The Entropy Rounding Method in Approximation Algorithms

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A general LP rounding problem

Problem:

- **Given:** $A \in \mathbb{R}^{n \times m}$, fractional solution $x \in [0,1]^m$
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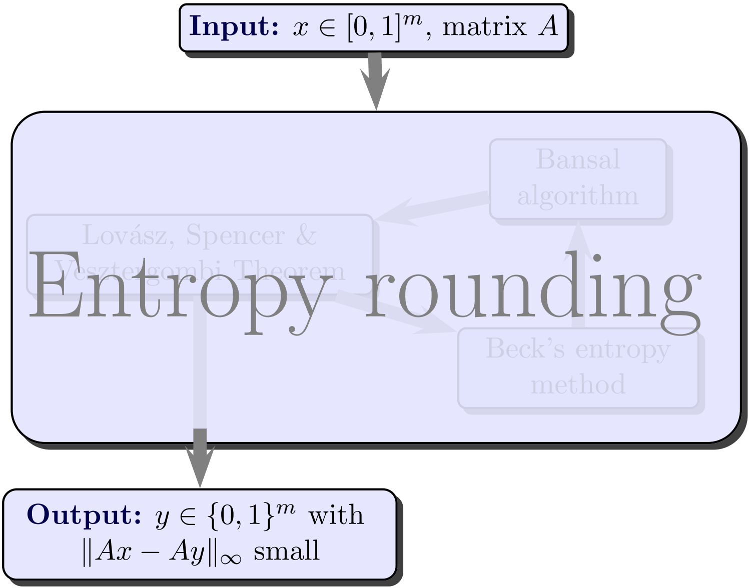
Try another way:

- "Entropy rounding method" based on discrepancy theory

A schematic view



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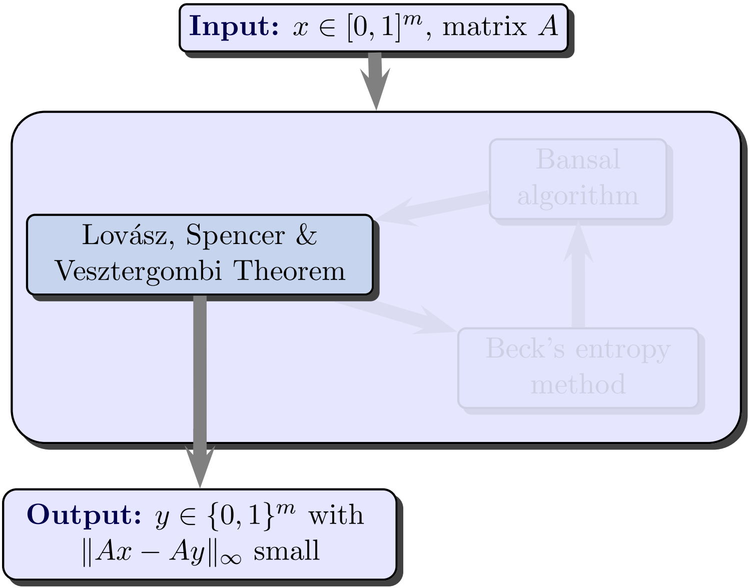

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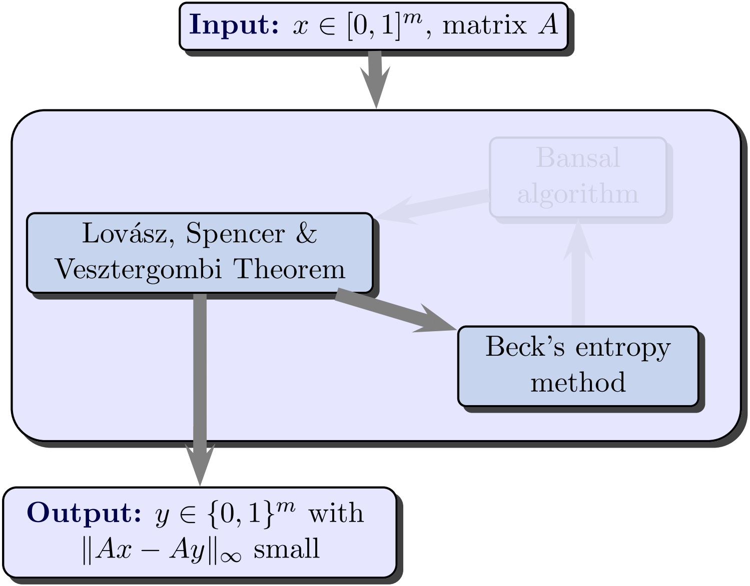

