A Logarithmic Additive Integrality Gap for Bin Packing

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Bin Packing

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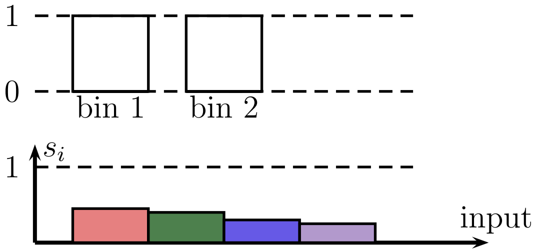
Goal: Pack items into minimum number of **bins** of size 1.



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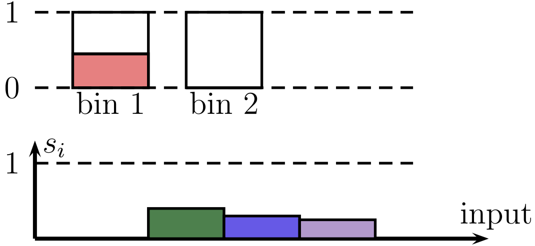

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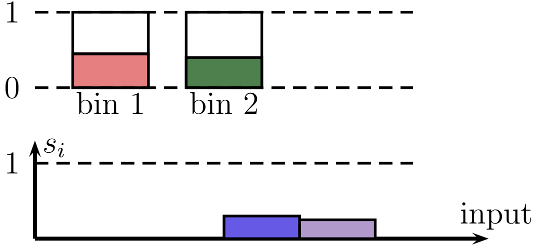

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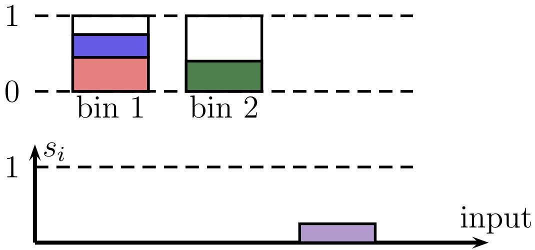

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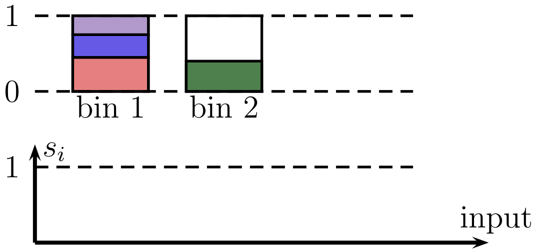

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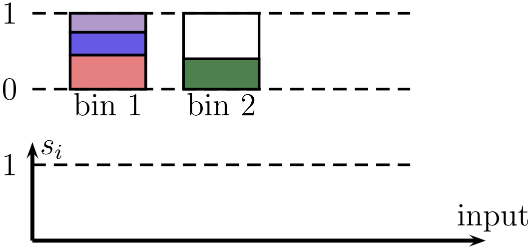

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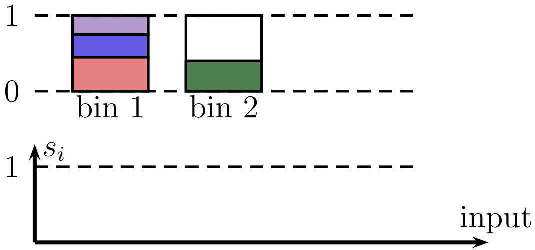

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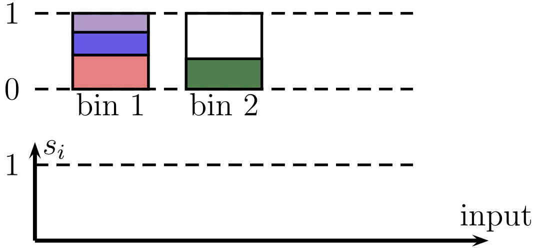

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- [de la Vega & Lücker '81] : $APX \leq (1 + \varepsilon) OPT + O(1/\varepsilon^2)$ in time $O(n) \cdot f(\varepsilon)$

The Gilmore Gomory LP relaxation

- $b_i = \#\text{items with size } s_i$
- **Feasible patterns**:

$$\mathcal{P} = \left\{ p \in \mathbb{Z}^n_{\geq 0} \mid \sum_{i=1}^{n} s_i p_i \leq 1 \right\}$$

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$$\begin{aligned} \min \sum_{p \in \mathcal{P}} x_p & & & \\ \sum_{p \in \mathcal{P}} p_i \cdot x_p & \geq & b_i & \quad \forall i \in [n] \\ x_p & \geq & 0 & \quad \forall p \in \mathcal{P} \end{aligned}$$

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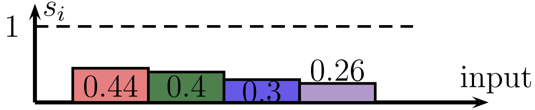
- Can find x with $\mathbf{1}^T x \leq OPT_f + \delta$ in time poly$(\|b\|_1, \frac{1}{\delta})$

The Gilmore Gomory LP - Example



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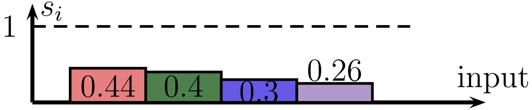

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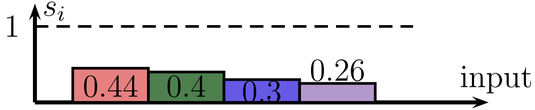

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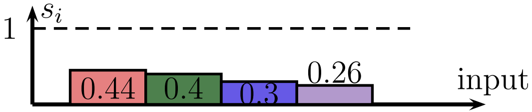

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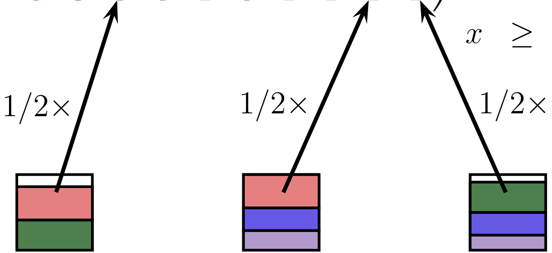

