An Average-Case Analysis for Rate-Monotonic Multiprocessor Real-time Scheduling

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DISCRETE OPTIMIZATION

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Real-time Scheduling

Given: synchronous tasks $\tau_1, \ldots, \tau_n$ where task τ_i

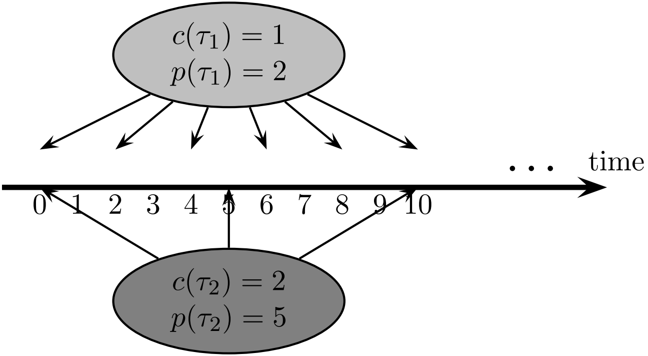
- is periodic with period $p(\tau_i)$
- has running time $c(\tau_i)$
- has implicit deadline

W.l.o.g.: Task τ_i releases job of length $c(\tau_i)$ at $0, p(\tau_i), 2p(\tau_i), \ldots$

Scheduling policy:

- multiprocessor
- fixed priority
- pre-emptive

Example



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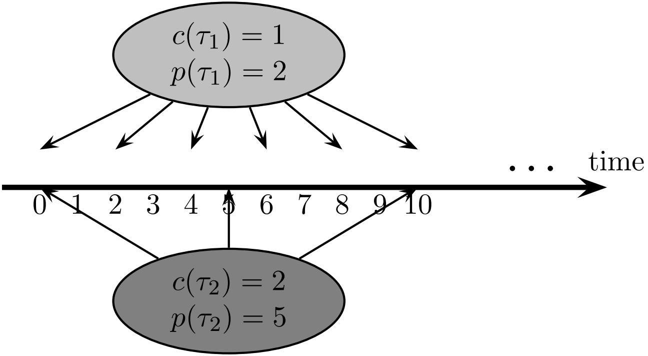
Theorem (Liu & Layland '73)

Optimal priorities: $\frac{1}{p(\tau_i)}$ *for* τ_i *(Rate-monotonic Schedule)*



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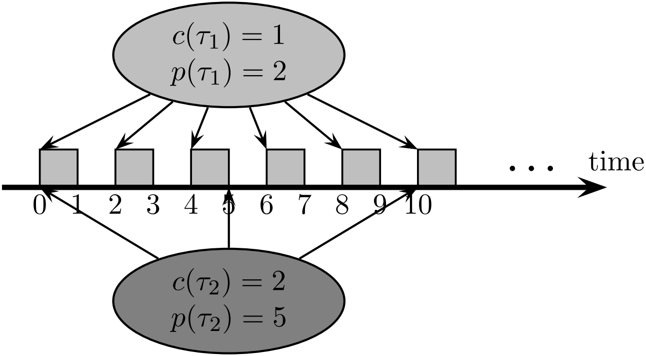

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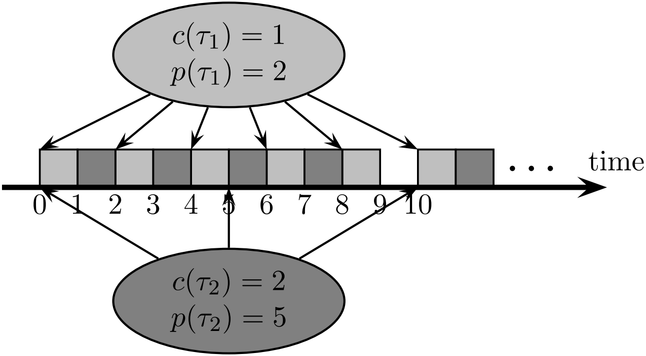

