Some 0/1 polytopes need exponential size extended formulations

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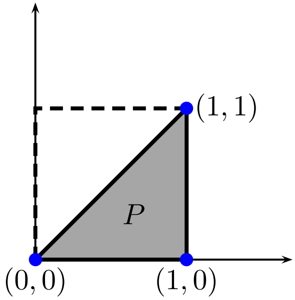
0/1 **polytopes**

$$P \quad = \quad \mathrm{conv}(X) \text{ with } X \subseteq \{0,1\}^n$$



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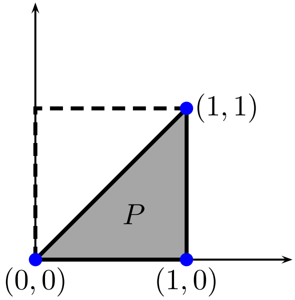

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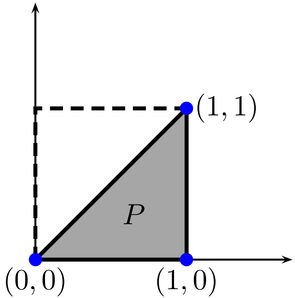

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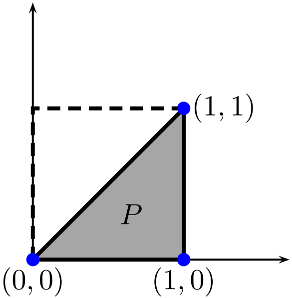

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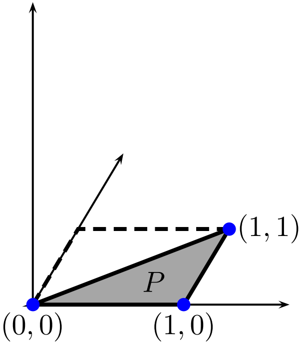

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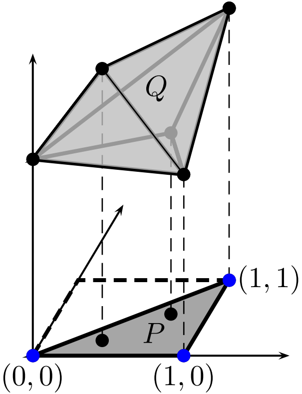
&= \{x \in \mathbb{R}^n \mid Ax \le b\} && \rightarrow \text{exponential!} \\
&= p(Q) && \rightarrow \text{can be polynomial}
\end{aligned}
$$

- $p : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n$ linear projection
- $Q = \{(x,y) \mid Bx + Cy \le c\}$

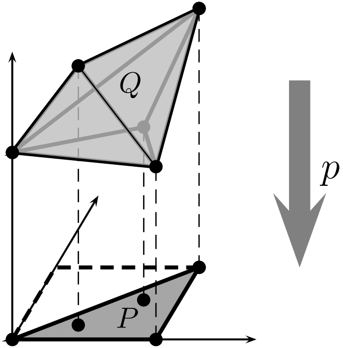


Extension complexity

Definition

Extension complexity:

$$\mathrm{xc}(P) := \min \left\{ \#\text{facets of } Q \;\middle|\; \begin{array}{c} Q \text{ polyhedron} \\ p \text{ linear projection} \\ p(Q) = P \end{array} \right\}$$



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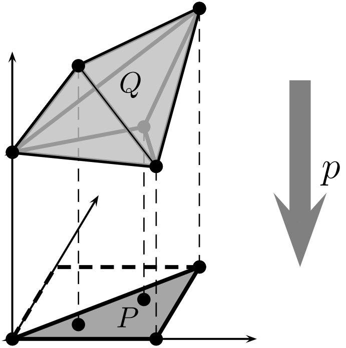
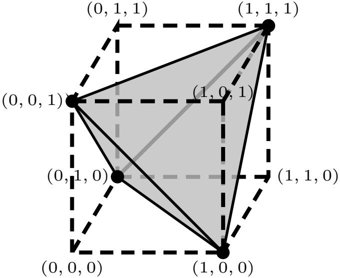
If $\mathrm{xc}(P) \leq poly(n) \Rightarrow (Q, p)$ **compact formulation** of P.