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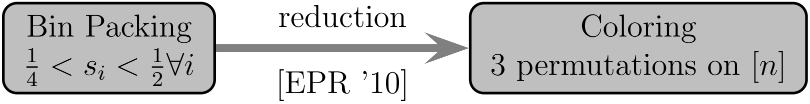

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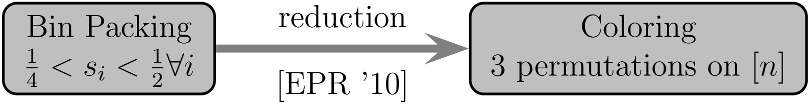
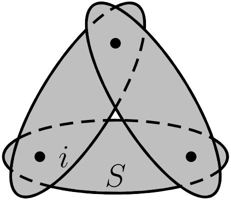


Discrepancy $\Theta(\log n)$
[Newman, Nikolov '11]

Discrepancy theory



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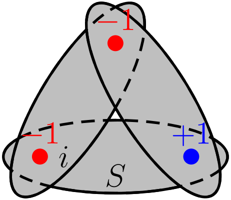

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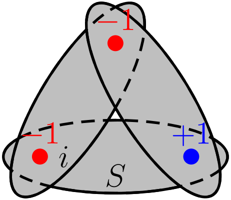

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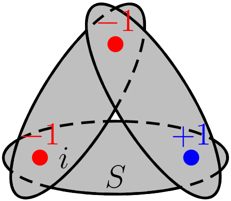

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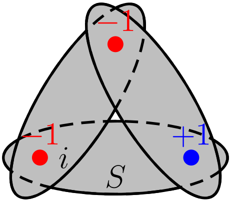

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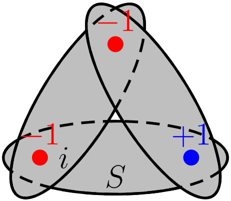

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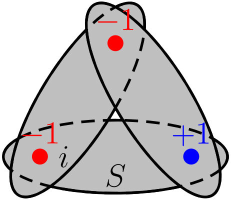

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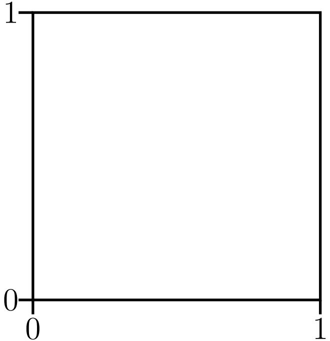
⇒ Partial coloring method

- Initially **non-constructive!**
  Algorithms by [Bansal '10, Lovett-Meka '12, R. '14, ES'14]

Constructive Partial Coloring Lemma

Lemma [Lovett-Meka '12]



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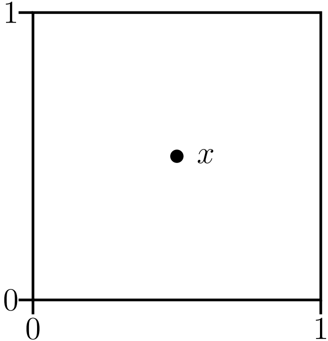
Given $x \in [0,1]^m$



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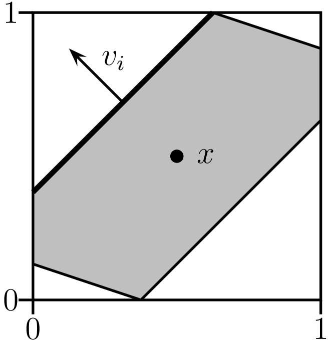
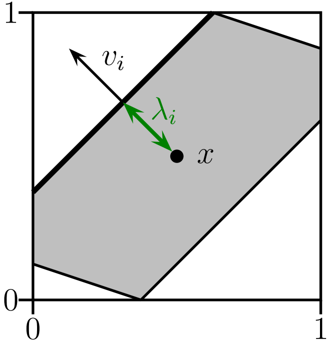


Constructive Partial Coloring Lemma

Lemma [Lovett-Meka '12]

Given $x \in [0,1]^m$, unit vectors v_i, parameters $\lambda_i \geq 0$

- $|\langle v_i, y - x \rangle| \leq \lambda_i \; \forall i$



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Lemma [Lovett-Meka '12]

Given $x \in [0,1]^m$, unit vectors v_i, parameters $\lambda_i \geq 0$

Then one can find $y \in [0,1]^m$ with

- $y_j \in \{0,1\}$ for at least half of the indices j
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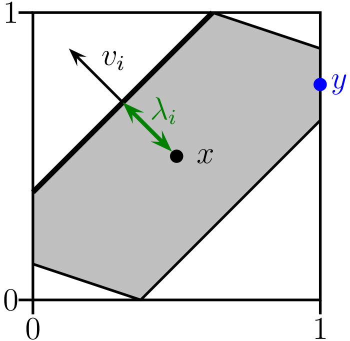

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Lemma [Lovett-Meka '12]

Given $x \in [0,1]^m$, unit vectors v_i, parameters $\lambda_i \geq 0$ s.t. $\sum_i e^{-\lambda_i^2/16} \leq \frac{m}{16}$. Then one can find $y \in [0,1]^m$ with

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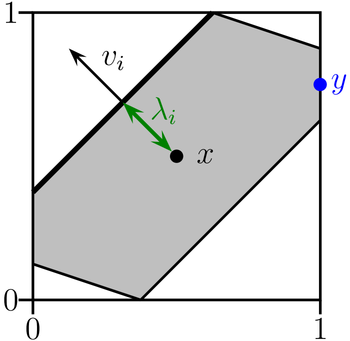

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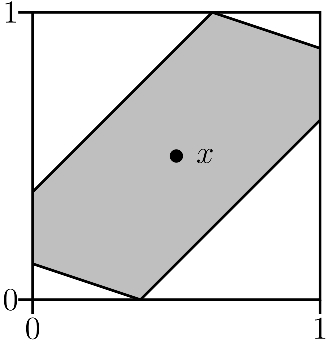

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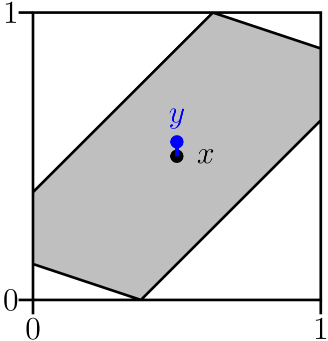

