Tutorial: The Lasserre Hierarchy in Approximation algorithms

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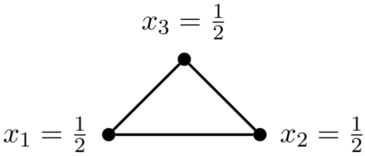
Motivation

Problem: Weak LP $K = \{x \in \mathbb{R}^n \mid Ax \geq b\}$

Example: Independent Set

- Relaxation: $K = \{\mathbb{R}^V \mid x_u + x_v \leq 1 \; \forall (u,v) \in E\}$



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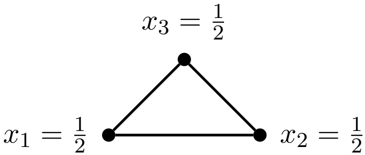

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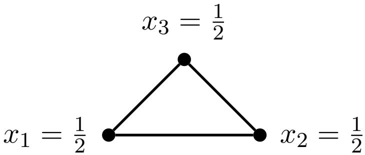

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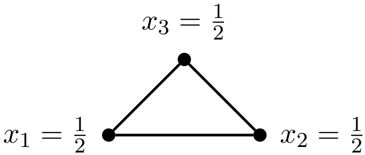
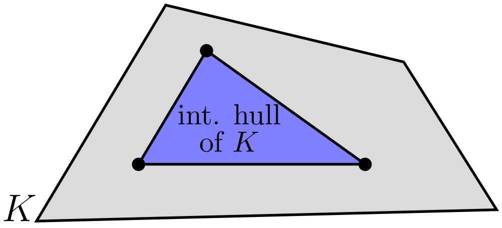


Try to convince you: Option II is better!

Lift-and-project methods



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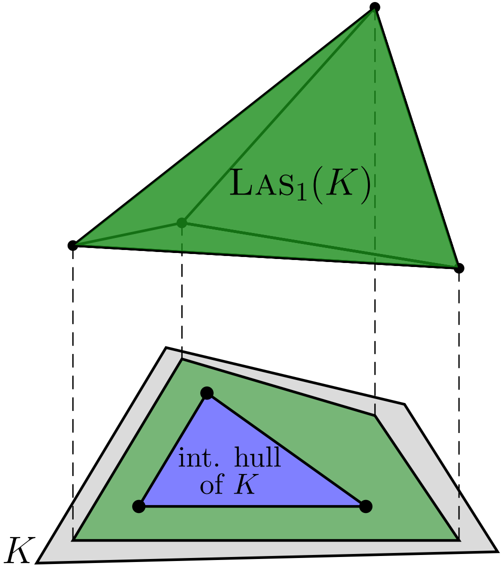

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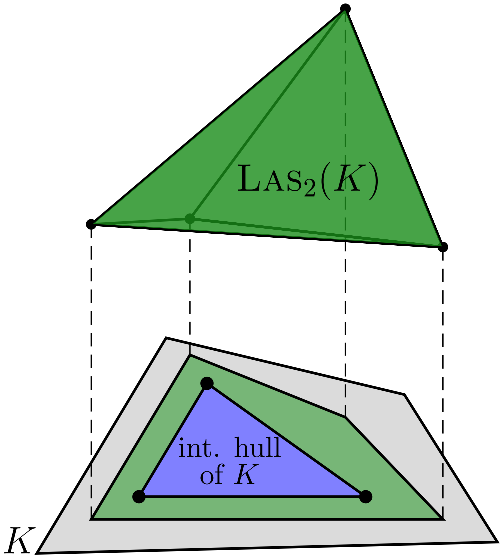

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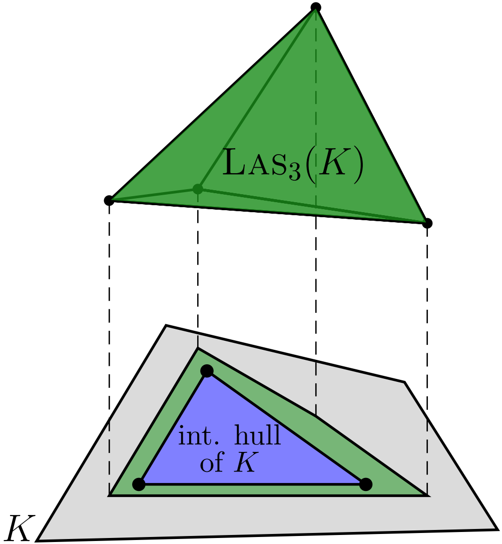

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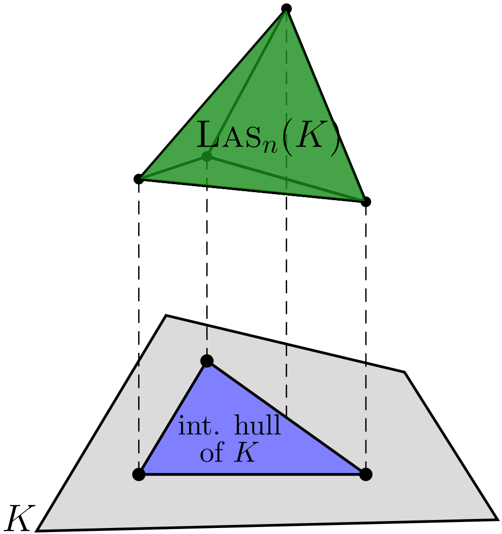

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- $\boldsymbol{x}^T M \boldsymbol{x} \geq 0 \; \forall \boldsymbol{x}$
- Any principal submatrix has det ≥ 0
- $\exists$ vectors $\boldsymbol{v}_i$ with $M_{ij} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle$