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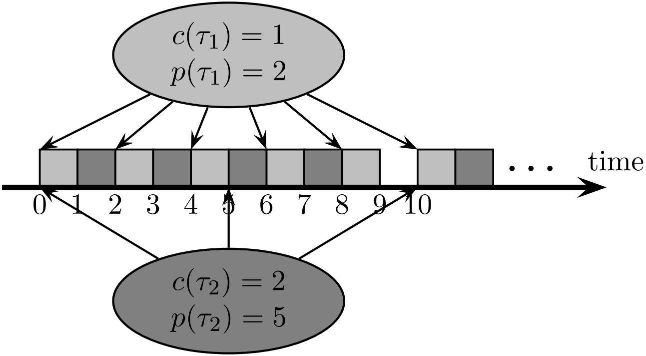

Definition

$u(\tau) = \frac{c(\tau)}{p(\tau)}$ = **utilization** of task τ

Feasibility test

Theorem (Lehoczky et al. '89)

If $p(\tau_1) \leq \ldots \leq p(\tau_n)$ *then the* **response time** $r(\tau_i)$ *in a RM-schedule is the smallest value s.t.*

$$c(\tau_i) + \sum_{j<i} \left\lceil \frac{r(\tau_i)}{p(\tau_j)} \right\rceil c(\tau_j) \leq r(\tau_i)$$

1 machine suffices $\Leftrightarrow \forall i : r(\tau_i) \leq p(\tau_i)$.

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Observation

If $p(\tau_1) \mid p(\tau_2) \mid \ldots \mid p(\tau_n)$: 1 machine suffices $\Leftrightarrow u(\mathcal{S}) \leq 1$

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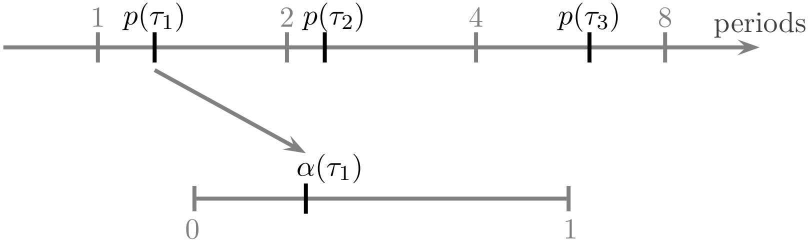

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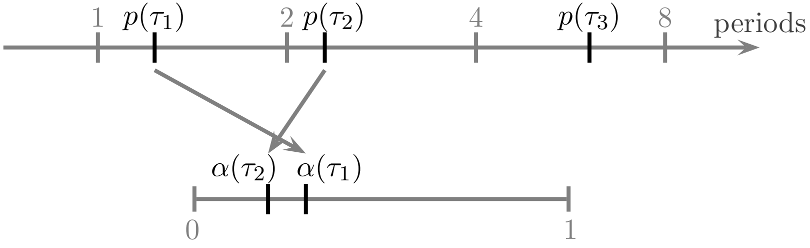

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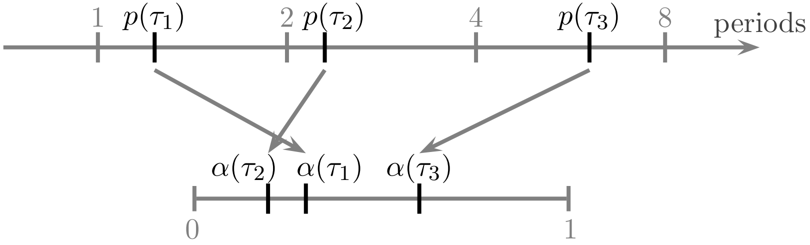