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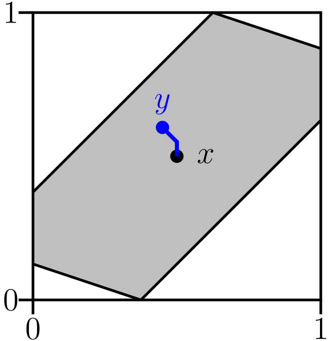

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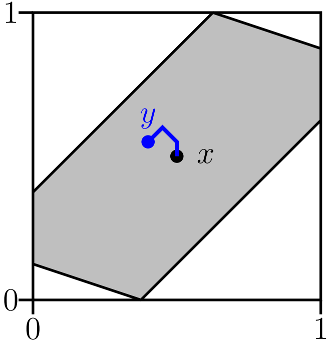

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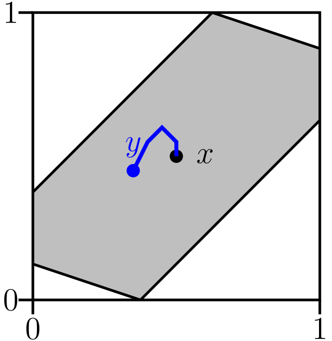

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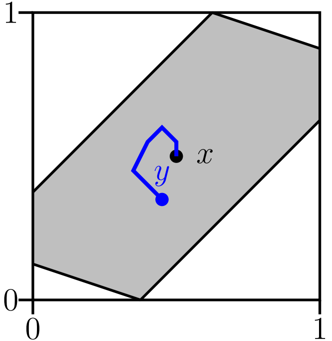

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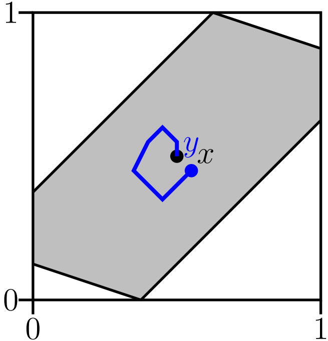

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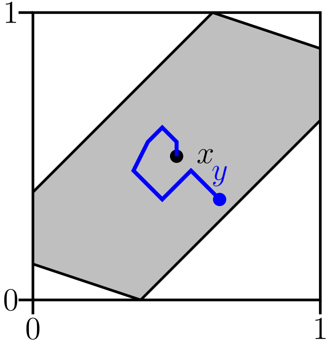

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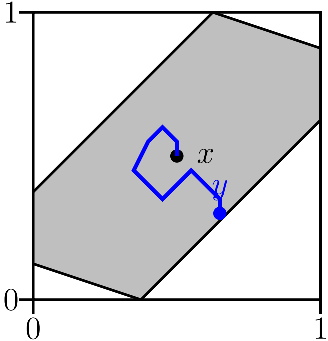

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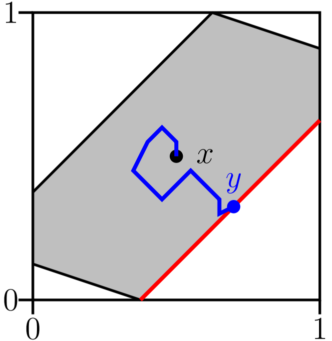

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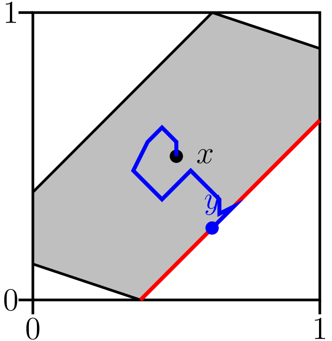

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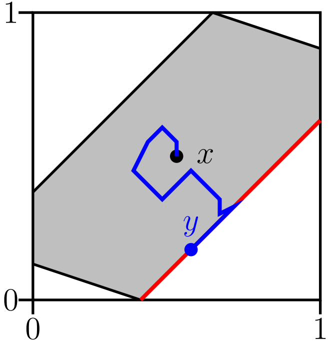

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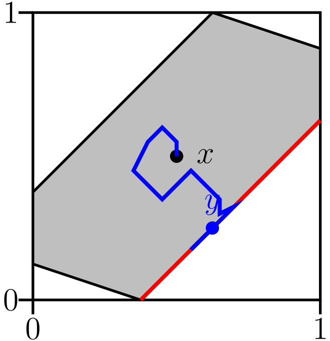

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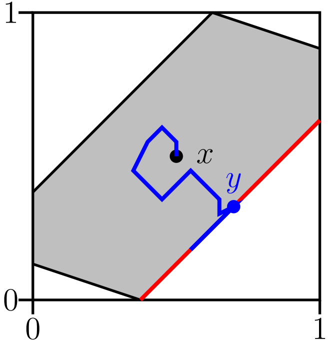

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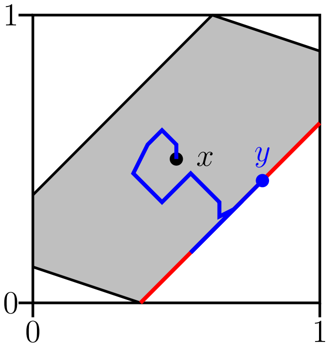

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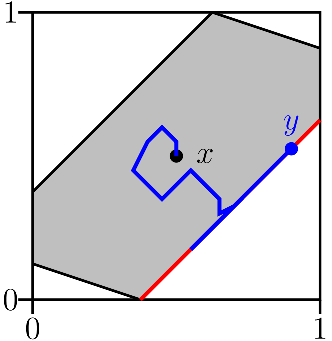

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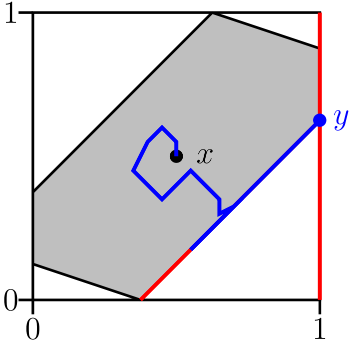

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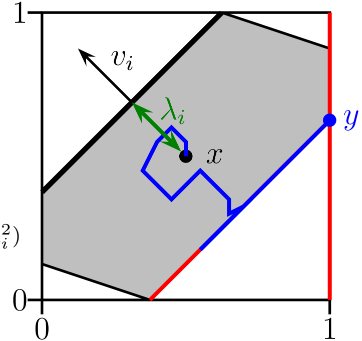