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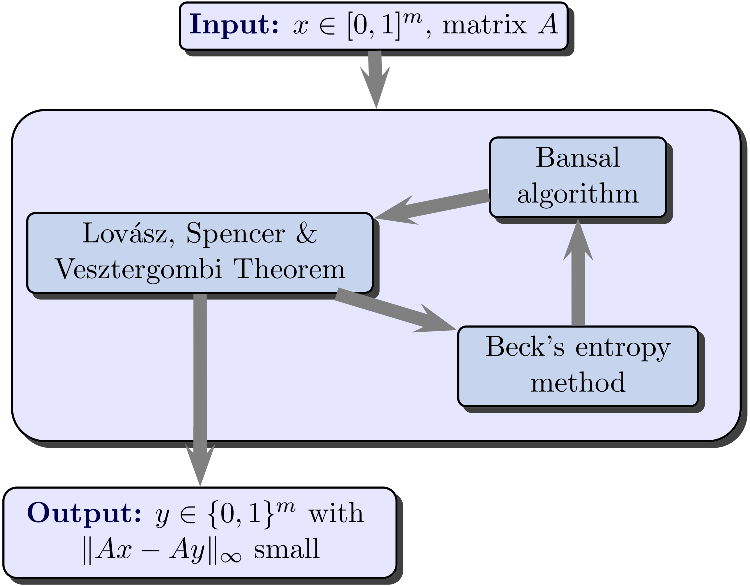
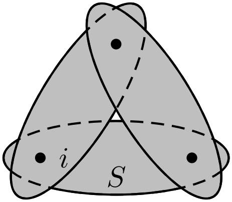


Discrepancy theory

- Set system $\mathcal{S} = \{S_1, \ldots, S_m\}, S_i \subseteq [n]$



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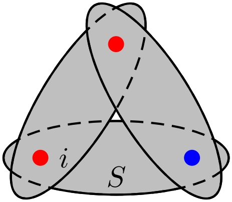

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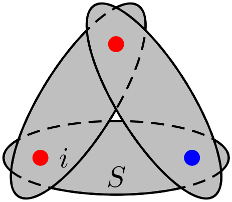

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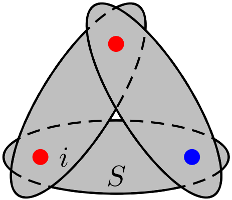

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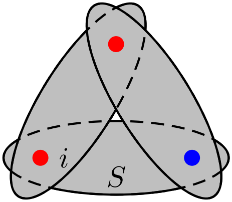

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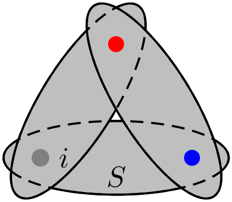

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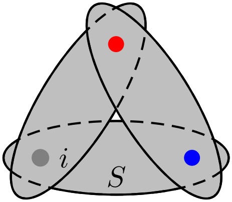

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- **Half coloring:** $\chi : [n] \to \{0, -1, +1\}$, $|\mathrm{supp}(\chi)| \geq n/2$
- For matrix A: $\mathrm{disc}(A) = \min_{\chi \in \{\pm 1\}^n} \|A\chi\|_\infty$

The LSV-Theorem

Theorem (Lovász, Spencer & Vesztergombi '86)