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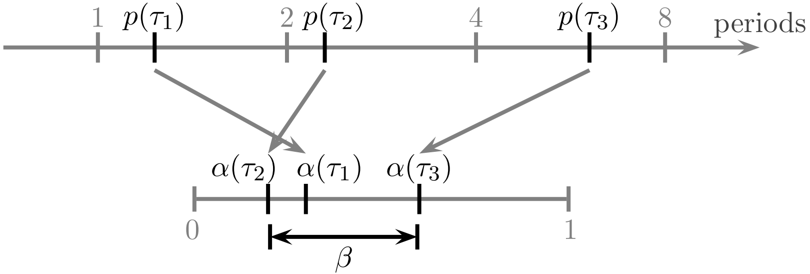

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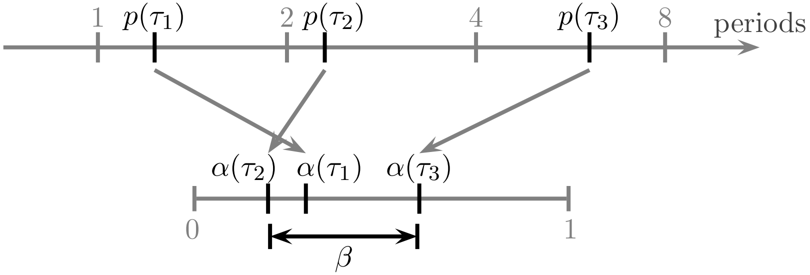

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Definition

Probability distribution:

- Adversary gives $p(\tau_i)$
- $u(\tau_i) \in [0,1]$ uniformly at random

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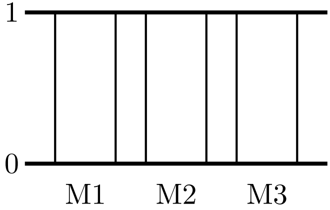

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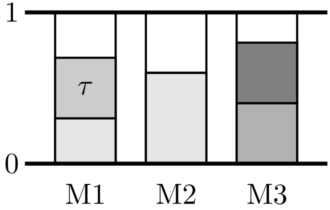

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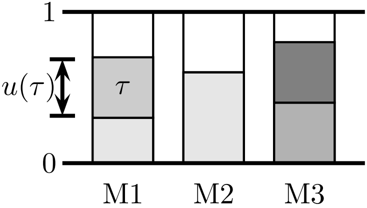

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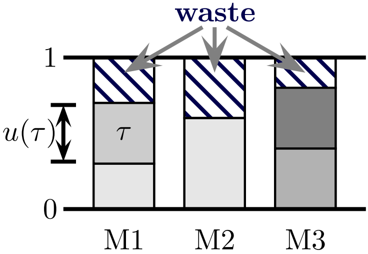

Our algorithm

First Fit Matching Periods (FFMP):

1. **Sort:** $0 \leq \alpha(\tau_1) \leq \alpha(\tau_2) \leq \ldots \leq \alpha(\tau_n) < 1$
2. **First Fit:** FOR $i = 1, \ldots, n$ DO
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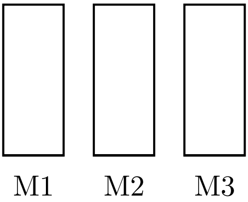

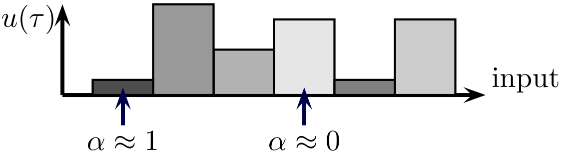

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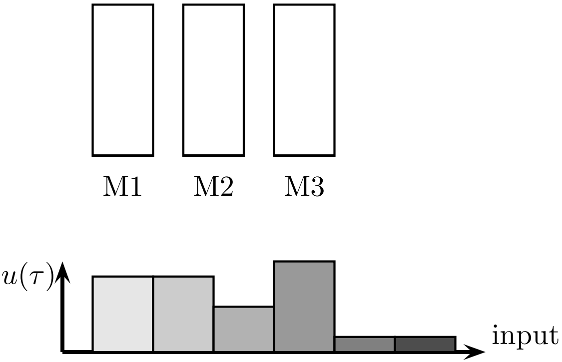