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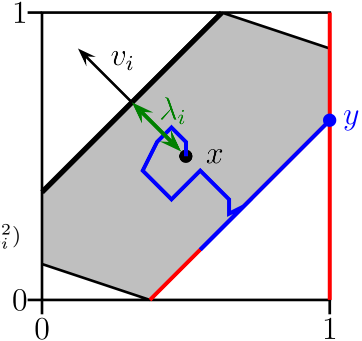

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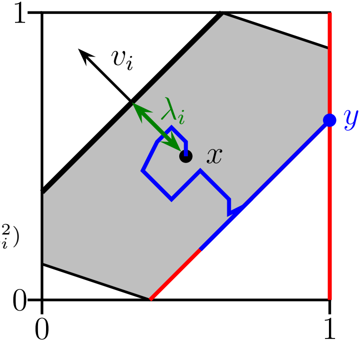

The algorithm

(1) Compute a fractional LP solution x

(2) FOR $\log n$ iterations DO

    (3) Rebuild incidence matrix

    (4) Apply Lovett-Meka rounding $x \to x'$

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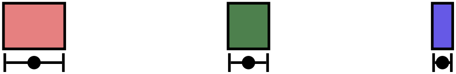
  Effect:

  - $|\mathrm{frac}(x')| \leq \frac{1}{2} \cdot |\mathrm{frac}(x)|$
  - $\mathrm{cost}(x') \leq \mathrm{cost}(x) + O(1)$

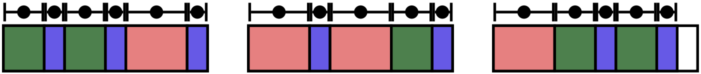
Assigning items to patterns

$b_1 = 2$ $b_2 = 1$ $b_3 = 7$

Items:

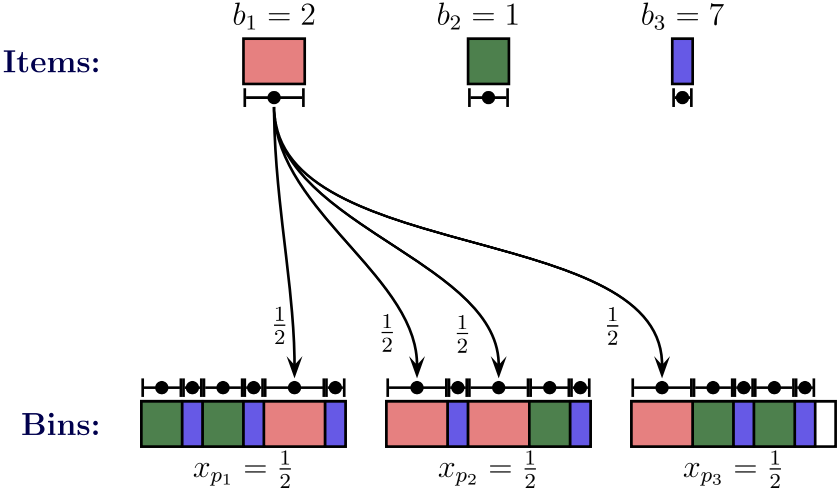


Bins:



$x_{p_1} = \frac{1}{2}$ $x_{p_2} = \frac{1}{2}$ $x_{p_3} = \frac{1}{2}$

Assigning items to patterns



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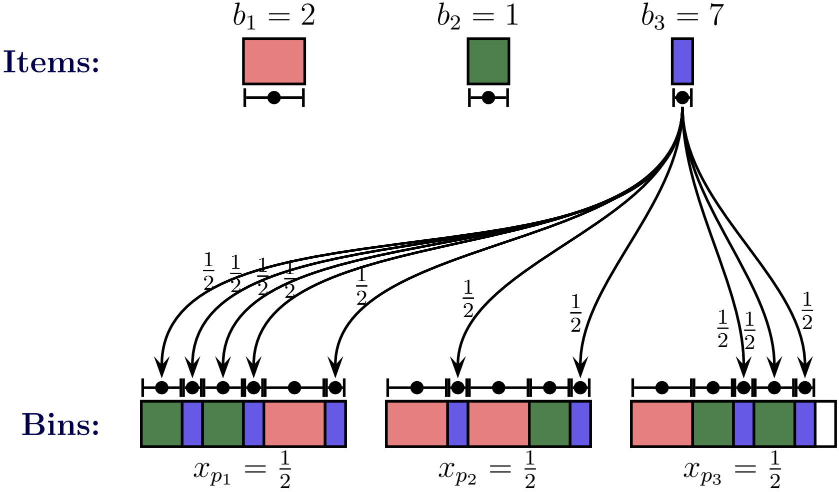

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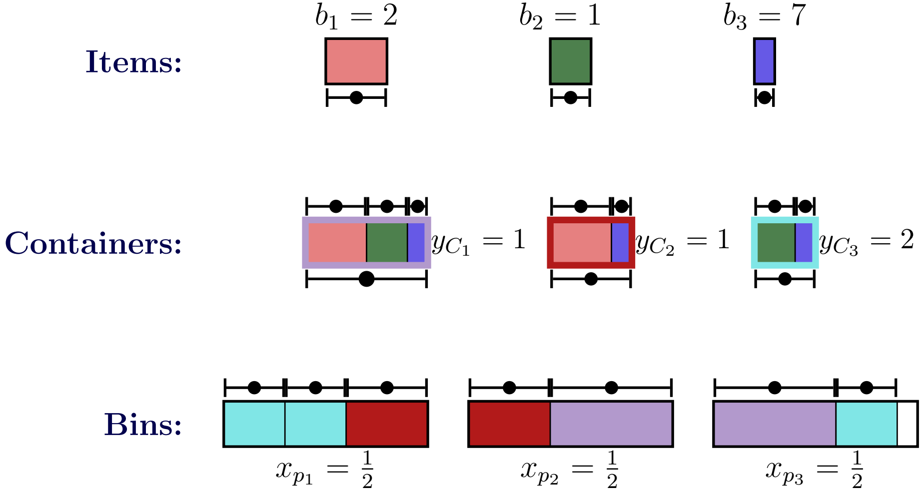