Given $A \in \mathbb{R}^{n \times m}, x \in [0,1]^m$.
Suppose for any $A' \subseteq A$, $\exists$ *coloring* $\chi : \|A'\chi\|_\infty \leq \Delta$.
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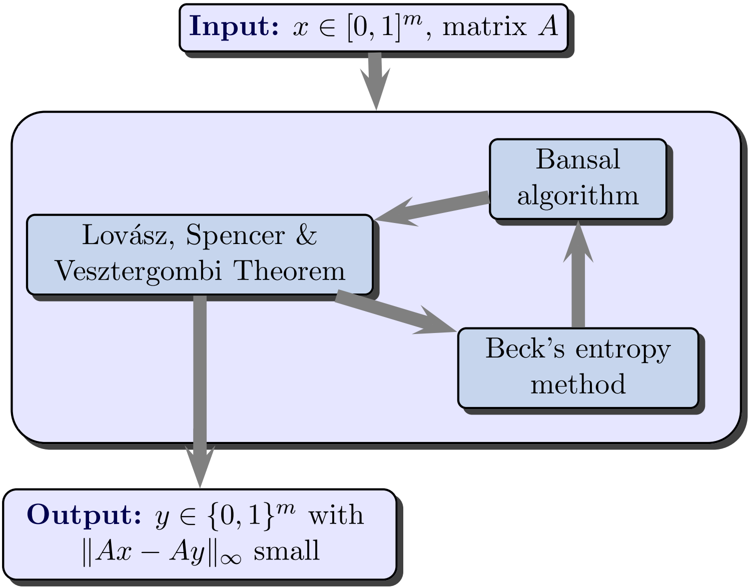

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For random variable Z, the **entropy** is

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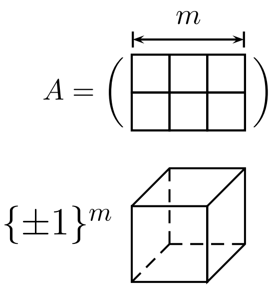
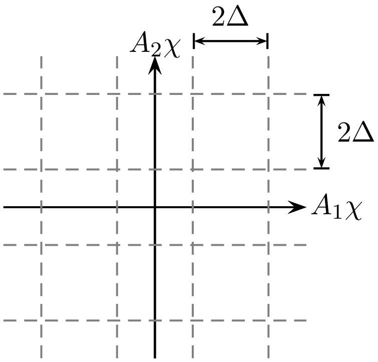
- *Subadditivity:* $H(f(Z, Z')) \leq H(Z) + H(Z')$.

Theorem [Beck's entropy method]

$$\underset{\chi_i \in \{\pm 1\}}{H}\left(\left\lceil \frac{A\chi}{2\Delta} \right\rfloor\right) \leq \frac{m}{5} \Rightarrow \exists \text{half-coloring } \chi^0 : \|A\chi^0\|_\infty \leq \Delta.$$





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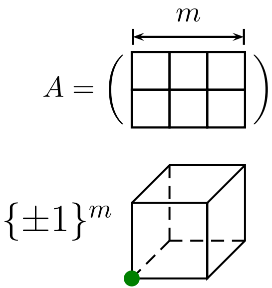

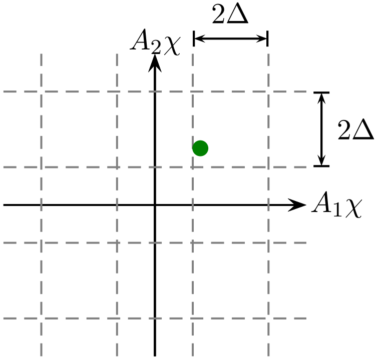

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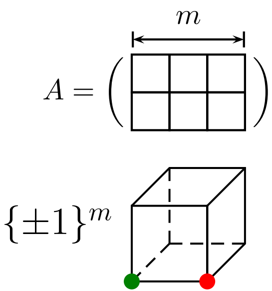

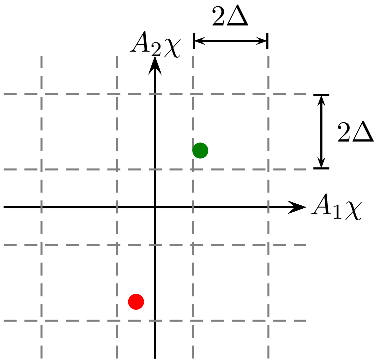

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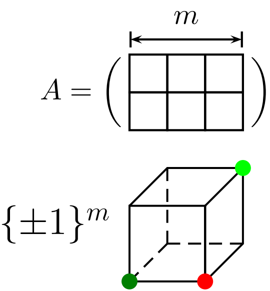