Example: Parity Polytope

$$P \quad = \quad \mathrm{conv}\{x \in \{0,1\}^n \mid \#\text{ ones in } x \text{ is odd}\}$$



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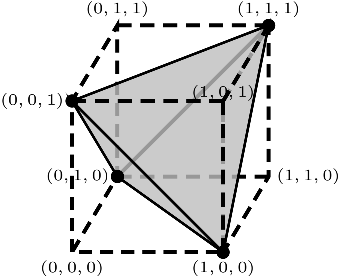

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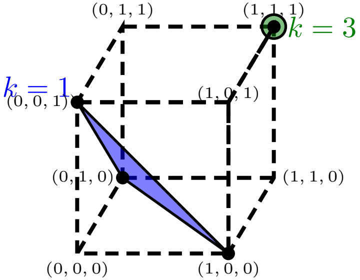
$$
\begin{aligned}
\mathbf{1}^T z^k &= k \qquad (k \text{ odd}) \\
\mathbf{0} \le z^k &\le \mathbf{1}
\end{aligned}
$$



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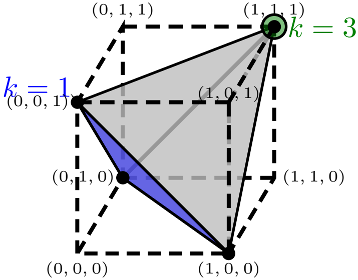
$$\begin{aligned} x &= \sum_{k \text{ odd}} z^k \\ \mathbf{1}^T z^k &= k \cdot \lambda_k \quad (k \text{ odd}) \\ \mathbf{0} \le z^k &\le \mathbf{1} \cdot \lambda_k \\ \mathbf{1}^T \lambda &= 1 \\ \lambda &\ge \mathbf{0} \end{aligned}$$



What's known?

Compact formulations:

- Perfect Matching in planar graphs [Barahona '93]
- Perfect Matching in bounded genus graphs [Gerards '91]
- $O(n \log n)$-size for Permutahedron [Goemans '10] (→ **tight**)
- $n^{O(1/\varepsilon)}$-size ε-apx for Knapsack Polytope [Bienstock '08]
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Theorem (Yannakakis)

No **symmetric** *compact formulation for* TSP Polytope *and* Perfect Matching Polytope*.*

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Theorem (Yannakakis)

No **symmetric** *compact formulation for* TSP POLYTOPE *and* PERFECT MATCHING POLYTOPE*.*

Theorem (Kaibel, Pashkovich & Theis '10)

Compact formulation for $\log n$ *size matchings, but no symmetric one.*

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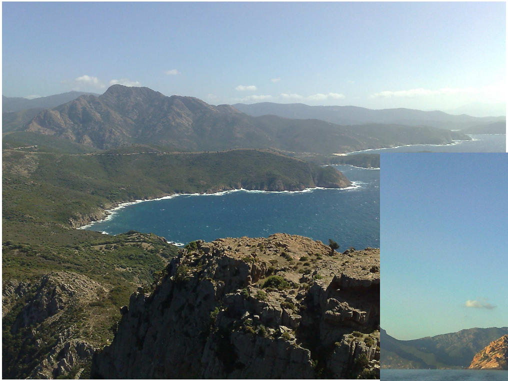

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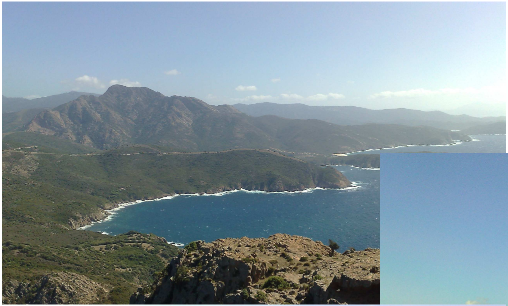