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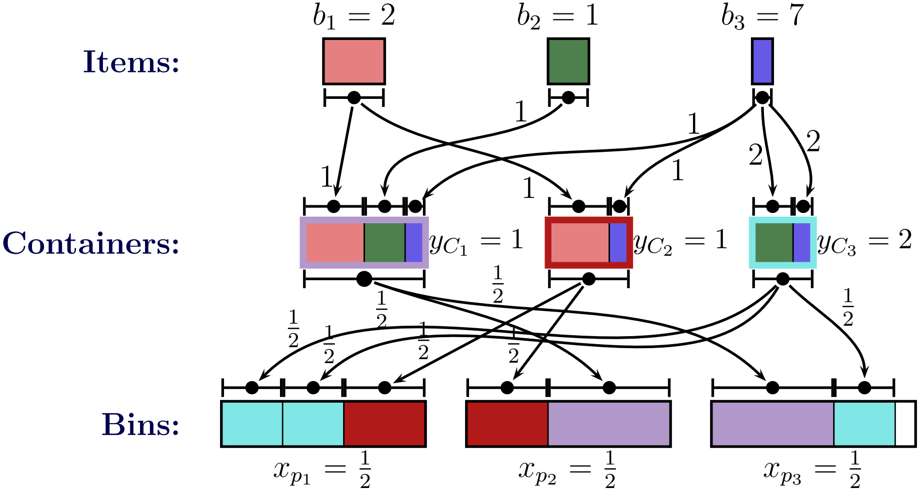

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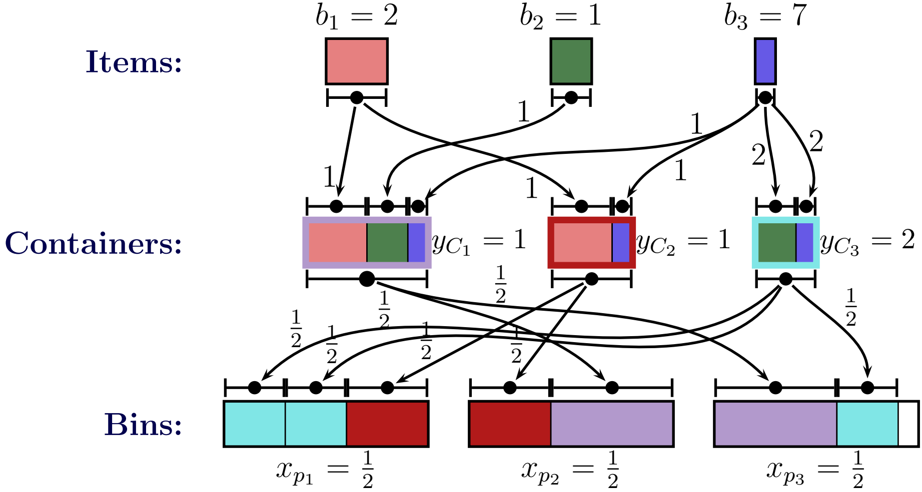
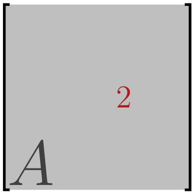


- deficiency = total size of non-assignable items+containers

Container building

- Consider fract. solution x at beginning of iteration



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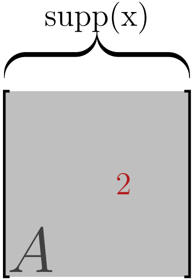

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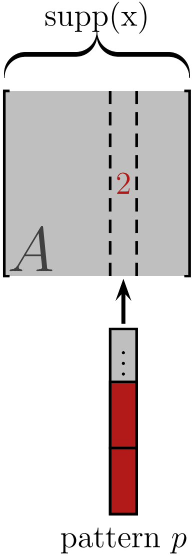

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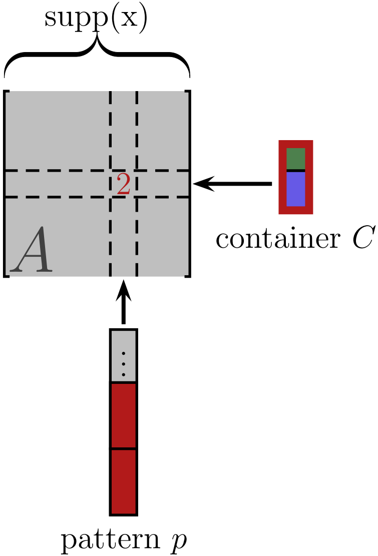

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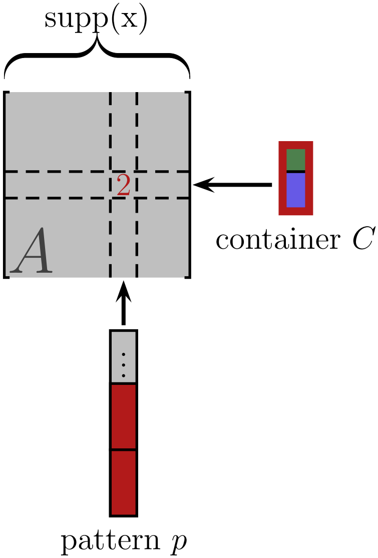
Lemma

Can reassign containers so that for container C of size $s_C \in [\frac{1}{k}, \frac{2}{k}]$:

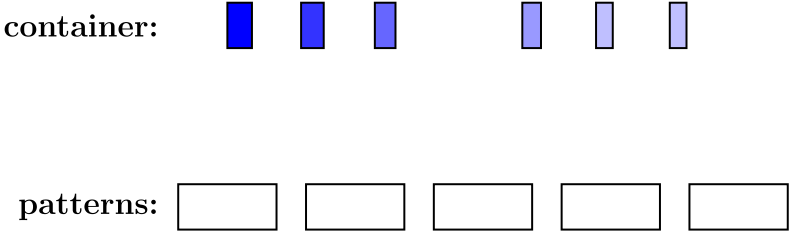
- Each row has* $\|A_C\|_1 \geq k^{1/2}$
- All entries $\leq k^{1/4}$

Deficiency increase is $O(1)$.