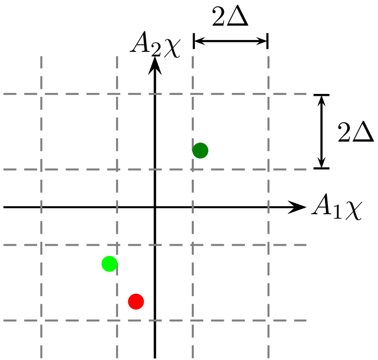

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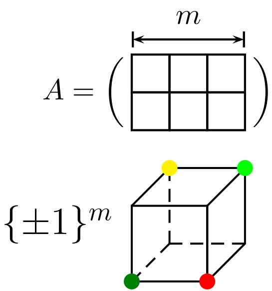

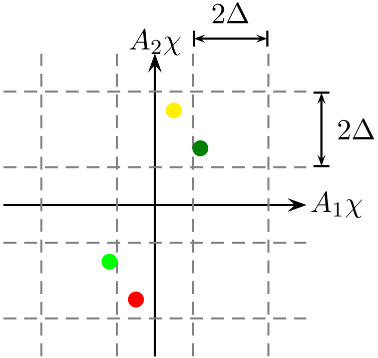

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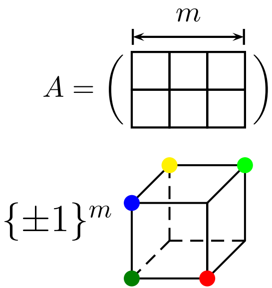

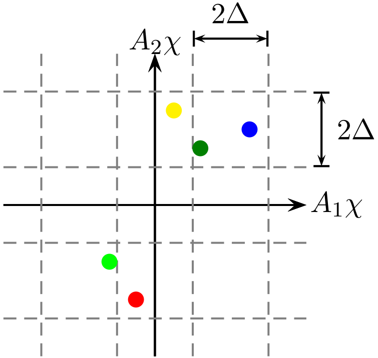

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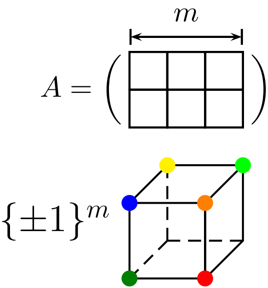

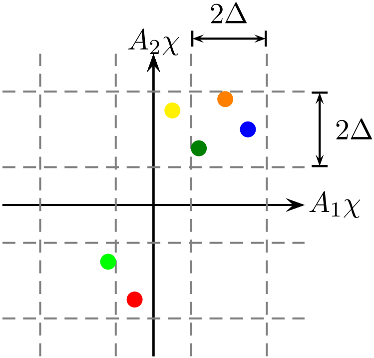

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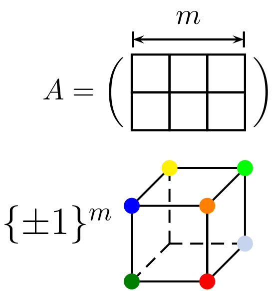

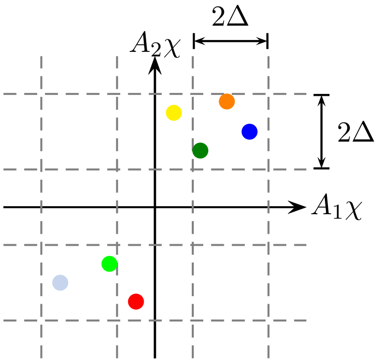

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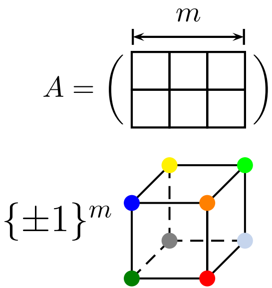