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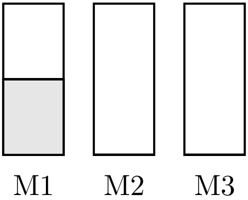

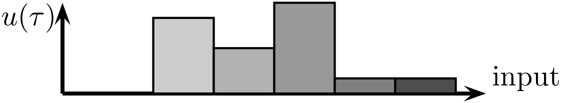

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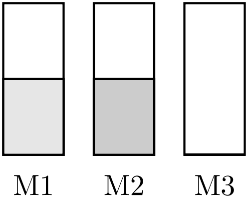

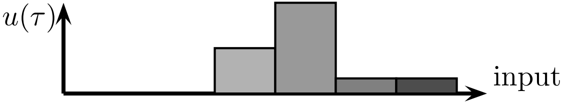

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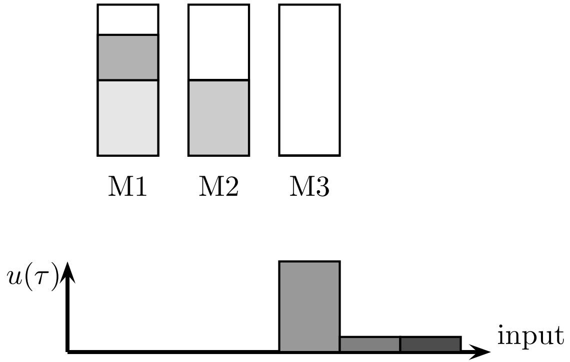

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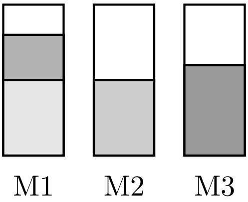

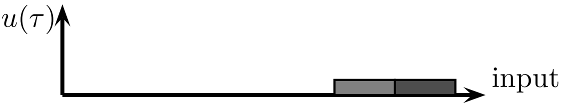

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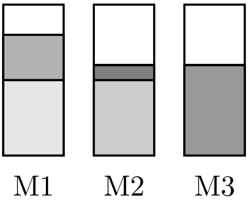

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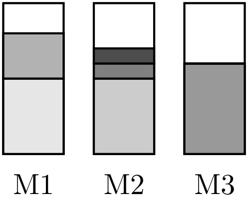

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Running time: $\mathcal{O}(n \log n)$

Theorem

For n tasks with $u(\tau_i) \in [0, 1]$ uniformly at random

$$E[\textit{waste of FFMP}] \le \mathcal{O}(n^{3/4} (\log n)^{3/8})$$

(average utilization $\to$ 100%)

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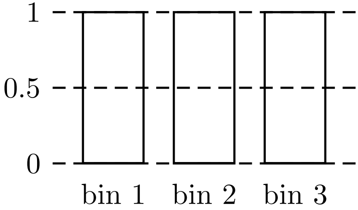

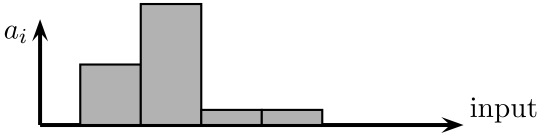

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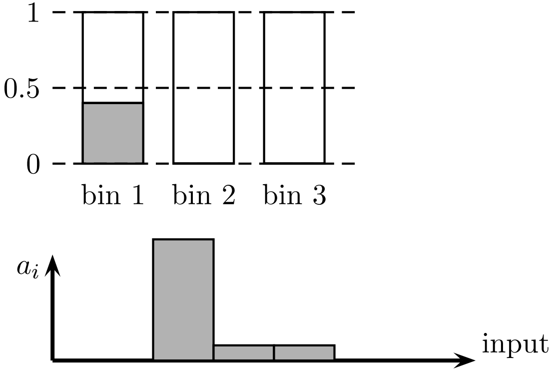