* modulo technicalities



Container building (2)



For each size class $[\frac{1}{k}, \frac{2}{k}]$ (starting with smallest items) do:

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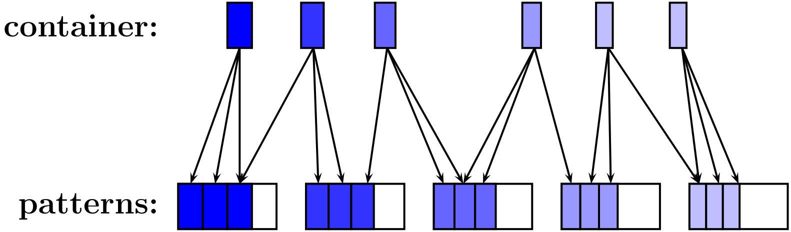

For each size class $[\frac{1}{k}, \frac{2}{k}]$ (starting with <u>smallest</u> items) do:

(1) Group containers so that $\|A_C\|_1 \geq k^{1/2}$ $\rightarrow$ cost $O(\frac{k^{1/2}}{k})$

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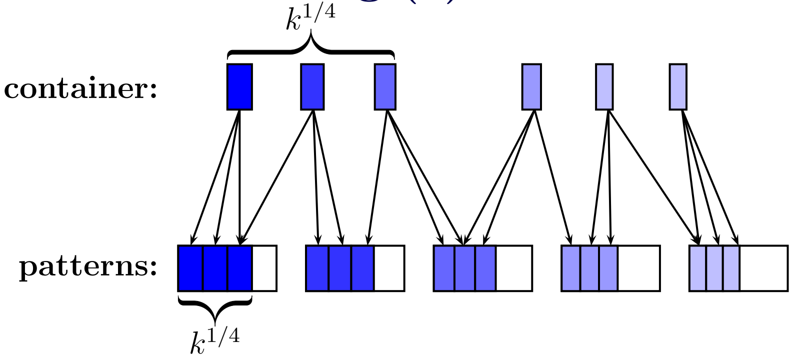

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Container building (2)

container:

patterns:

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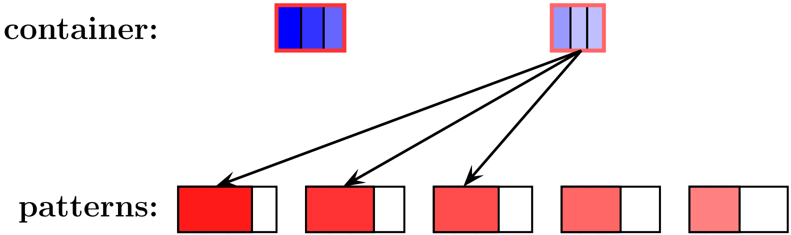

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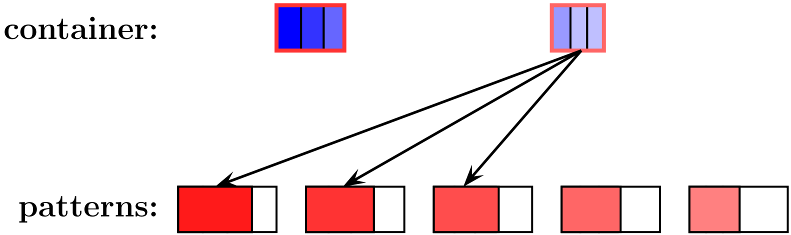

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- **Deficiency increase:** $O(k^{1/2} \cdot \frac{1}{k}) = \Theta(\frac{1}{k^{1/2}})$ for class $\frac{1}{k}$.
- Over all k: $\sum_k \Theta(k^{-1/2}) = O(1)$

Applying the Partial Coloring Lemma

$$A = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

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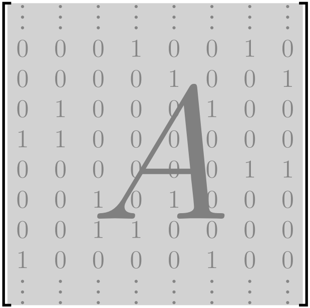

- **Given:** x. **Find:** y with $|(\sum_{j \le i} A_j)(x - y)|$ small

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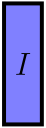
- **Given:** x. **Find:** y with $|(\sum_{j \leq i} A_j)(x - y)|$ small
- Suppose $\frac{1}{k} \leq s_i \leq \frac{2}{k}$

Applying the Partial Coloring Lemma

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$



- **Given:** x. **Find:** y with $|(\sum_{j \le i} A_j)(x - y)|$ small
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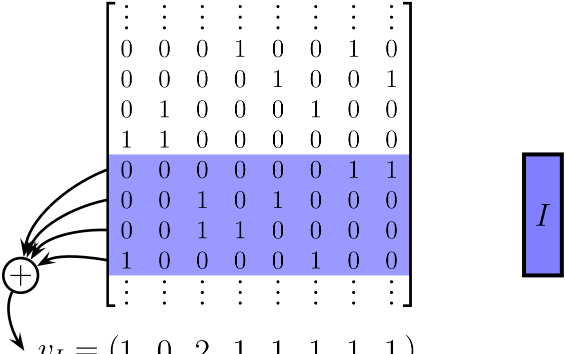
Applying the Partial Coloring Lemma



$$v_I = \begin{pmatrix} 1 & 0 & 2 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

- **Given:** x. **Find:** y with $|(\sum_{j \le i} A_j)(x - y)|$ small
- Suppose $\frac{1}{k} \le s_i \le \frac{2}{k}$
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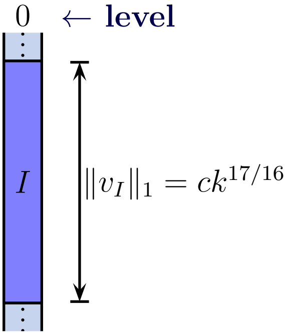

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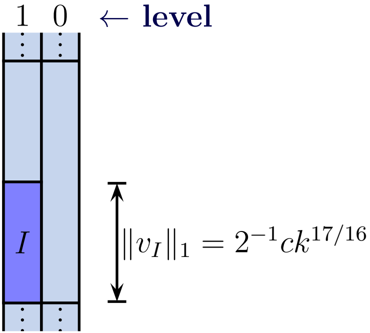