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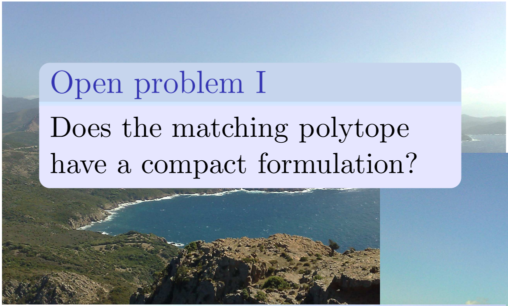

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Does the matching polytope have a compact formulation?

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Open problem II (V. Kaibel)

Is there any 0/1 polytope without a compact formulation?

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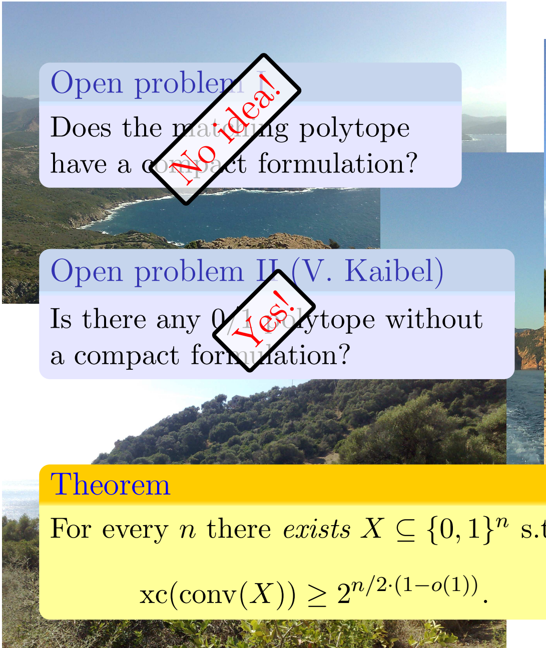

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Open problem II (V. Kaibel)

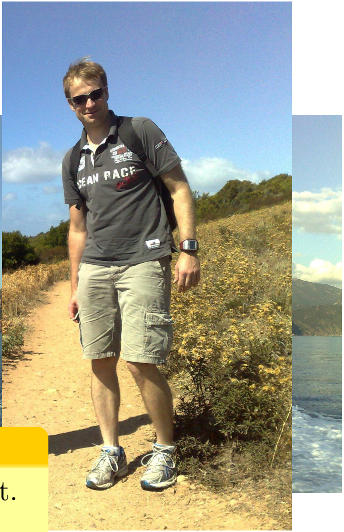
Is there any 0/1 polytope without a compact formulation?

Yes!

Theorem

For every n there *exists* $X \subseteq \{0,1\}^n$ s.t.

$$\mathrm{xc}(\mathrm{conv}(X)) \geq 2^{n/2\cdot(1-o(1))}.$$



Proof strategy

$X \subseteq \{0, 1\}^n$

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$$X \subseteq \{0,1\}^n \quad \Longrightarrow \quad \overbrace{\left\{ x \mid \exists y : \boxed{A}\, x + \boxed{B}\, y \le \boxed{b} \right\}}^{\mathrm{conv}(X)}$$

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$$X \subseteq \{0,1\}^n \xrightarrow{\text{injective map}} \overbrace{\left\{ x \mid \exists y : \boxed{A}\, x + \boxed{B}\, y \leq \boxed{b} \right\}}^{\mathrm{conv}(X)}$$

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$$=$$

$$2^{2^n}$$

Proof strategy

irrational entries

$$X \subseteq \{0,1\}^n \xrightarrow{\text{injective map}} \overbrace{\left\{ x \mid \exists y : \boxed{A}\, x + \boxed{B \; \boxed{\sqrt{2}}}\, y \leq \boxed{b} \right\}}^{\mathrm{conv}(X)}$$

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Doesn't work!