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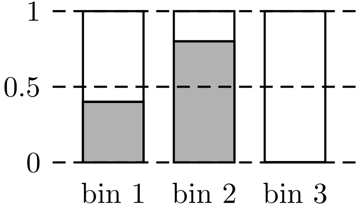

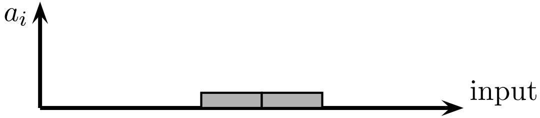

Bin Packing

Given: n items with **sizes** $a_i \in [0, 1]$
Find: Assignment to minimum number of **bins** of size ≤ 1

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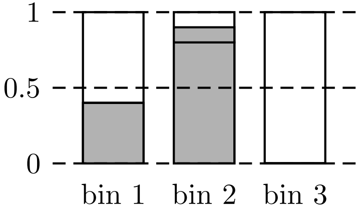

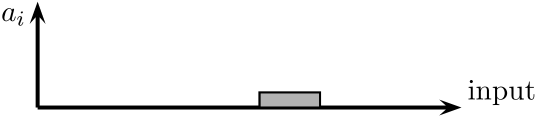

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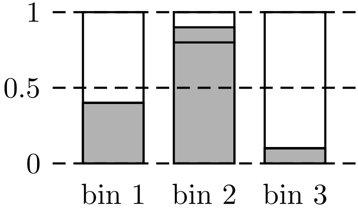

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Theorem (Shor '84)

First Fit is **monotone**.

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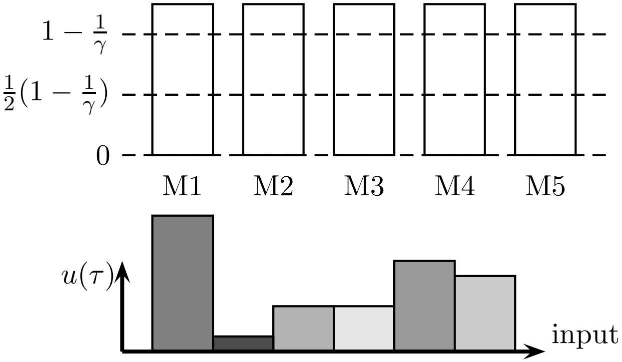

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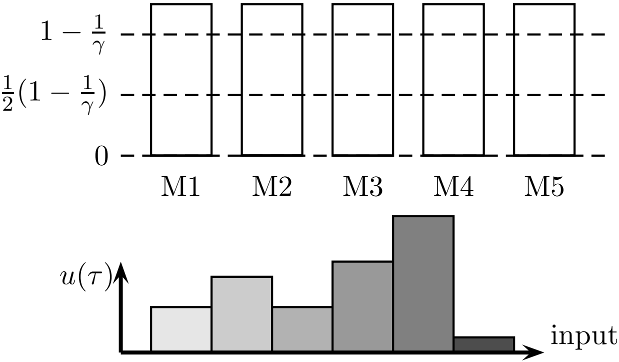

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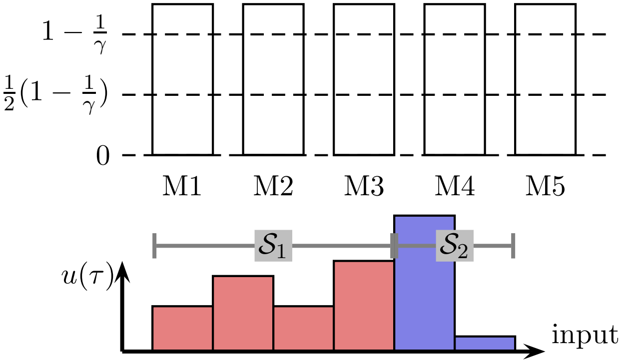

