Approximating Connected Facility Location Problems via Random Facility Sampling and Core Detouring

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DISCRETE OPTIMIZATION

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Connected Facility Location

Given:

- graph $G = (V, E)$, with edge costs $c : E \to \mathbb{Q}^+$
- *facilities* $\mathcal{F} \subseteq V$, with *opening cost* $f : \mathcal{F} \to \mathbb{Q}^+$
- a set of *demands* $\mathcal{D} \subseteq V$
- a parameter $M \geq 1$,

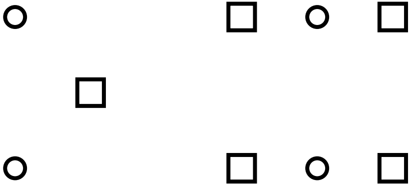
Goal:

- open facilities $F \subseteq \mathcal{F}$
- find Steiner tree T spanning opened facilities

minimizing

$$\underbrace{\sum_{i \in F} f(i)}_{\text{opening cost } O} \quad + \quad \underbrace{M \sum_{e \in T} c(e)}_{\text{Steiner cost } S} \quad + \quad \underbrace{\sum_{j \in \mathcal{D}} \ell(j, F)}_{\text{connection cost } C}$$

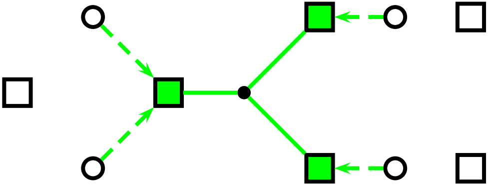
The Problem



□ = facility

○ = demand

The Problem



= facility

= demand

= open facility

= connection path

= Steiner tree edge

Previous Results

- APX-hard (reduction from Steiner tree)
- 10.66 worst-case approximation based on LP-rounding [Gupta et al STOC'01].
- primal-dual 8.55 worst-case approximation [Swamy, Kumar APPROX'02].

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$\Rightarrow$ Here: 4-approximation

Single-Sink Rent-or-Buy

Def: CFL with $\mathcal{F} = V$ and $f(i) = 0$

Known Results:

- 9.01 worst-case approximation based on LP-rounding [Gupta et al STOC'01].
- primal-dual 4.55 worst-case approximation [Swamy, Kumar APPROX'02].
- 3.55 expected approximation based on random-sampling [Gupta, Kumar, Roughgarden STOC'03].
- 4.2 worst-case approximation based on the derandomized of GK&R [Gupta, Srinivasan, Tardos APPROX'04].
- 4.0 worst-case approximation based on the derandomized of GK&R [van Zuylen, Williamson ORL].

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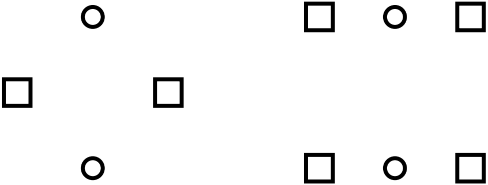
$\Rightarrow$ Here: 2.92-approximation

The Algorithm



- - = FL approximate solution

= sampled demand

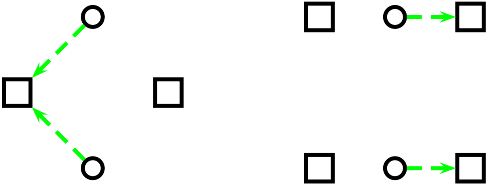
= sampled facility

— = Steiner tree in APX

- - = connections in APX

The Algorithm

1. Run a (black-box) algo for (unconnected) facility location.



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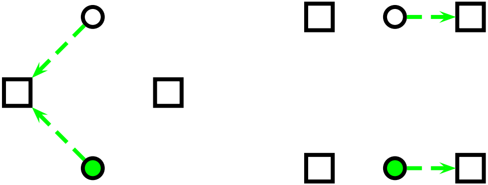
The Algorithm

1. Run a (black-box) algo for (unconnected) facility location.
2. Mark each demand independently with prob. $p \approx 1/M$
3. Mark a random demand



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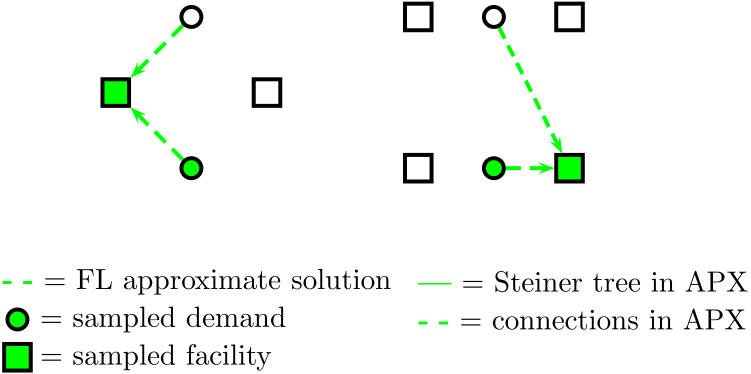
4. Open the facilities, serving sampled demand.