Slack-matrix

Write: $P = \mathrm{conv}(\{x^1, \ldots, x^v\}) = \underbrace{\{x \in \mathbb{R}^n \mid Ax \leq b\}}_{\text{non-redundant}}$

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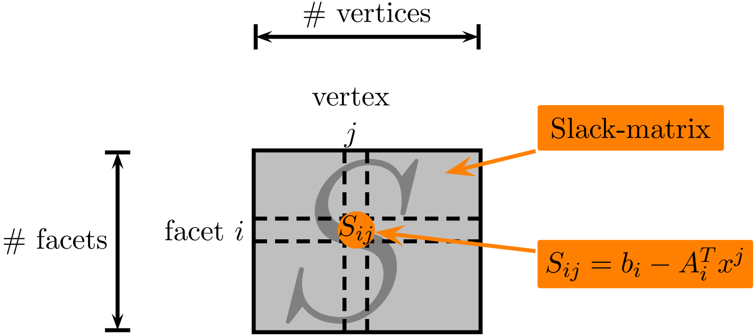

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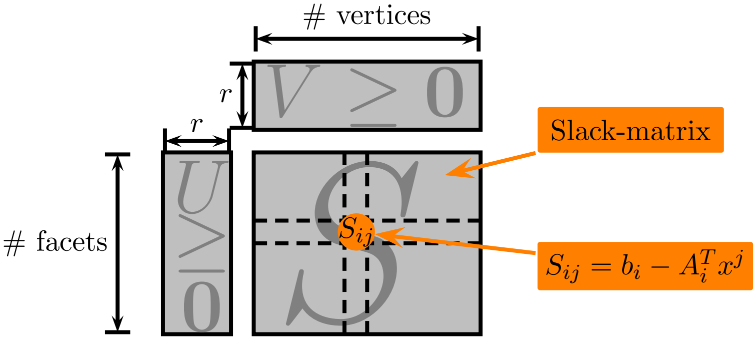
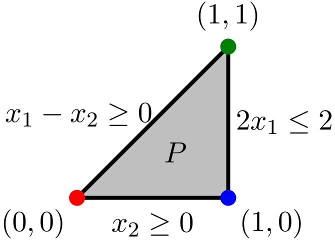


Non-negative rank:

$$\mathrm{rk}_+(S) = \min\{r \mid \exists U \in \mathbb{R}_{\geq 0}^{f \times r}, V \in \mathbb{R}_{\geq 0}^{r \times v} : S = UV\}$$

Example for slack-matrix



$$S = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Yannakakis' Theorem

Theorem (Yannakakis '91)

Let S be **slackmatrix** *for P:*

- $xc(P) = rk_+(S)$.
- *For any non-neg. factorization $S = UV$:*

$$P = \{x \in \mathbb{R}^n \mid \exists y \geq \mathbf{0} : Ax + Uy = b\}$$

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- $A_i x > b_i \implies A_i x + \underbrace{U_i y}_{\geq 0} > b_i$.

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- $A_i x > b_i \Longrightarrow A_i x + \underbrace{U_i y}_{\geq 0} > b_i$.
- "$\geq$" follows from an application of duality.

Controlling the coefficients

- Fix $X \subseteq \{0,1\}^n$, $P := \mathrm{conv}(X) = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ and Slack-matrix $S = UV$

Valid assumption:

- A, b integral with $\|A\|_\infty, \|b\|_\infty \leq 2^{n \log(2n)}$

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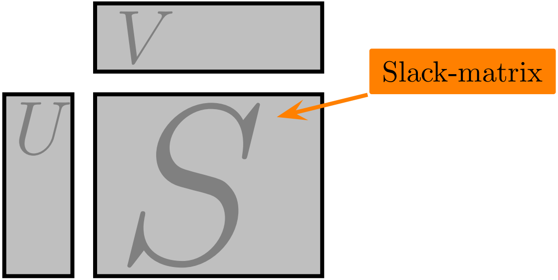