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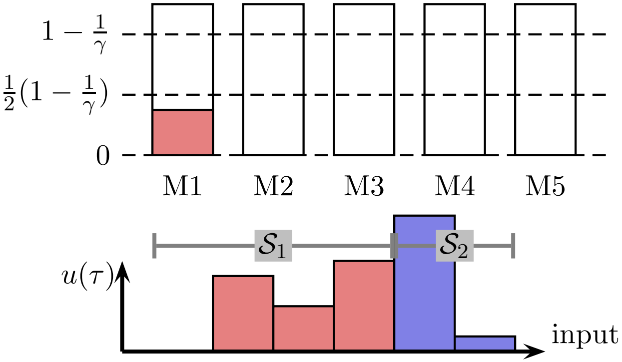

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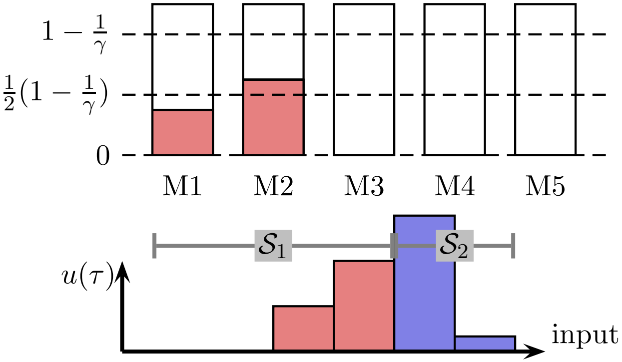

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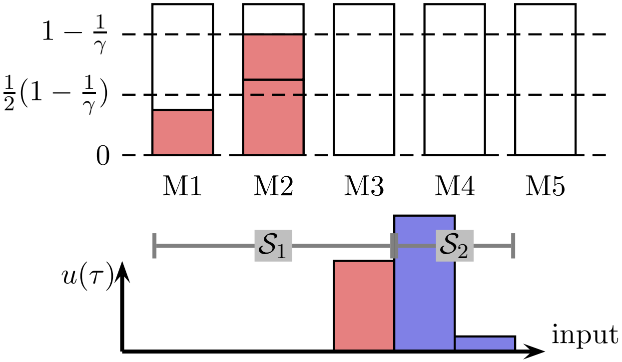

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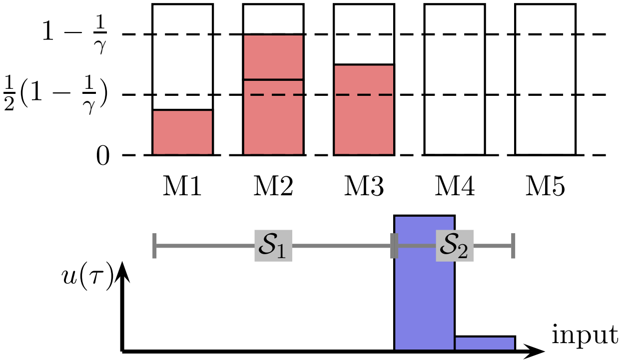

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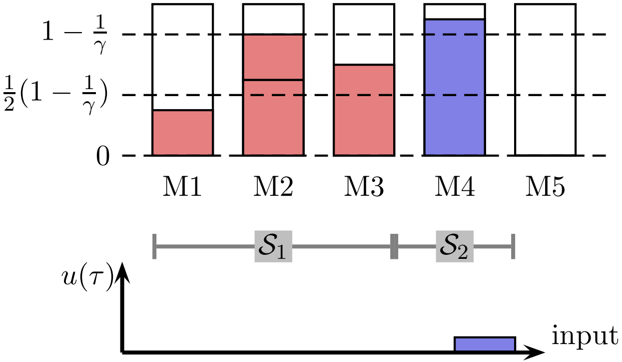