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$$\begin{aligned} (y_{I \cup J})_{|I|,|J| \leq t} &\succeq 0 \\ \left(\sum_{i \in [n]} A_{\ell i} y_{I \cup J \cup \{i\}} - b_\ell y_{I \cup J} \right)_{|I|,|J| \leq t} &\succeq 0 \quad \forall \ell \in [m] \\ y_\emptyset &= 1 \end{aligned}$$

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- Solvable in time $n^{O(t)} m^{O(1)}$

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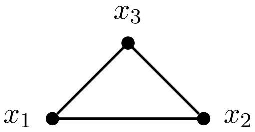
Moment matrix for $t = 1$:

$$M_1(y) = \begin{array}{cc} \begin{array}{cccc} \emptyset & \{1\} & \{2\} & \{3\} \end{array} & \\ \begin{pmatrix} 1 & y_1 & y_2 & y_3 \\ y_1 & y_1 & y_{12} & y_{13} \\ y_2 & y_{12} & y_2 & y_{23} \\ y_3 & y_{13} & y_{23} & y_3 \end{pmatrix} & \begin{array}{c} \emptyset \\ \{1\} \\ \{2\} \\ \{3\} \end{array} \end{array}$$



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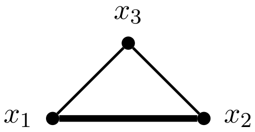


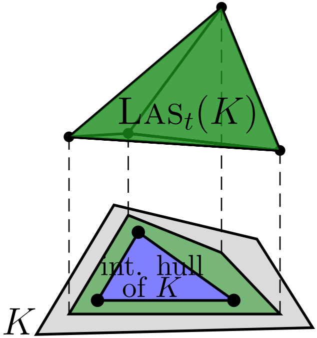
Moment matrix for edge $(1, 2)$ for $t = 1$:

$$\begin{array}{cc} \begin{matrix} \emptyset & \{1\} & \{2\} & \{3\} \end{matrix} & \\ \begin{pmatrix} 1 - y_1 - y_2 & y_1 - y_1 - y_{12} & y_2 - y_{12} - y_2 & y_3 - y_{13} - y_{23} \\ y_1 - y_1 - y_{12} & y_1 - y_1 - y_{12} & y_{12} - y_{12} - y_{12} & y_{13} - y_{13} - y_{123} \\ y_2 - y_{12} - y_2 & y_{12} - y_{12} - y_{12} & y_2 - y_{12} - y_2 & y_{23} - y_{123} - y_{23} \\ y_3 - y_{13} - y_{23} & y_{13} - y_{13} - y_{123} & y_{23} - y_{123} - y_{23} & y_3 - y_{13} - y_{23} \end{pmatrix} & \begin{matrix} \emptyset \\ \{1\} \\ \{2\} \\ \{3\} \end{matrix} \end{array}$$

Basic properties (1)

Lemma

$conv(K \cap \{0,1\}^n) \subseteq \text{LAS}_t^{proj}(K)$



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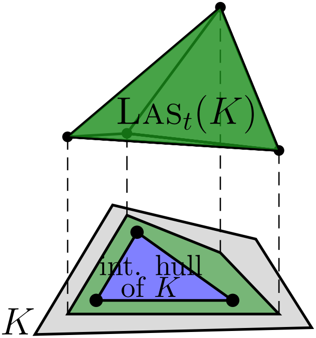

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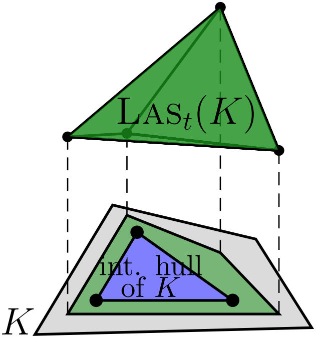
- Then

$$(M_t(y))_{I,J} = y_{I \cup J} = y_I \cdot y_J = (yy^T)_{I,J}$$

and $yy^T \succeq 0$.



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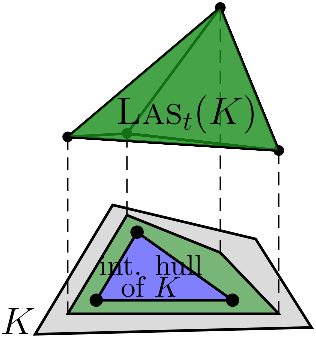

- Similarly $\sum_{i=1}^n a_i y_{I \cup J \cup \{i\}} - \beta y_{I \cup J} = (ax - \beta) \cdot y_I \cdot y_J$ and $(ax - \beta) \cdot yy^T \succeq 0$. □

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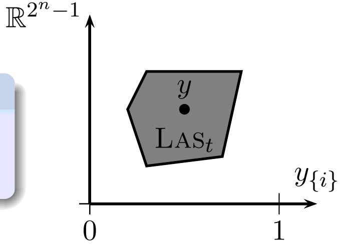

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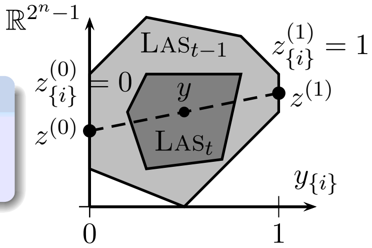

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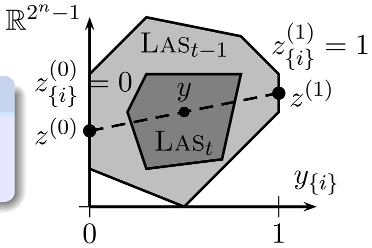