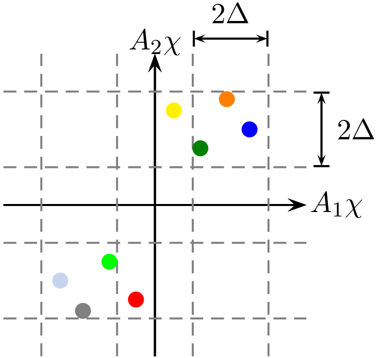

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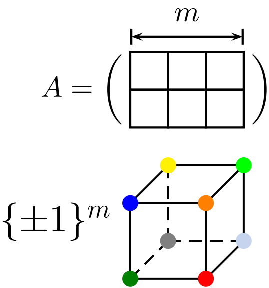

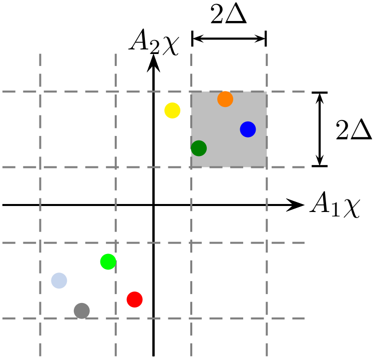

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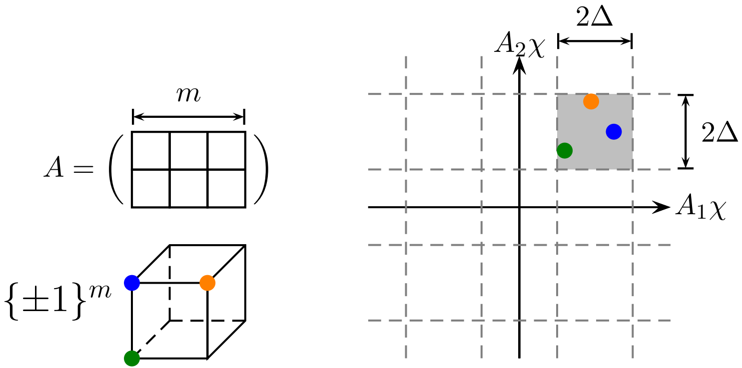

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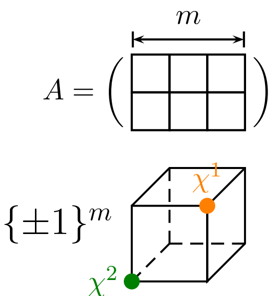

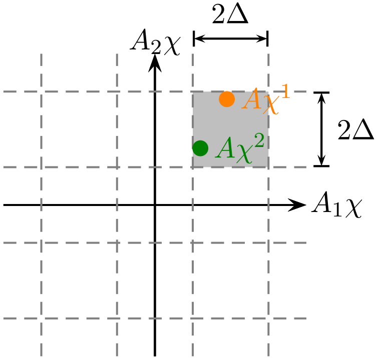

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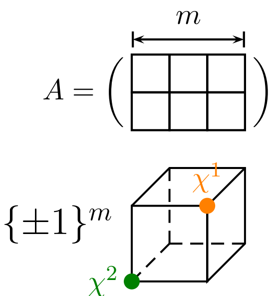

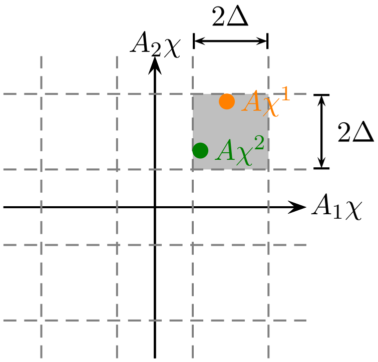

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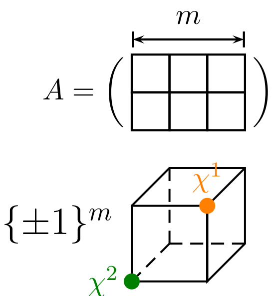
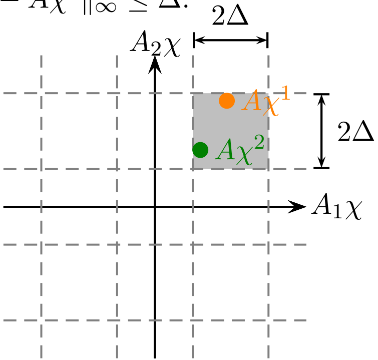
- Then $\|A\chi^0\|_\infty \leq \frac{1}{2}\|A\chi^1 - A\chi^2\|_\infty \leq \Delta$. □