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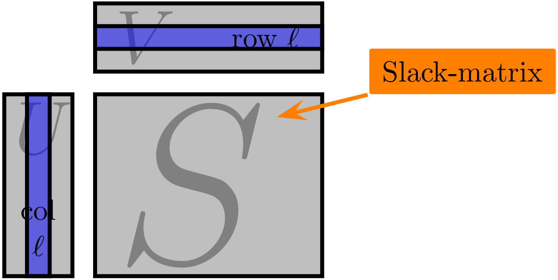

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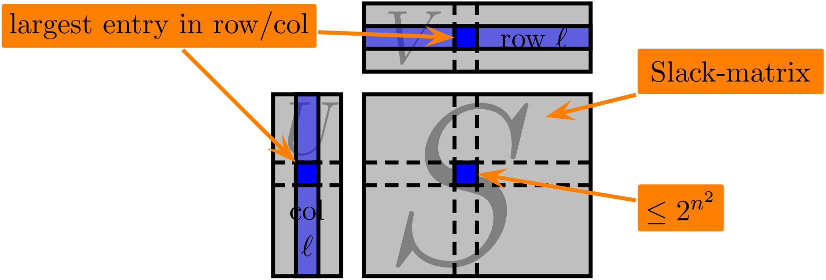
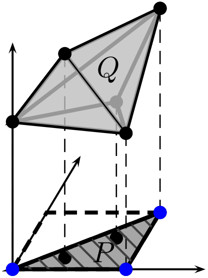
  - But $\|S\|_\infty \geq \|U^\ell\|_\infty \cdot \|V_\ell\|_\infty$



Rounding entries



$$\color{blue}{X} = \left\{ x \in \{0,1\}^n \mid \exists \mathbf{0} \le y \le \infty : A \cdot x + U \cdot y = b \right\}$$

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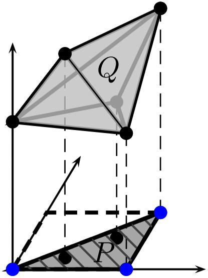

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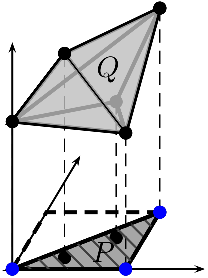

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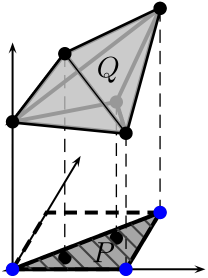

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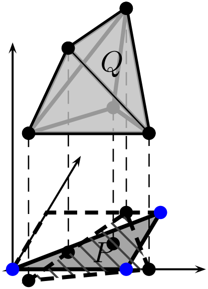

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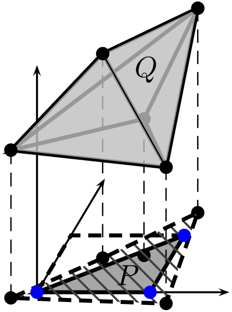
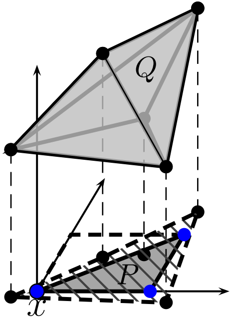


$$X = \left\{ x \in \{0,1\}^n \mid \exists \mathbf{0} \le y \le 2^{n^2} : A \cdot x + U \cdot y = b \pm \frac{1}{2^{n^2}} \right\}$$

Rounding entries

- Pick $\boxed{\text{rows } I}$ max. $|\det([A_I, U_I])|$
- Round $U_{ij} \to U'_{ij} \in \mathbb{Z} \cdot (\frac{1}{2})^{n^3}$
- Consider vertex $x \in X$: rounding error $\leq \|U - U'\|_\infty \cdot \|y\|_1 \leq \ldots \leq (\frac{1}{2})^{n^2}$