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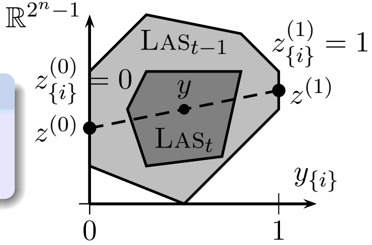

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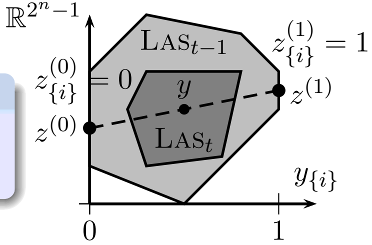

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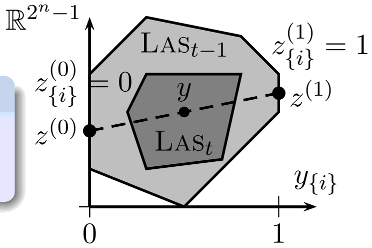

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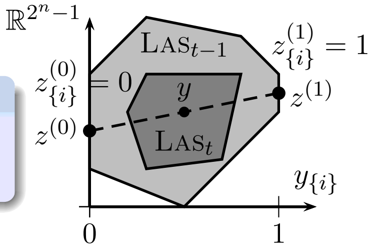

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- Take $\boldsymbol{v}_I$ with $\langle \boldsymbol{v}_I, \boldsymbol{v}_J\rangle = y_{I\cup J}$ for $|I|,|J| \leq t$.
- Then

$$\left\langle \frac{\boldsymbol{v}_{I\cup\{i\}}}{\sqrt{y_i}}, \frac{\boldsymbol{v}_{J\cup\{i\}}}{\sqrt{y_i}} \right\rangle = \frac{y_{I\cup J\cup\{i\}}}{y_i} = z^{(1)}_{I\cup J}$$

and $M_{t-1}(z^{(1)}) \succeq 0$.

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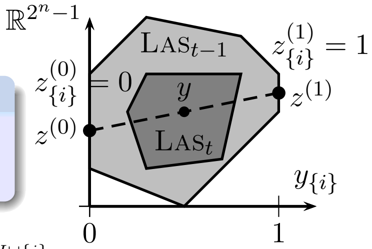

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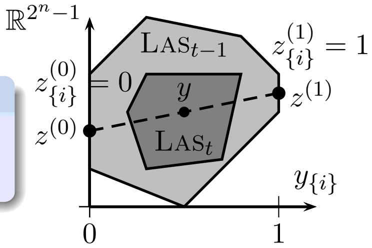
- Moreover $z_i^{(0)} = 0, z_i^{(1)} = 1$. Remains to show psdness.
- Take $\boldsymbol{v}_I$ with $\langle \boldsymbol{v}_I, \boldsymbol{v}_J\rangle = y_{I\cup J}$ for $|I|,|J| \leq t$.
- Moreover

$$\left\langle \frac{\boldsymbol{v}_I - \boldsymbol{v}_{I\cup\{i\}}}{\sqrt{1-y_i}}, \frac{\boldsymbol{v}_J - \boldsymbol{v}_{J\cup\{i\}}}{\sqrt{1-y_i}} \right\rangle = \frac{\boldsymbol{v}_I\boldsymbol{v}_J - \boldsymbol{v}_I\boldsymbol{v}_{J\cup\{i\}} - \boldsymbol{v}_J\boldsymbol{v}_{I\cup\{i\}} + \boldsymbol{v}_{I\cup\{i\}}\boldsymbol{v}_{J\cup\{i\}}}{1-y_i}$$

$$= \frac{y_{I\cup J} - y_{I\cup J\cup\{i\}}}{1-y_i} = z^{(0)}_{I\cup J}$$

Thus $M_{t-1}(z^{(0)}) \succeq 0$.

Inducing on one variable



Lemma

For $y \in \text{LAS}_t(K)$, $t \geq 1$, $i \in [n]$,
$y \in conv\{z \in \text{LAS}_{t-1}(K) \mid z_i \in \{0,1\}\}$.

- Define $z^{(1)}_I := \frac{y_{I\cup\{i\}}}{y_i}$ and $z^{(0)}_I := \frac{y_I - y_{I\cup\{i\}}}{1-y_i}$
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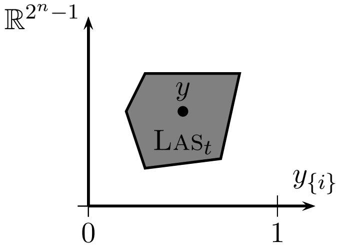
- Similar for slack moment matrices.

Consequences (1)

Operation: "Induce on $x_i = 1$"



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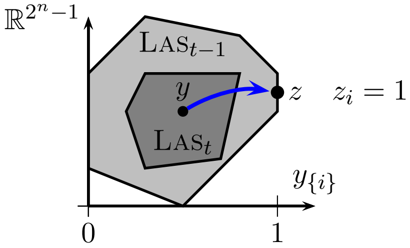

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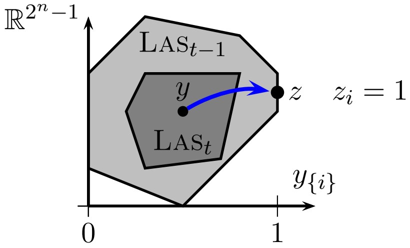

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For $y \in \textsc{Las}_t(K)$, pick a set $|S| \leq t$:

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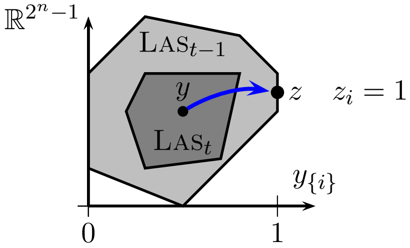