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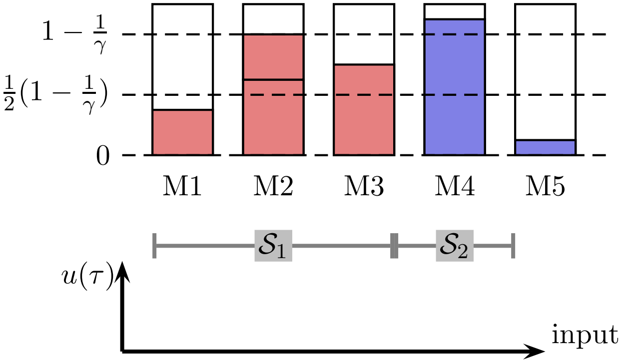

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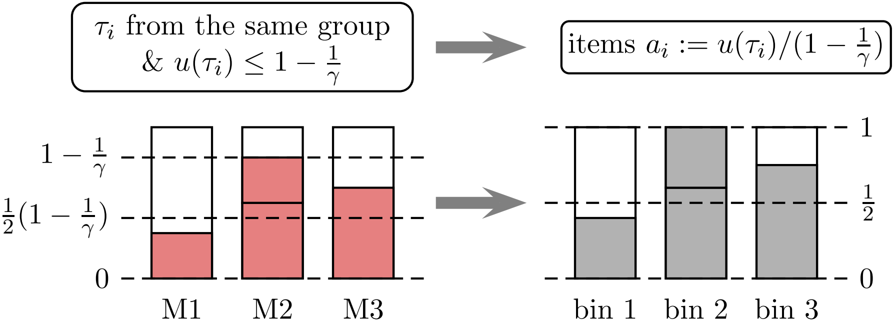

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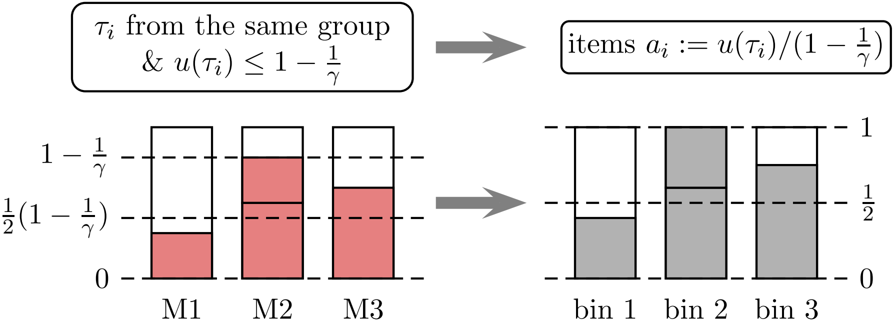

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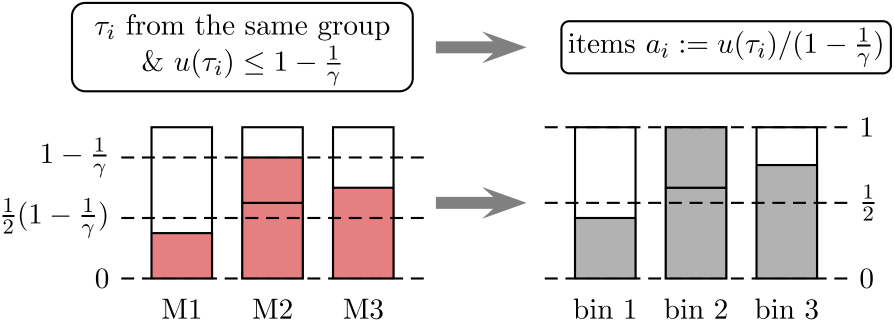

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Corollary

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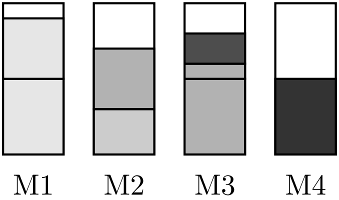