A bound on the entropy

- Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$.
- Consider $Z := \left\lceil \frac{\sum_j \chi(j) \cdot \alpha_j}{\lambda \cdot \|\alpha\|_2} \right\rfloor$ with $\chi(j) \in \{\pm 1\}$ at random

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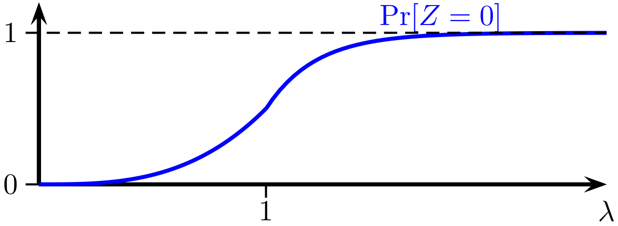

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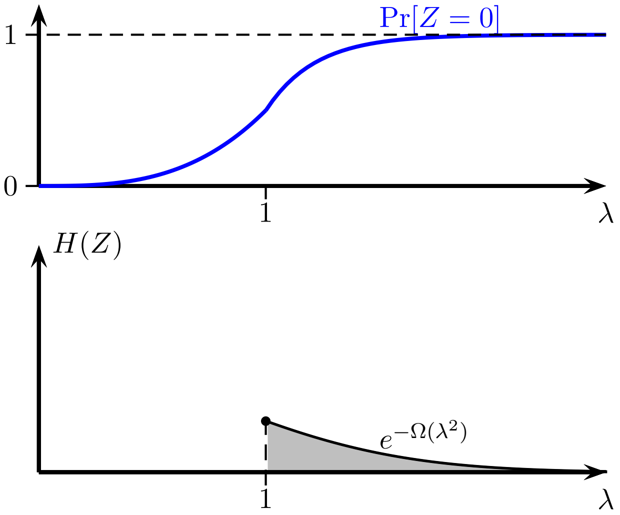

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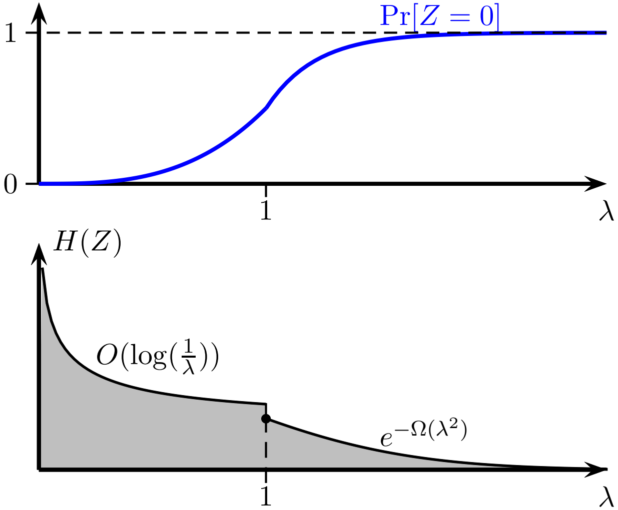