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- Explicit formula for each z via **inclusion exclusion formula**
- Gap closed after n rounds

Consequences (2)

Lemma (Locally consistent probability distribution)

For $y \in \textsc{Las}_t(K)$ and $|S| \le t$, there is a random variable $X \in \{0,1\}^S$ with

$$\Pr\Big[\bigwedge_{i \in I} (X_i = 1)\Big] = y_I \quad \forall I \subseteq S$$

- For example for Independent Set, for $|S| \le t$ nodes, y gives a probability distribution of independent sets in $G[S]$

Application 1:

Scheduling On 2 Machines With Precedence Constraints

$$P2 \mid \text{prec}, p_j = 1 \mid C_{\max}$$

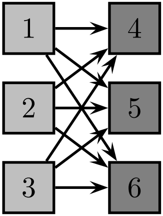
Source: [Svensson, unpublished, 2011]

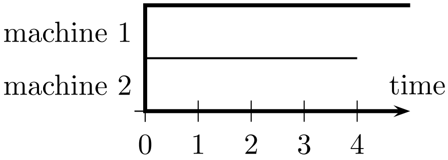
$P2 \mid \textbf{prec}, p_j = 1 \mid C_{\max}$

Input:

- **jobs** J with **unit processing time**
- **precedence constraints**
- 2 identical machines

Goal: minimize makespan





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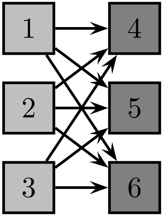

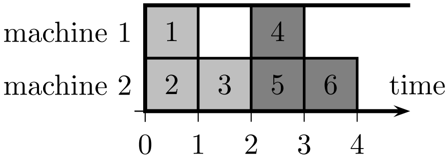

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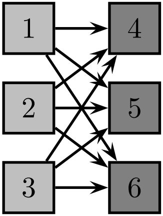

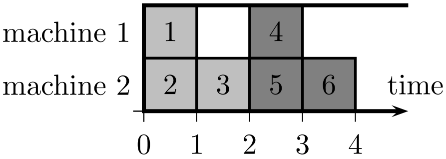

Known results:

- **NP**-hard for general m

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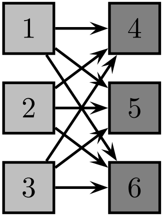

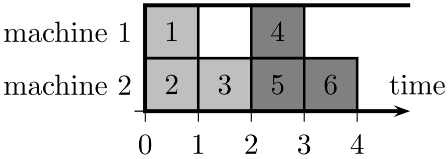

Known results:

- **NP**-hard for general m
- poly-time for 2 machines [Coffman, Graham '72]

Time indexed LP

$$
\begin{array}{rcll}
\sum_{t=1}^{T} x_{jt} & = & 1 & \forall j \in J \\
\sum_{j \in J} x_{jt} & \leq & 2 & \forall t \in [T] \\
\sum_{t' \leq t} x_{it'} & \geq & \sum_{t' \leq t+1} x_{jt'} & \forall i \prec j \;\; \forall t \in [T] \\
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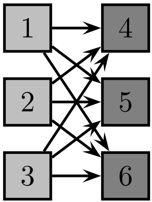
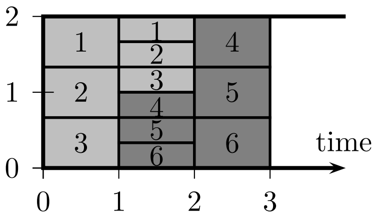
- Integrality gap is $\geq \frac{4}{3}$





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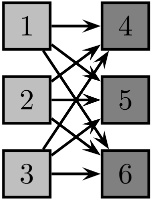

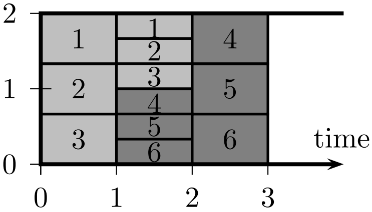

- **Claim:** $y \in \textsc{Las}_1(LP(T)) \Rightarrow \exists$ schedule of length T

Scheduling algorithm

(1) Find $y \in \textsc{Las}_1(LP(T))$

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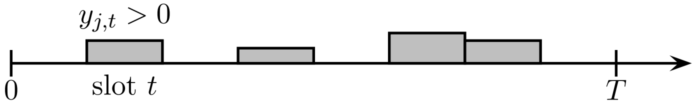
Fractional schedule for j:



Scheduling algorithm

(1) Find $y \in \textsc{Las}_1(LP(T))$

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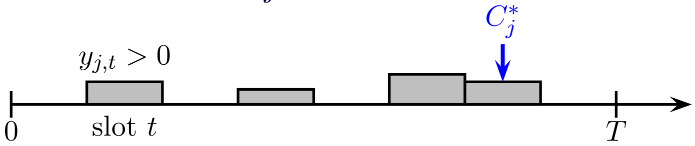

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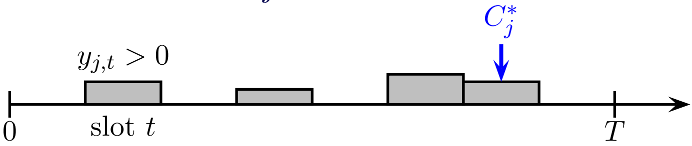

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(4) Run list schedule w.r.t. ordering $\Rightarrow \sigma : J \to \mathbb{N}$