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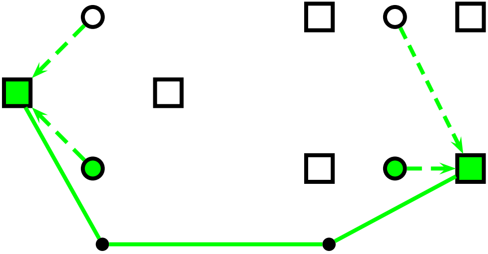
5. Compute 1.55-approx. Steiner Tree T spanning opened facilities



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What to do...

We will bound separately

- Opening cost
- Steiner tree cost
- Connection cost

Each cost type will bounded by combination of

- O^* = opening cost
- S^* = Steiner tree cost
- C^* = connection cost

} in opt. solution

- O_{fl} = opening cost
- C_{fl} = connection cost

} in approx. facility location solution

We use

- $O^* + S^* + C^* = OPT$
- $O_{fl} + C_{fl} \leq 1.52 \cdot OPT_{fl} \leq 1.52 \cdot OPT$

Assume $|\mathcal{D}|/M \gg 1$ (otherwise we have a PTAS)

Analysis: Opening Cost

Lemma

The opening cost O_{apx} satisfies $O_{apx} \leq O_{fl}$.

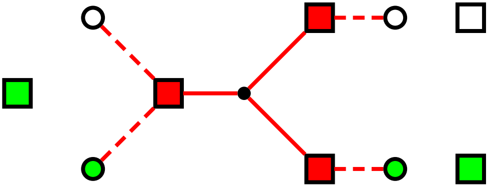
O_{fl} and C_{fl} denote the opening and connection costs in the approximate (unconnected) facility location solution computed in the first step of the algorithm.

Analysis: Steiner Cost

Lemma

The Steiner cost S_{apx} satisfies

$$E[S_{apx}] \leq \rho_{st}(S^* + pM(1 + o(1)) \cdot C^*) + pM(1 + o(1)) \cdot C_{fl})$$



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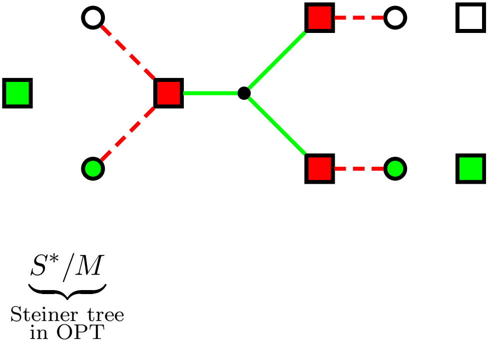

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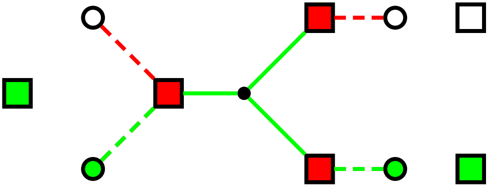

$$\underbrace{S^*/M}_{\text{Steiner tree in OPT}} + \underbrace{\left(p + \frac{1}{|\mathcal{D}|}\right) C^*}_{\text{paths sampled demands - Steiner tree}}$$

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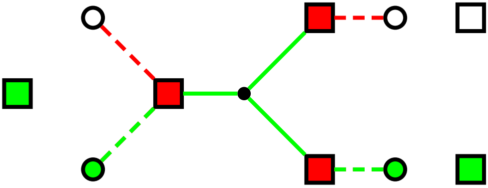

$$\underbrace{\rho_{St}}_{<1.55} \cdot \Big(\underbrace{S^*/M}_{\text{Steiner tree in OPT}} + \underbrace{(p + \frac{1}{|\mathcal{D}|})\, C^*}_{\text{paths sampled demands - Steiner tree}} \Big)$$