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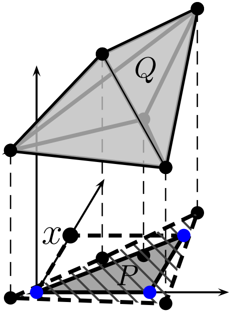

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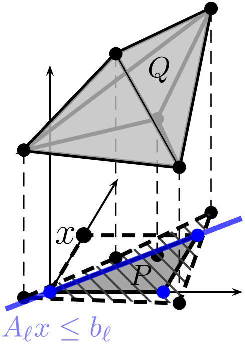

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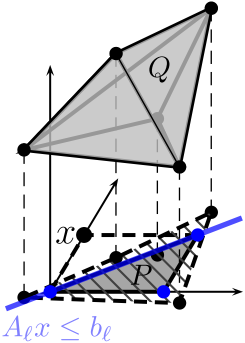

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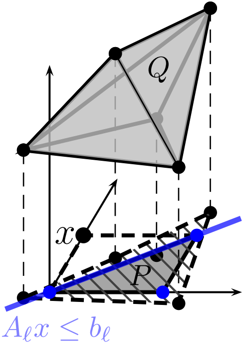

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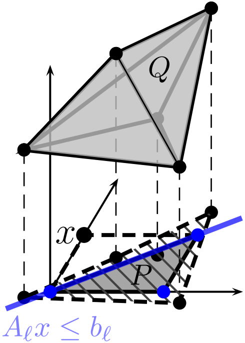

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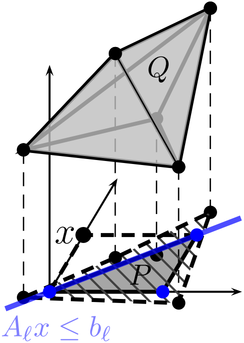

- One $\boxed{i \in I}$ violated by
  $\frac{1}{|I|} - \|U - U'\|_\infty \cdot \|y\|_1 \gg (\frac{1}{2})^{n^2}$
  $\Rightarrow x$ not feasible!

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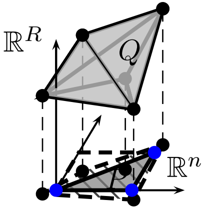

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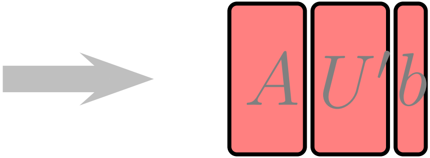

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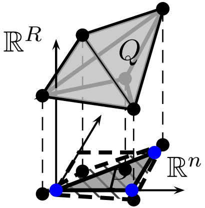

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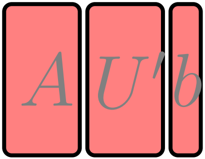

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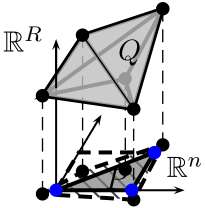

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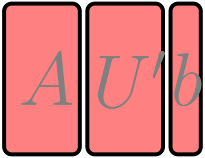

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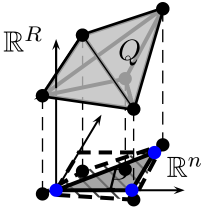

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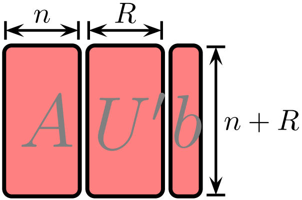

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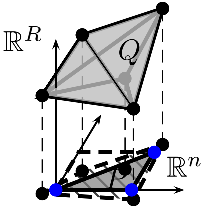