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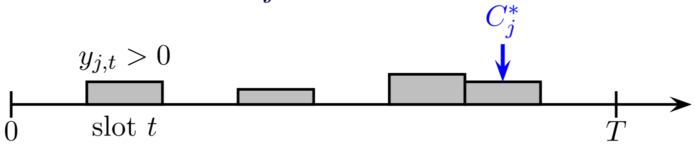

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For any job $\sigma_j \leq C_j^*$.

schedule σ:

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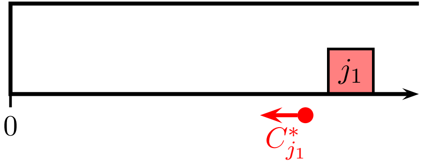

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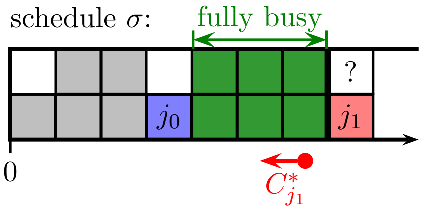

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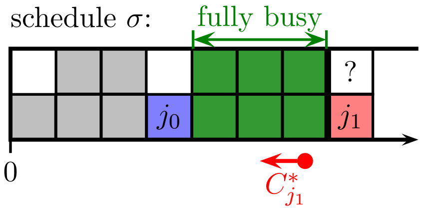

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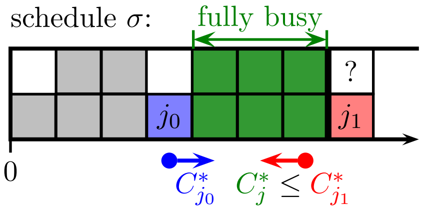

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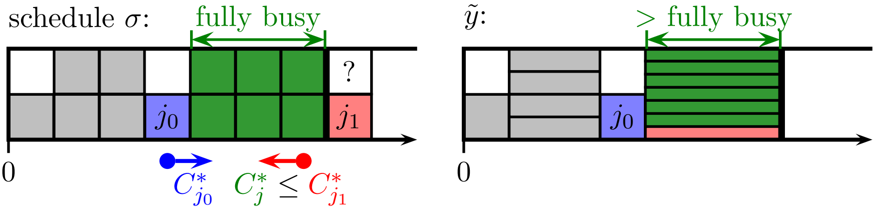

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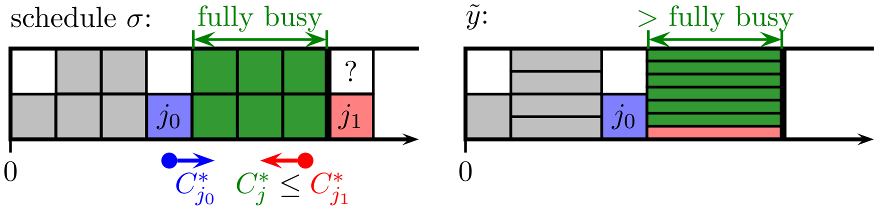
- For LP infeasible! **Contradiction!**



A PTAS for 3 machines?

Open problem

For $m = 3$ machines, is the gap for $f(\varepsilon)$-round Lasserre at most $1 + \varepsilon$?

- Neither known to be **NP**-hard, nor is a PTAS known!

Application 2:

Set Cover

Source: [Chlamtac, Friggstad, Georgiou 2012]

Subexp. Set Cover Approximation

Set Cover:

- **Input:** Family of sets $S_1, \dots, S_m \subseteq [n]$ with cost c_{S_j}
- **Goal:** Cover elements at minimum cost

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- $\ln(n)$-apx [Chvátal '79]

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- Reduction SAT to SET COVER with gap $(1-\varepsilon)\ln(n)$ in time $n^{O(1/\varepsilon)}$ [Dinur, Steurer '13]

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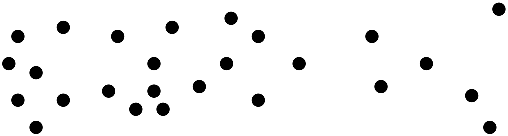
Theorem

There is a $(1-\varepsilon)\ln(n)$-apx in time $2^{\tilde{O}(n^{\varepsilon})}$.

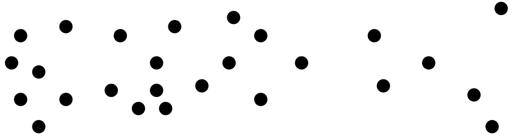
The algorithm



The algorithm

(1) Compute $y \in \textsc{Las}_{n^\varepsilon}(K)$ with

$$K := \left\{ x \in \mathbb{R}^m_{\geq 0} \mid \sum_{i:j \in S_i} x_i \geq 1 \ \forall \text{element } j;\ c^T x \leq OPT \right\}$$



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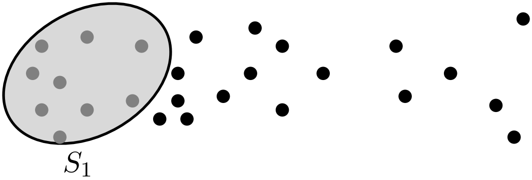
(2) FOR $i = 1, \ldots, n^\varepsilon$ DO

  (3) Find set S_i covering most new elements



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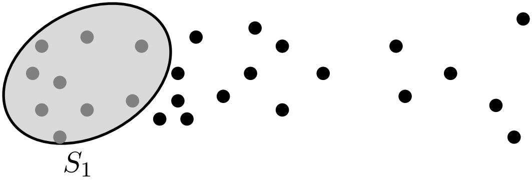

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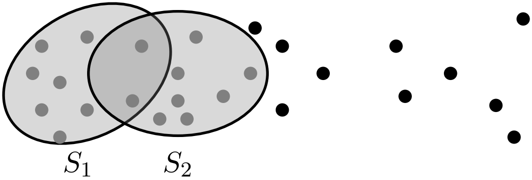