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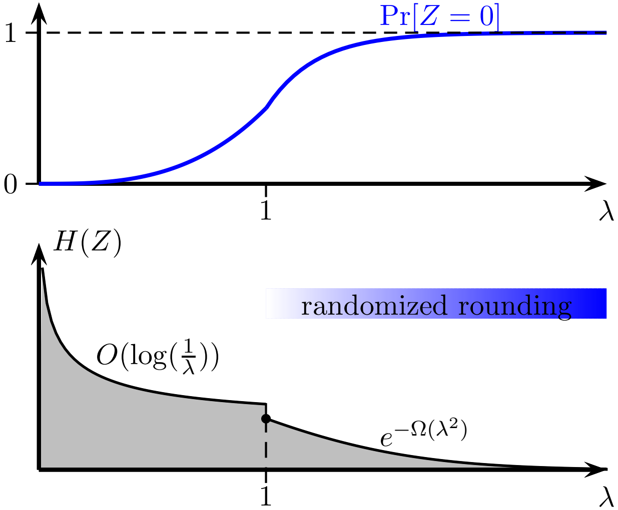
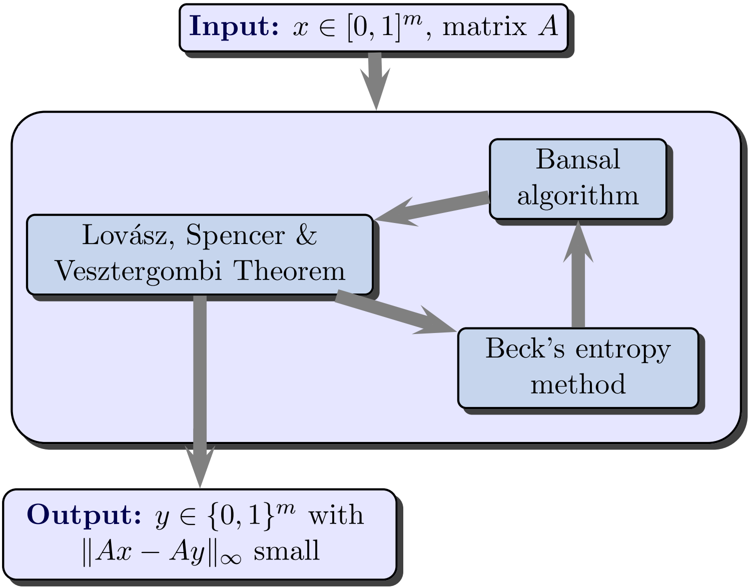


A schematic view



A general rounding theorem

Theorem

Input:

- *matrix* $A \in [-1,1]^{n \times m}$, $\Delta_i > 0$ *satisfying entropy condition for all submatrices*
- *vector* $x \in [0,1]^m$
- *row weights* $w(i)$ $(\sum_i w(i) = 1)$

There is a random variable $y \in \{0,1\}^m$ *with*

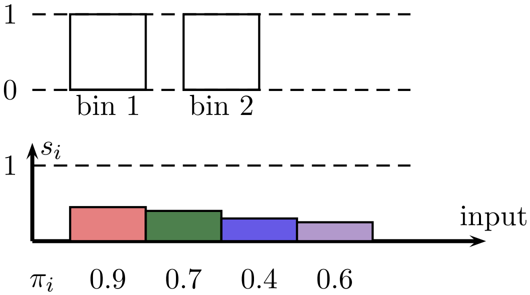
- *Bounded difference:*
  - $|A_i x - A_i y| \leq O(\log(m)) \cdot \Delta_i$
  - $|A_i x - A_i y| \leq O(\sqrt{1/w(i)})$
- *Preserved expectation:* $E[y_i] = x_i$.

Bin Packing With Rejection

Input:

- Items $i \in \{1, \ldots, n\}$ with **size** $s_i \in [0, 1]$, and **rejection penalty** $\pi_i \in [0, 1]$

Goal: Pack or reject. Minimize # **bins** + rejection cost.



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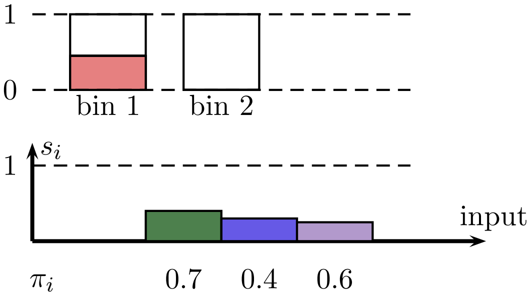