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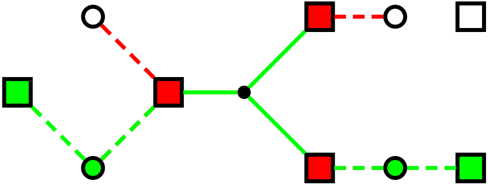

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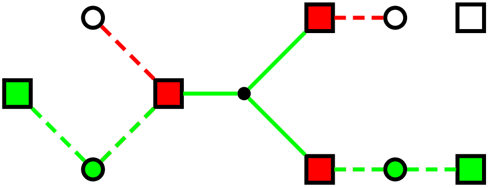


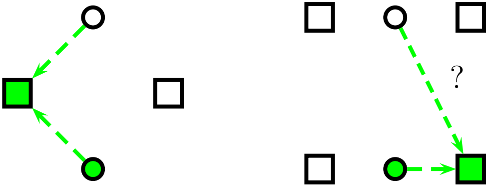
$$M \cdot (\underbrace{\rho_{St}}_{<1.55} \cdot (\underbrace{S^*/M}_{\text{Steiner tree in OPT}} + \underbrace{(p + \frac{1}{|\mathcal{D}|})\, C^*}_{\text{paths sampled demands - Steiner tree}}) + \underbrace{(p + \frac{1}{|\mathcal{D}|})\, C_{fl}}_{\text{paths sampled demands - sampled facilities}})$$

Analysis: Connection Cost

Lemma

The connection cost C_{apx} satisfies

$$E[C_{apx}] \leq 2C^* + C_{fl} + S^*/(p\,M).$$



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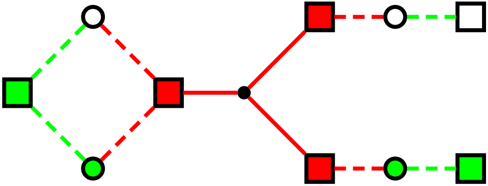

■ = facility in OPT

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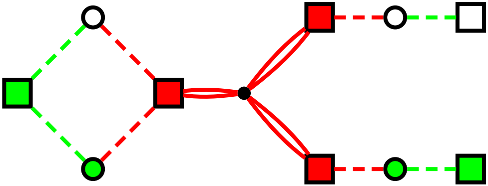

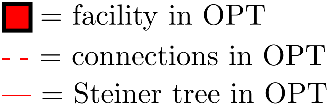

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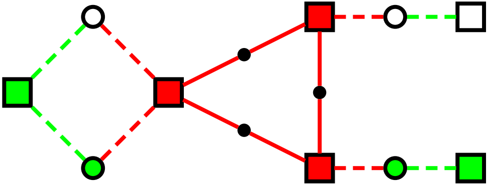

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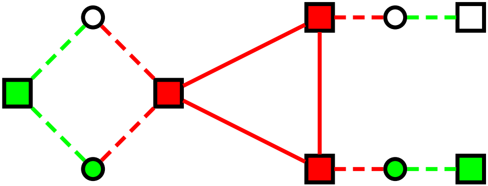

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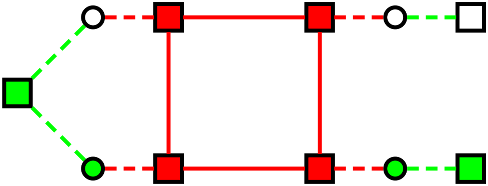

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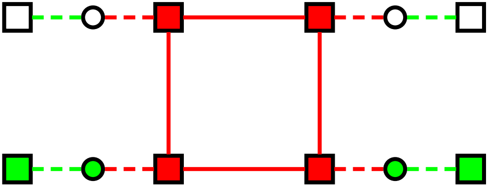

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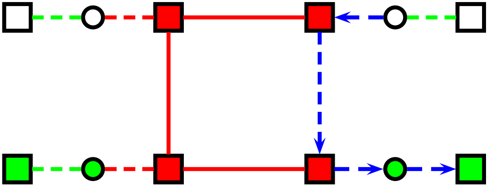

- Connect each demand to the closest open facility w.r.t. number of edges in the auxiliary graph above

Analysis: Connection Cost

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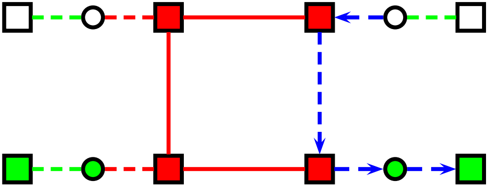

By symmetry:

$E[$flow on  $] \leq 2$

$E[$flow on  $] \leq 1$

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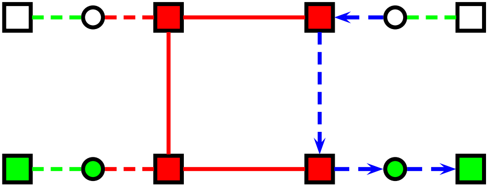

- $X_i :=$ # of cycle edges used by $i \in D$
- #demands = #cycle edges
- By symmetry: $E[\text{flow on cycle edge}] = E[X_i]$

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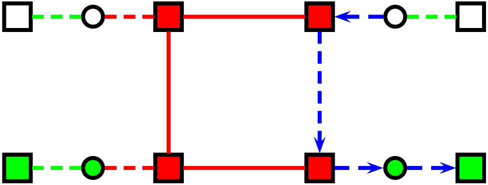


$$\Pr[X_i > k] \leq (1-p)^{2k+1}$$

$$E[X_i] = \sum_{k \geq 1} \Pr[X_i \geq k] = \sum_{k \geq 1} (1-p)^{2k-1} \leq \frac{1}{2} \cdot \frac{1}{p}$$

$$E[\text{cost of used cycle edges}] \leq \frac{1}{2} \cdot \frac{1}{p} \cdot \frac{2S^*}{M} = S^*/(pM)$$

Total Cost

Theorem

There is an expected 4.55 *approximation algorithm for CFL.*

Proof:

$$
\begin{aligned}
& E[APX] \\
\leq\ & O_{fl} + \rho_{st}(S^* + pMC^*) + pMC_{fl} + 2C^* + C_{fl} + \frac{S^*}{pM} \\
\leq\ & (1+pM)(O_{fl} + C_{fl}) + \rho_{st}(S^* + pMC^*) + 2C^* + \frac{S^*}{pM} \\
\overset{O^*_{fl}+C^*_{fl}\leq O^*+C^*}{\leq}\ & (1+pM)\rho_{fl}(O^* + C^*) + \rho_{st}(S^* + pMC^*) + 2C^* + \frac{S^*}{pM} \\
\overset{p=0.334/M}{\leq}\ & 2.03\,O^* + 4.55\,C^* + 4.55\,S^* \\
\leq\ & 4.55\,OPT
\end{aligned}
$$

Refinements

Theorem

There is an expected 4 approximation algorithm for CFL.

- Using bi-factor facility location, for a parameter $\delta \geq 1$, we obtain:

$$C_{fl} + O_{fl} \leq (1.11 + \ln \delta) O^* + (1 + 0.78/\delta) C^*.$$

- Using flow cancelling over the Euler tour, we can show that (for $|\mathcal{D}|/M \gg 1$)

$$C_{apx} \leq 2C^* + C_{fl} + 0.807 \frac{S^*}{pM}.$$

Refinements

Theorem

There is a 4.23 worst-case approximation algorithm for CFL.

Proof: Use idea of van Zuylen and Williamson to estimate expected Steiner cost

Corollary

There is an expected 2.92 and worst-case 3.28 approximation algorithm for SROB.

Proof: Same analysis with $C_{fl} = O_{fl} = 0$.

Summarizing

More results

Problem	**Now**	**Previous best**	
CFL	4.00*	8.55	Swamy, Kumar '02
	4.23		
SROB	2.92*	3.55*	Gupta, Kumar, Roughgarden '03
	3.28	4	van Zuylen, Williamson '07
k-CFL	6.85*	15.55*	Swamy and Kumar '02
	6.98		
tour-CFL	4.12*	5.83*	Ravi, Salman '99 (special case)
soft-CFL	6.33*	–	

The * indicates randomized results

A CFL PTAS for $|\mathcal{D}|/M = O(1)$

Algorithm:

1. Guess the best choice of $\leq k$ facilities F
2. Compute an optimum Steiner tree spanning F

A CFL PTAS for $|\mathcal{D}|/M = O(1)$

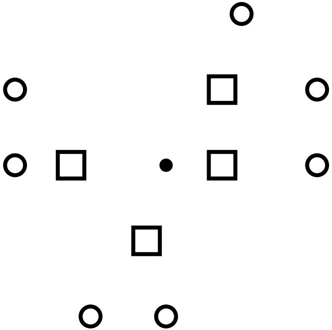
Algorithm:

1. Guess the best choice of $\leq k$ facilities F
2. Compute an optimum Steiner tree spanning F

Lemma

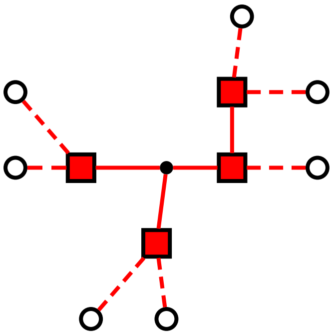
For $k := \frac{2|\mathcal{D}|}{\varepsilon M}$ the algorithm yields a solution of cost $\leq (1+\varepsilon)OPT$.