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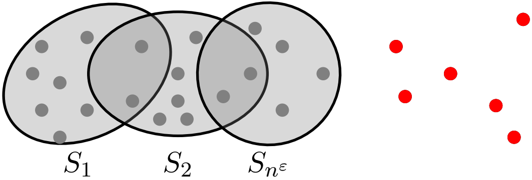

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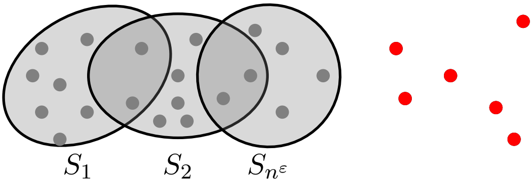

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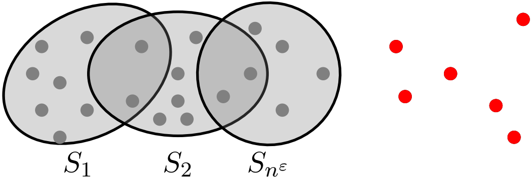
Final $\tilde{y}$ has:

- $\tilde{y}_1 = \ldots = \tilde{y}_{n^\varepsilon} = 1$



The algorithm

(1) Compute $y \in \textsc{Las}_{n^\varepsilon}(K)$ with

$$K := \left\{ x \in \mathbb{R}^m_{\geq 0} \mid \sum_{i:j \in S_i} x_i \geq 1 \ \forall \text{element } j;\ c^T x \leq OPT \right\}$$

(2) FOR $i = 1, \dots, n^\varepsilon$ DO

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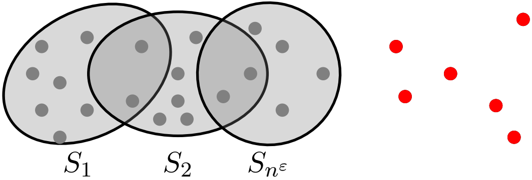

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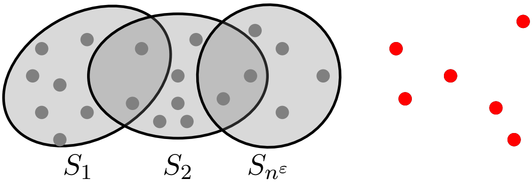

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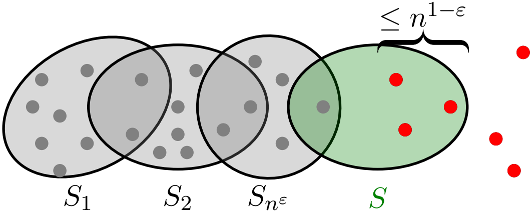

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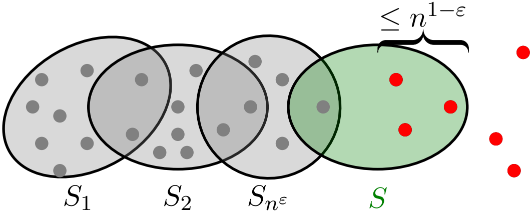

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- **Apx-ratio:** $\ln(n^{1-\varepsilon}) = (1-\varepsilon)\ln(n)$

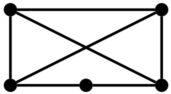
Application 3:

Max Cut

Source: [Goemans, Williamson '95]

MaxCut

MaxCut: Given $G = (V, E)$. Maximize $|\delta(S)|$



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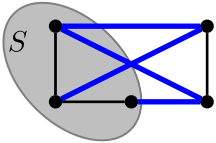

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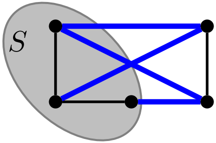

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$$\max\left\{\sum_{e\in E} z_e \mid z_{ij} \leq \min\{x_i + x_j, 2 - x_i - x_j\}\ \forall (i,j) \in E\right\}$$

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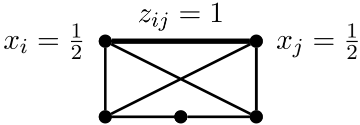

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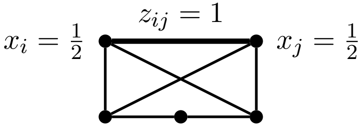

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- Always $OPT_f = |E|$
- Consider $(x, z) \in \textsc{Las}_3(K)$

Vector representation

Observation

For $y \in \textsc{Las}_t(K)$, there are $\boldsymbol{v}_I$ with $\langle \boldsymbol{v}_I, \boldsymbol{v}_J \rangle = y_{I \cup J}$ for $|I|, |J| \leq t$.

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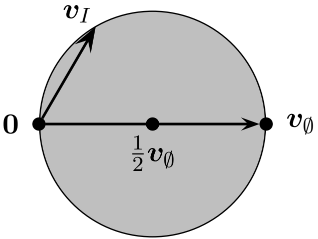
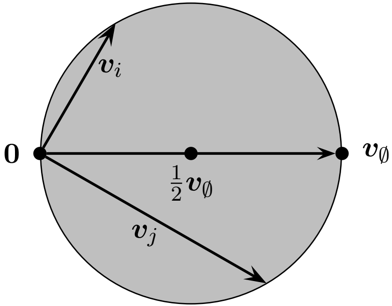


