx 1.39 \cdot M.$ $\square$

Open problems

Open Problem I

$1.01 \leq$ Steiner tree approximability ≤ 1.39

Open problems

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Open Problem II

Is there an iterative randomized rounding approach for Facility Location or k-Median?

Open problems

Open Problem III

Is there an iterative randomized rounding approach for ATSP?

(1) Solve Held-Karp relaxation:

$$\begin{aligned} \min c^T x \\ x(\delta^+(S)) &\geq 1 \quad \forall \emptyset \subset S \subset V \\ x(\delta^+(v)) = x(\delta^-(v)) &= 1 \quad \forall v \in V \\ x_e &\geq 0 \quad \forall e \in E \end{aligned}$$

(2) Sample a collection of cycles $\mathcal{C}$ from x^*.

(3) Show $E[c(\mathcal{C})] \leq 1000 \cdot OPT$

(4) Show $E[OPT \text{ after contracting } \mathcal{C}] \leq 0.999 \cdot OPT$.

This would yield a $O(1)$-apx.

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Thanks for your attention