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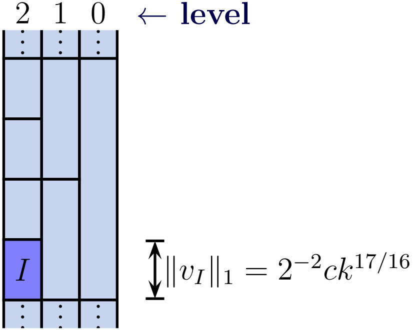

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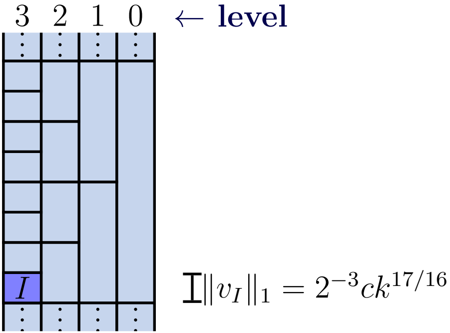

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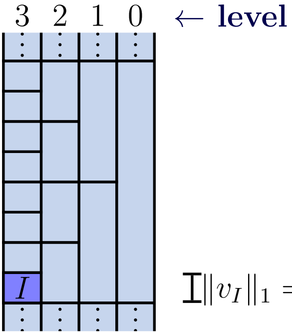

$\|v_I\|_1 = 2^{-3}ck^{17/16}$

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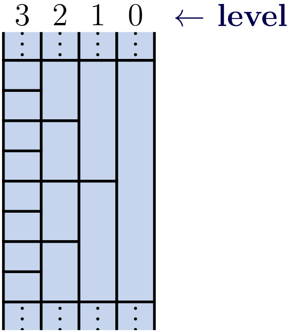
- Suppose $\frac{1}{k} \leq s_i \leq \frac{2}{k}$
- For interval $I \subseteq [n]$: $v_I := \sum_{i\in I} A_i$, $\lambda_I :=$ level(I)

Applying the Partial Coloring Lemma

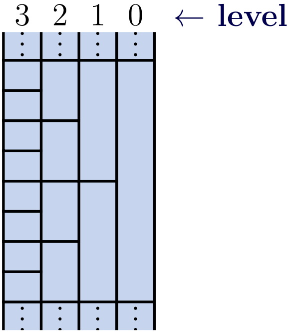
$$\underbrace{\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}}_{m \text{ columns}}$$



- Run Partial coloring with $v_I := \sum_{i \in I} A_i$ and $\lambda_I := \text{level}(I)$

Applying the Partial Coloring Lemma

$$\underbrace{\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}}_{m \text{ columns}}$$



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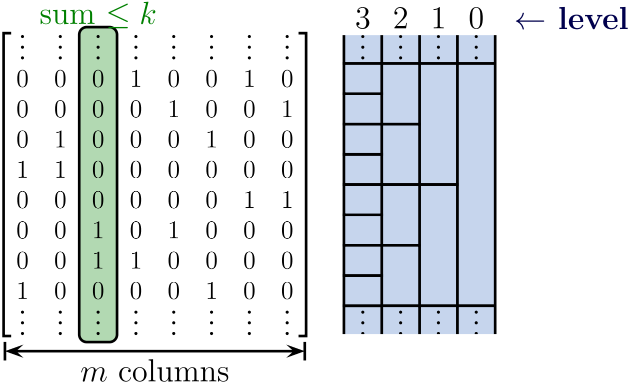
$$\sum_I e^{-\lambda_I^2/16}$$

Applying the Partial Coloring Lemma



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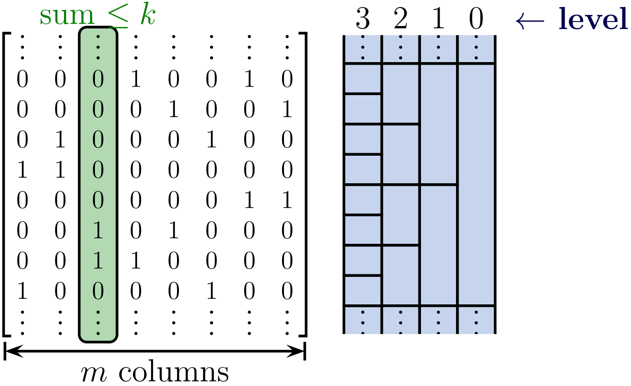
$$\sum_I e^{-\lambda_I^2/16} \leq \sum_{\ell \geq 0} \frac{k \cdot m}{ck^{17/16} 2^{-\ell}} \cdot e^{-\ell^2/16}$$

Applying the Partial Coloring Lemma



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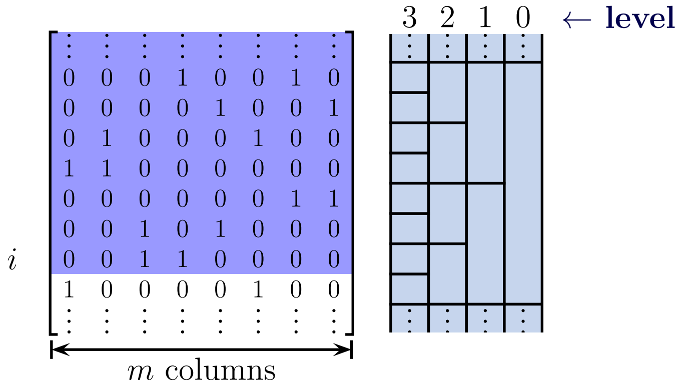
$$\sum_I e^{-\lambda_I^2/16} \leq \sum_{\ell \geq 0} \frac{k \cdot m}{ck^{17/16} 2^{-\ell}} \cdot e^{-\ell^2/16} \leq \frac{m}{100 \cdot k^{1/16}}$$