- $\boldsymbol{v}_I$ lies on the **sphere** with radius $\frac{1}{2}$ and center $\frac{1}{2}\boldsymbol{v}_\emptyset$ (since $\|\boldsymbol{v}_I - \frac{1}{2}\boldsymbol{v}_\emptyset\|_2^2 = \|\boldsymbol{v}_I\|_2^2 - 2 \cdot \frac{1}{2}\boldsymbol{v}_I\boldsymbol{v}_\emptyset + \frac{1}{4}\|\boldsymbol{v}_\emptyset\|_2^2 = \frac{1}{4}$)

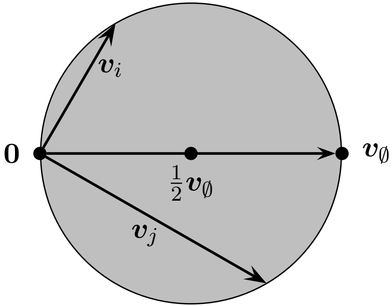
Vector representation for MaxCut

MaxCut: $\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = x_{\{i,j\}}$



Vector representation for MaxCut

MaxCut: $\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = x_{\{i,j\}}$ and $z_{ij} = x_i + x_j - 2x_{\{i,j\}}$



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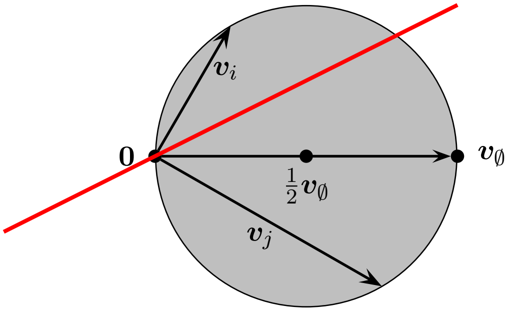

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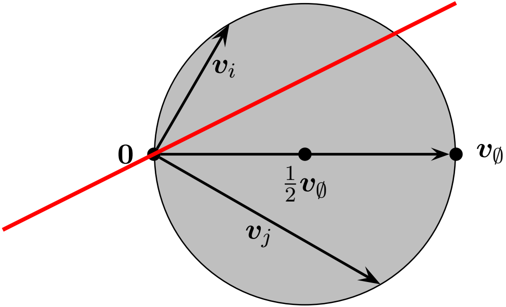
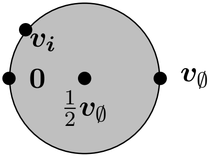


- Now **Hyperplane rounding**?
- **Problem:** Angles are in $[0^o, 90^o]$, $\Pr[\text{cut } (i,j)] \leq \frac{1}{2}$

Vector transformation



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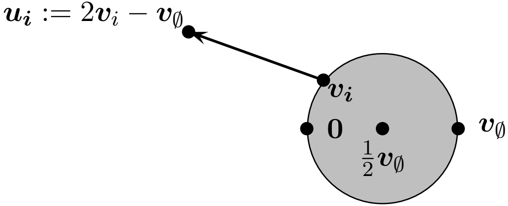

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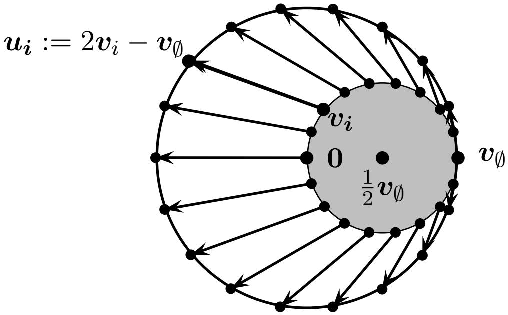
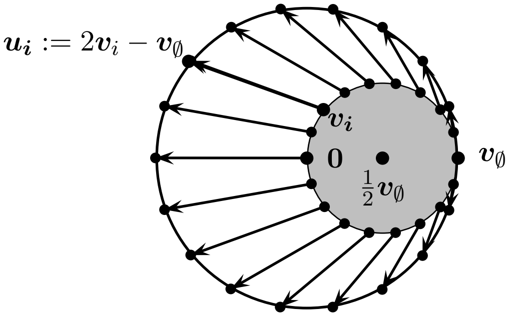


Observations:

- u_i are unit vectors

Vector transformation



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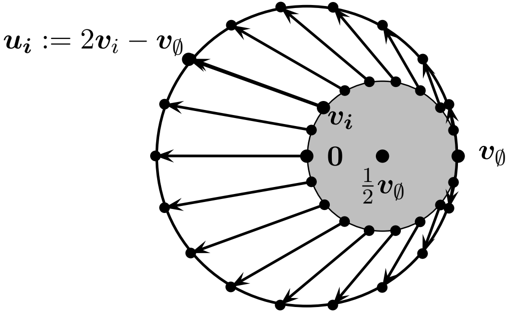

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- $\boldsymbol{u}_i$ are unit vectors
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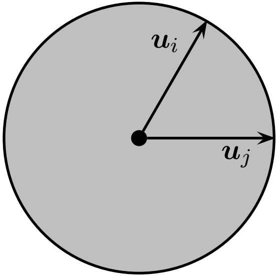
- $\boldsymbol{u}_i$'s are solutions to **Goemans Williamson SDP**

$$\max \Big\{ \sum_{(i,j)\in E} c_{ij} \cdot \frac{1 - \boldsymbol{u}_i \boldsymbol{u}_j}{2} \mid \|\boldsymbol{u}_i\|_2 = 1 \ \ \forall i \in V \Big\}$$

Hyperplane rounding



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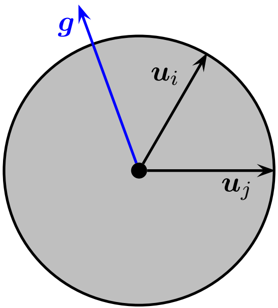