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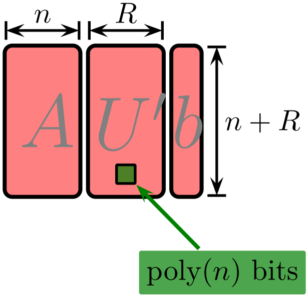

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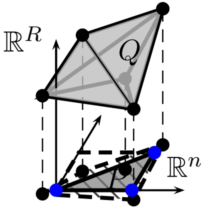

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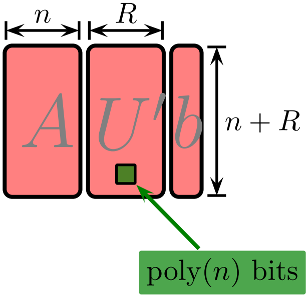

$$2^{2^n} \qquad \leq \qquad 2^{poly(n)\cdot R^2}$$

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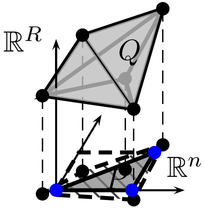

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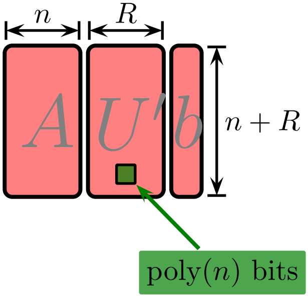

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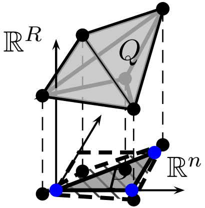

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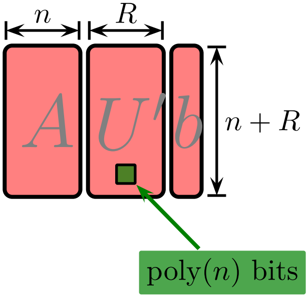

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Consequences for matroids

- **Fact:** There are $2^{2^n/poly(n)}$ many matroids on n elements [Duke '03].

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Theorem

For every n there exists a matroid $\mathcal{M} = ([n], \mathcal{I})$ such that

$$\mathrm{xc}(\mathrm{conv}(\chi(\mathcal{I})) \geq 2^{n/2\cdot(1-o(1))}.$$

Consequences for TSP

Theorem (Folkore)

$\mathbf{NP} \not\subseteq \mathbf{P}_{/\mathbf{poly}} \Rightarrow$ no compact formulation for

$$P_{TSP} = \text{conv}\{\chi(C) \mid C \text{ is Hamiltonian in complete graph}\}$$

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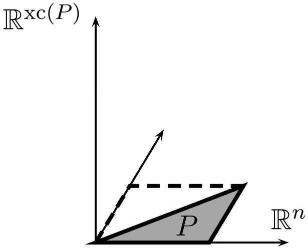

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Given instance $G = (V, E)$ for Hamiltonian Cycle. Optimize

$$c_e := \begin{cases} 1 & e \in E \\ 0 & \text{otherwise} \end{cases}$$

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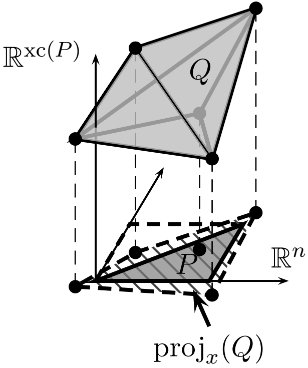

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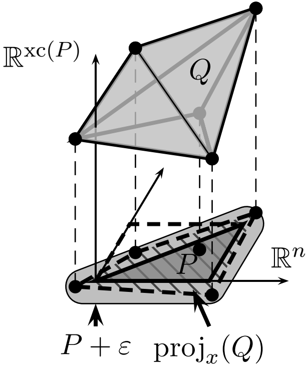
- $\exists$HC $\Rightarrow OPT \geq n$
- NO HC $\Rightarrow OPT \leq n - 1 + \varepsilon n \leq n - \frac{1}{2}$



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Thanks for your attention