Proof



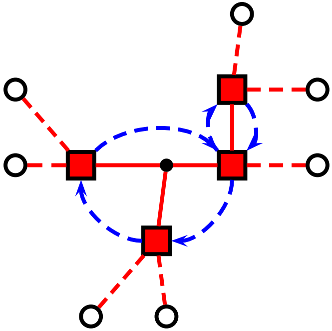
Proof

- T^* := Steiner tree in opt solution



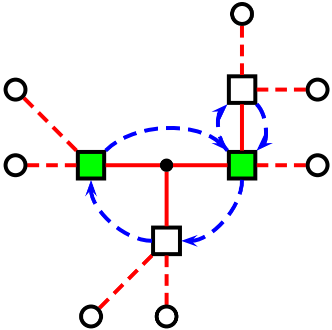
Proof

- T^* := Steiner tree in opt solution
- C := Euler tour (on opt. facilities) of cost $\leq 2c(T^*)$



Proof

- $T^* :=$ Steiner tree in opt solution
- $C :=$ Euler tour (on opt. facilities) of cost $\leq 2c(T^*)$
- Mark facility "each" distance of $\frac{2c(T^*)}{k} \to F$
- k marked facilities F: $\forall i \in F^* : \ell(i, F) \leq \frac{2c(T^*)}{k}$



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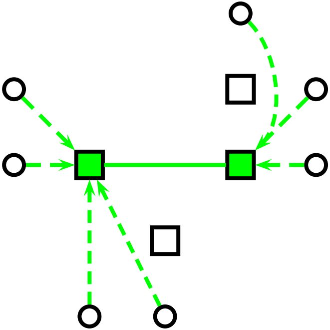

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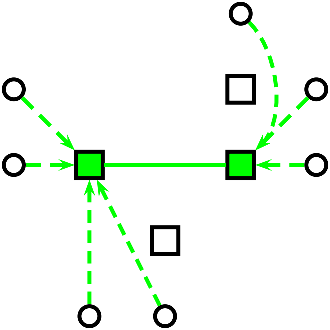
- k marked facilities F: $\forall i \in F^* : \ell(i, F) \leq \frac{2c(T^*)}{k}$
  - Opening costs: $\leq O^*$



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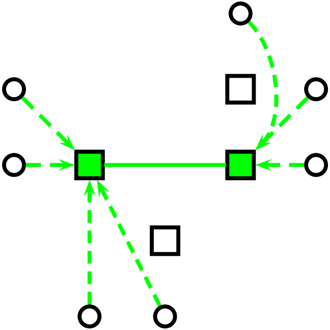

- Opening costs: $\leq O^*$
- Steiner costs: $\leq S^*$

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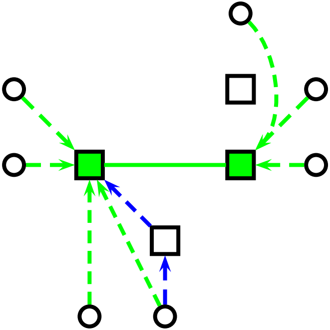

- Opening costs: $\leq O^*$
- Steiner costs: $\leq S^*$
- Connection costs:

$$\begin{aligned} &\leq \quad C^* + \frac{2c(T^*)}{k} \cdot |\mathcal{D}| \\ &\leq \quad C^* + \frac{2|\mathcal{D}|}{M \cdot k} \cdot S^* \\ &\leq \quad C^* + \varepsilon S^* \end{aligned}$$

Open Problems

- Some of the best known approximation algorithms for network design are based on random sampling:
  - Single-Sink Buy-at-Bulk
  - Multi-Commodity Rent-or-Buy
  - Virtual Private Network Design
  - ...
- Can the improved bound on the connection cost help for these problems?

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  - Single-Sink Buy-at-Bulk
  - Multi-Commodity Rent-or-Buy
  - Virtual Private Network Design
  - ...
- Can the improved bound on the connection cost help for these problems?

Thanks for your attention