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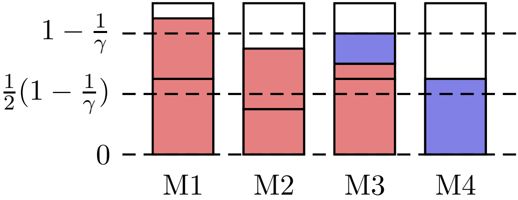

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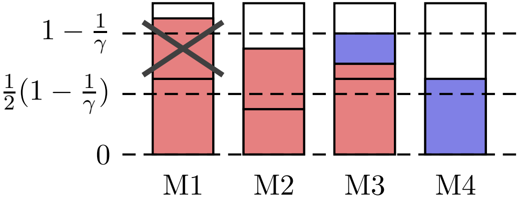

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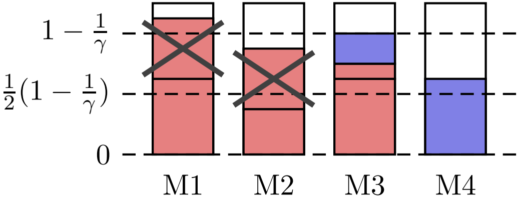

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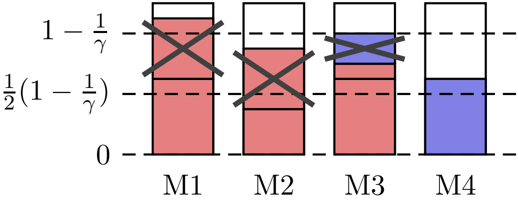

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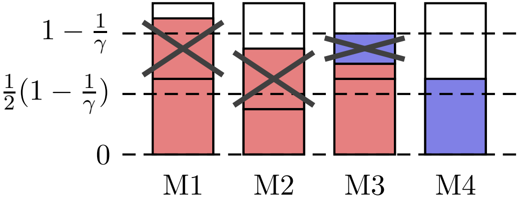

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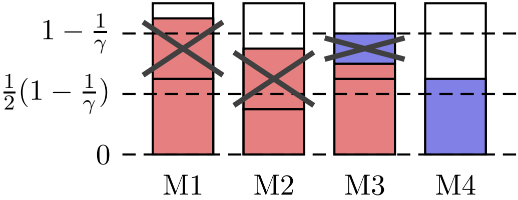

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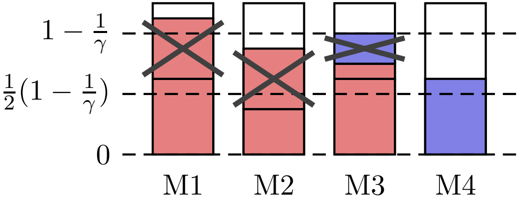

□

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$$\begin{aligned} E[\text{waste of FFMP}^*(\mathcal{S}_j)] &\leq |\mathcal{S}_j| \cdot \frac{1}{\gamma} + E[\text{waste of FirstFit}(I_j)] \\ &\leq \frac{|\mathcal{S}_j|}{\gamma} + f(|I_j|) \leq \frac{|\mathcal{S}_j|}{\gamma} + f(|\mathcal{S}_j|) \end{aligned}$$

$E[\text{waste of FFMP}^*(\mathcal{S})]$

Main result

Theorem

$E[\textit{waste of FFMP}^*] \leq \mathcal{O}(n^{3/4}(\log n)^{3/8})$

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$$E[\text{waste of FFMP}^*(\mathcal{S})] \leq \sum_{j=1}^{\gamma} \left(\frac{|\mathcal{S}_j|}{\gamma} + f(|\mathcal{S}_j|) \right)$$

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$$\overset{\text{Jensen}}{\leq} \frac{n}{\gamma} + \gamma \cdot f(n/\gamma) \overset{\gamma := n^{1/4}/(\log n)^{3/8}}{\leq} \mathcal{O}(n^{3/4}(\log n)^{3/8}) \quad \square$$

Open problems

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Thanks for your attention