Algorithm:

(1) Pick a random Gaussian $\boldsymbol{g}$

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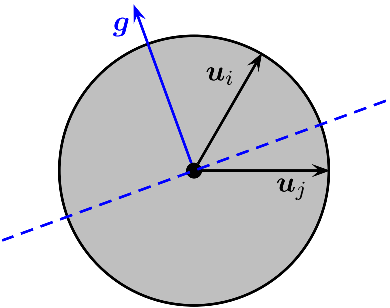

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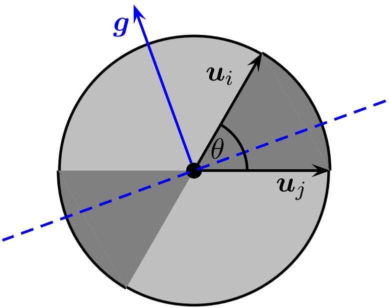

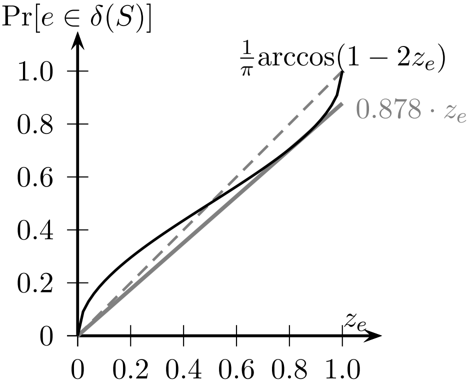

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Analysis:

$$\Pr[(i,j) \in \delta(S)] = \frac{\text{angle of } \boldsymbol{u}_i \text{ and } \boldsymbol{u}_j}{\pi} = \frac{\arccos(1-2z_e)}{\pi} \geq 0.87 \cdot z_e$$

Theory:

Global Correlation Rounding

Source:

- [Barak, Raghavendra, Steurer '11]
- [Guruswami, Sinop '11]

Global Correlation Rounding

Rand. Var. $X_1, X_2 \in \{0,1\}$ are **uncorrelated / independent**

$$\Leftrightarrow$$

$$\mathrm{Cov}[X_1, X_2] = \Pr[X_1 = X_2 = 1] - \Pr[X_1 = 1] \cdot \Pr[X_2 = 1] \stackrel{!}{=} 0$$

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Theorem

For any $y \in \textsc{Las}_t(K)$ can induce on $\leq O(\frac{1}{\varepsilon^3})$ variables to obtain $y' \in \textsc{Las}_{t-O(1/\varepsilon^3)}(K)$ s.t.

$$\Pr_{i,j \in [n]} \left[\left| y'_i \cdot y'_j - y'_{\{i,j\}} \right| \geq \varepsilon \right] \leq \varepsilon$$

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- Consider random variable $X \in \{0,1\}^n$.

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- Cannot go on for more than $O(\frac{1}{\varepsilon^3})$ rounds
- Variance also exists for Lasserre!

Application 4:

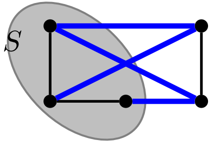
Max Cut in dense Graphs

Source: [de la Vega, Kenyon-Mathieu '95]

PTAS for MaxCut in dense graphs

Problem:

- Given $G = (V, E)$ with $|E| \geq \varepsilon n^2$.
- Maximize $|\delta(S)|$



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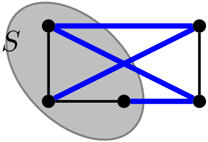

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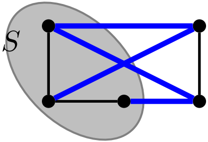

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- Compute in $n^{O(1/\varepsilon^4)}$ uncorrelated $(x, z) \in \textsc{Las}_3(K)$:

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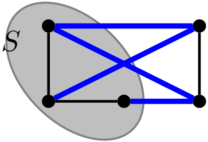

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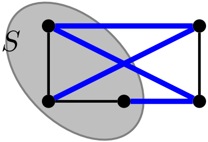

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- Take $S \subseteq V$ with $\Pr[i \in S] = x_i$ **independently!!**

PTAS for MaxCut in dense graphs

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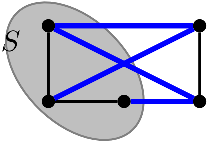

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- Compute in $n^{O(1/\varepsilon^4)}$ uncorrelated $(x, z) \in \textsc{Las}_3(K)$:

$$\Pr_{i,j \in V}[|x_i x_j - x_{\{i,j\}}| \geq \varepsilon] \leq \varepsilon^2$$

- **Observe:** $1 - \varepsilon$ fraction of edges $\leq \varepsilon$ correlated.
- Take $S \subseteq V$ with $\Pr[i \in S] = x_i$ **independently!!**
- Then

$$\Pr[(i,j) \in \delta(S)] = x_i + x_j - 2x_i x_j \overset{(i,j)\text{ uncor.}}{\approx} x_i + x_j - 2x_{\{i,j\}} = z_e$$

The end

Open problems:

- $f(\varepsilon, k)$ rounds solve Unique Games?
- $O(1)$ rounds give a $O(\log n)$-apx for coloring 3-colorable graphs?
- $f(\varepsilon)$-rounds give PTAS for $P3 \mid \text{prec}, p_j = 1 \mid C_{\max}$?
- $O(1)$ rounds give $(2 - \varepsilon)$-apx for Unrelated Machine Scheduling $Q \mid p_{ij} \mid C_{\max}$?

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Thanks for your attention

Slides and lecture notes can be found under
`http://www-math.mit.edu/~rothvoss/lecturenotes.html`