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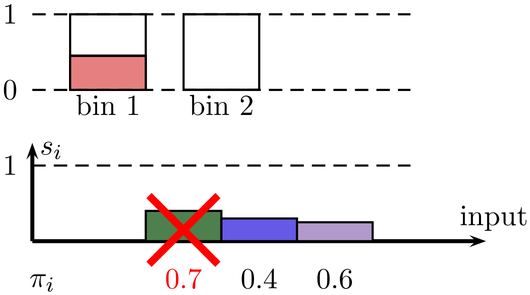

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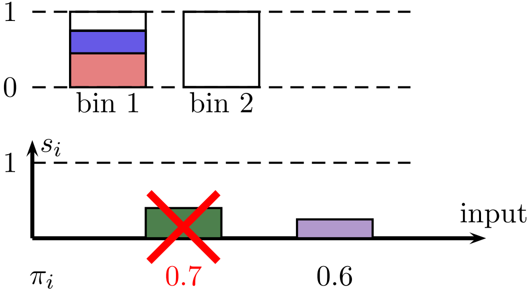

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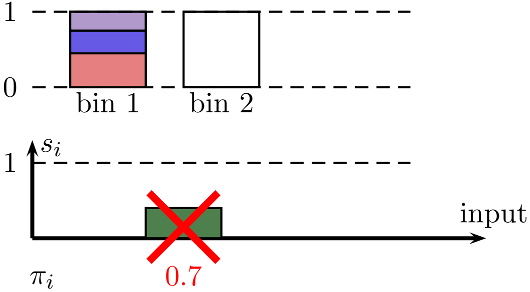


The column-based LP

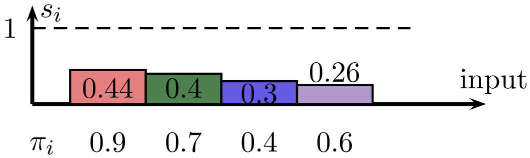
Set Cover formulation:

- **Bins:** Sets $S \subseteq [n]$ with $\sum_{i \in S} s_i \leq 1$ of cost $c(S) = 1$
- **Rejections:** Sets $S = \{i\}$ of cost $c(S) = \pi_i$

LP:

$$
\begin{aligned}
\min \sum_{S \in \mathcal{S}} c(S) \cdot x_S & & & \\
\sum_{S \in \mathcal{S}} \mathbf{1}_S \cdot x_S & \geq & \mathbf{1} & \\
x_S & \geq & 0 & \quad \forall S \in \mathcal{S}
\end{aligned}
$$

The column-based LP - Example



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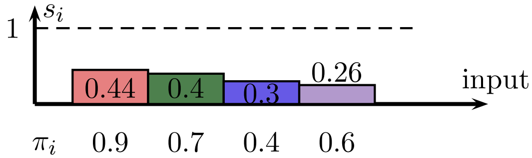

$$\min \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & | & .9 & .7 & .4 & .6 \end{pmatrix} x$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & | & 0 & 0 & 0 & 1 \end{pmatrix} x \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

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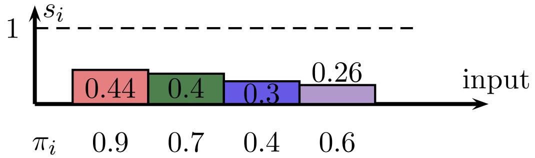

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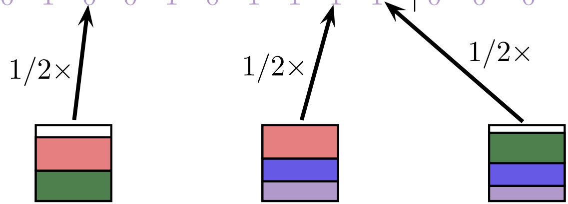

Our results

Theorem

There is a randomized poly-time approximation algorithm for **Bin Packing With Rejection** with

$$APX \leq OPT_f + O(\log^2 OPT_f)$$

(with high probability).

- Previously best known: $APX \leq OPT_f + \frac{OPT_f}{(\log OPT_f)^{1-o(1)}}$ [Epstein & Levin '10]
- As good as classical **Bin Packing** [Karmarkar & Karp '82]

The end

Open question

Are there more (convincing) applications?

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Thanks for your attention