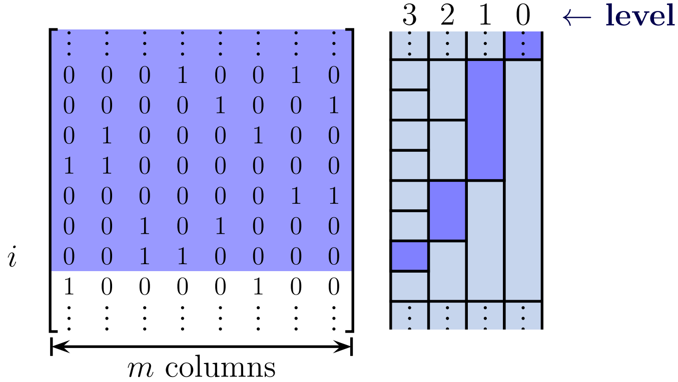
Applying the Partial Coloring Lemma



- Bound error for item i:

$$|(\sum_{j \leq i} A_j)(x - y)| \leq$$

Applying the Partial Coloring Lemma



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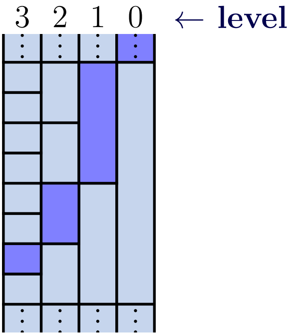

- Bound error for item i:

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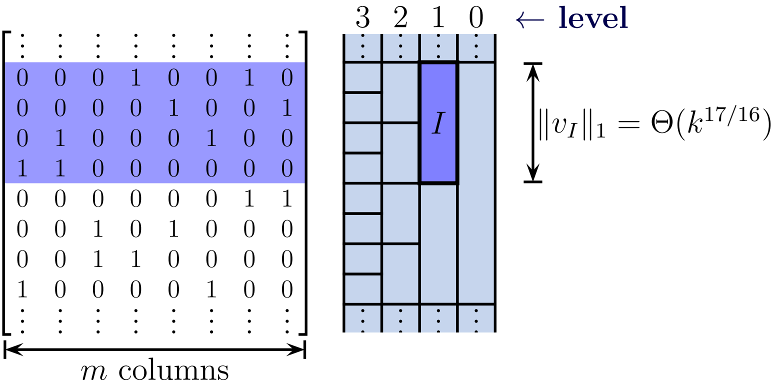
Applying the Partial Coloring Lemma



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$$|(\sum_{j\le i} A_j)(x-y)| \le \sum_{\ell\ge 0} \ell \cdot \|v_{I \text{ on level } \ell}\|_2 \le O(1)\cdot \|v_I\|_2$$

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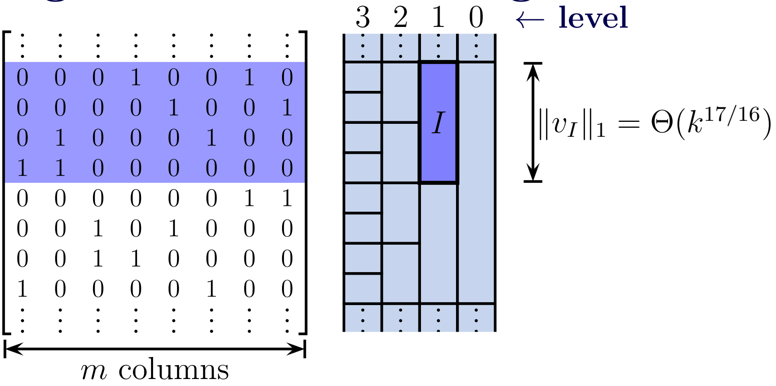

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$$\overset{\text{Hölder}}{\le} \quad \|v_I\|_1 \cdot \sqrt{\frac{k^{1/4}}{k^{1/2}}}$$

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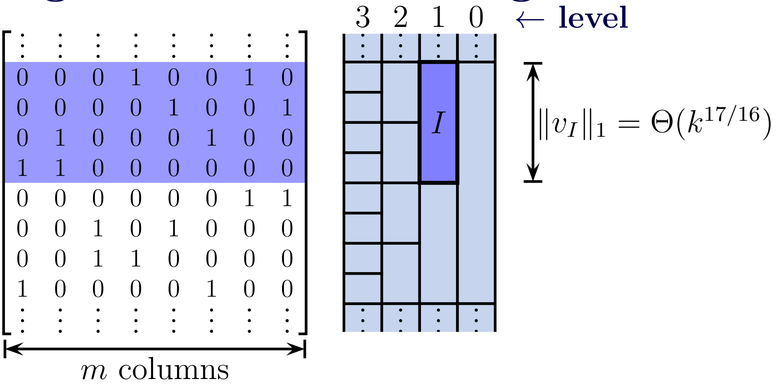

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$$\overset{\text{Hölder}}{\le} \quad \|v_I\|_1 \cdot \sqrt{\frac{k^{1/4}}{k^{1/2}}} \le O(k^{15/16})$$

The algorithm

(1) Compute a fractional LP solution x

(2) FOR $\log n$ iterations DO

  (3) Rebuild incidence matrix
  Effect: Rows of A get small $\|\cdot\|_2$-norm

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  Effect: $|\text{frac}(x')| \leq \frac{1}{2} \cdot |\text{frac}(x)|$

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- Total deficiency increase: $O(\log n)$.

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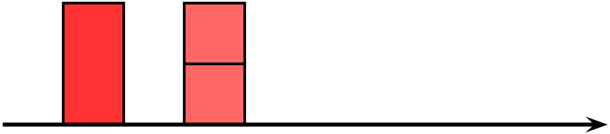

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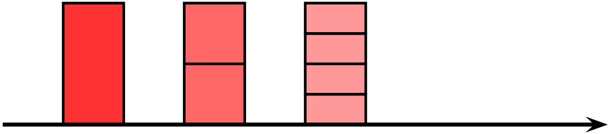

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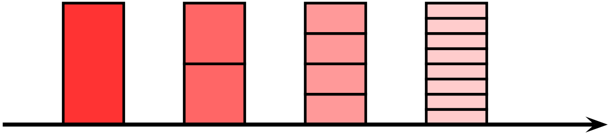

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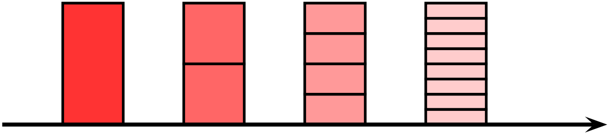

- **Solution:** Use slack!

The end

Open question I

Close the gap

$$1 \leq \text{additive integrality gap} \leq O(\log n)$$

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Thanks for your attention