Constructive Discrepancy Minimization for Convex Sets

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Discrepancy theory

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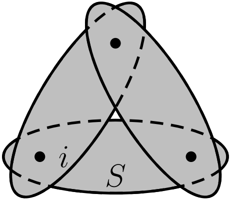

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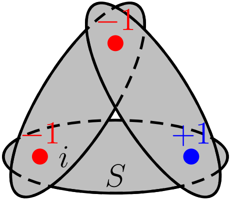

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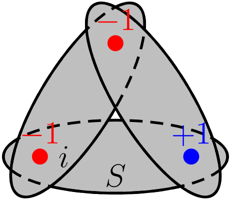

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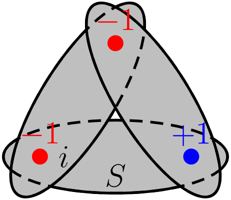

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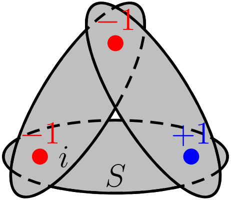

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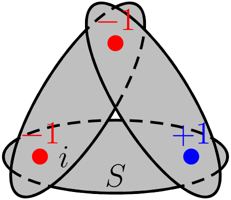

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Main method: Find a **partial coloring** $\chi : [n] \to \{0, \pm 1\}$

- low discrepancy $\max_{S\in\mathcal{S}} |\chi(S)|$
- $|\mathrm{supp}(\chi)| \geq \Omega(n)$

Discrepancy theory (2)

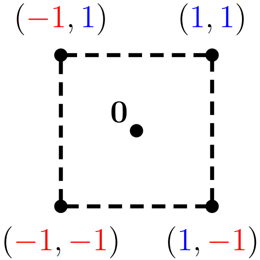
Lemma (Spencer)

For m set on $n \leq m$ elements there is a **partial coloring** of discrepancy $O(\sqrt{n \log \frac{2m}{n}})$.

- Run argument $\log n$ times
- Total discrepancy is

$$\lesssim \sqrt{n} + \sqrt{n/2} + \sqrt{n/2^2} + \ldots + 1 = O(\sqrt{n})$$

Thm of Spencer-Gluskin-Giannopolous



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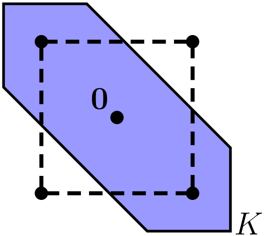

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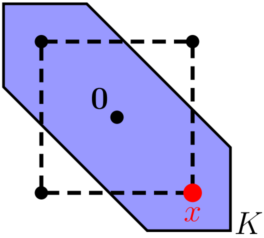

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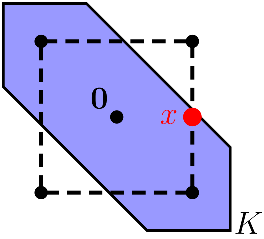

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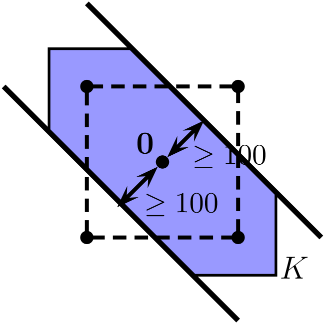

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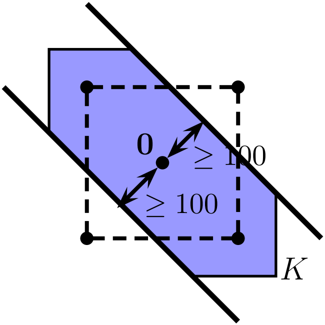

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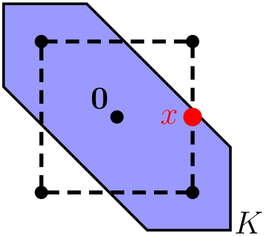

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- Counting argument: any such K admits partial coloring

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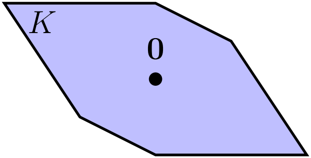
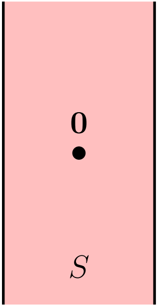
Lemma (Sidak-Kathri '67)

For K convex and symmetric and strip S,

$$\gamma_n(K \cap S) \geq \gamma_n(K) \cdot \gamma_n(S)$$





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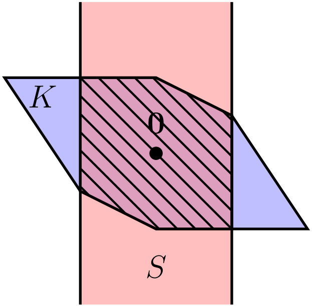

Partial coloring approaches

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- **Bansal '10:** SDP-based random walk for Spencer's Thm
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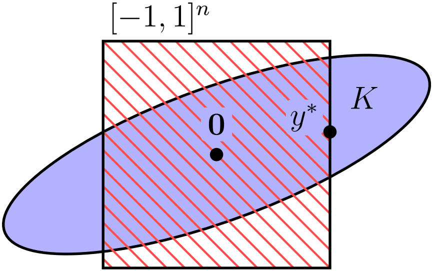
- **Lovett-Meka '12:**
  - (+) poly-time
  - (+) simple and elegant
  - (+/−) Works for any $K = \{x : |\langle x, v_i \rangle| \leq \lambda_i \forall i \in [m]\}$ with $\sum_{i=1}^{m} e^{-\lambda_i^2/16} \leq \frac{n}{16}$

Our main result

Theorem (R. 2014)

For a **convex symmetric set** $K \subseteq \mathbb{R}^n$ with $\gamma_n(K) \geq e^{-\delta n}$, one can find a $y \in K \cap [-1,1]^n$ with $|\{i : y_i = \pm 1\}| \geq \varepsilon n$ in **poly-time**.



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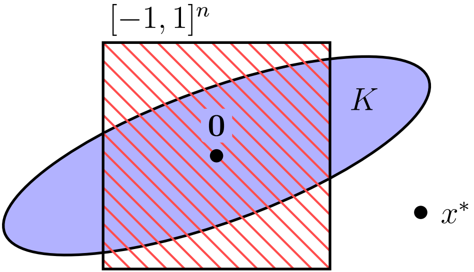

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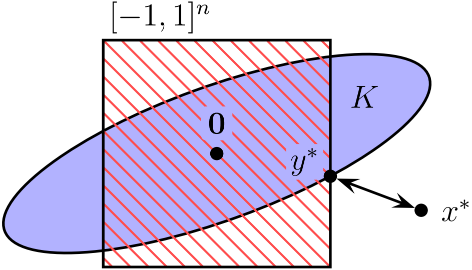
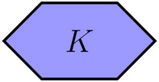


Isoperimetric inequality

- For set K



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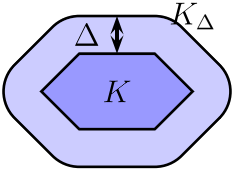
- For set K, let $K_\Delta := \{x : d(x, K) \leq \Delta\}$



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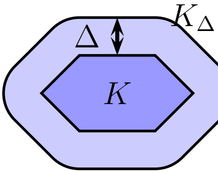

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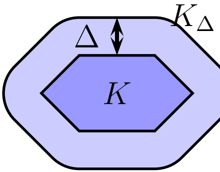

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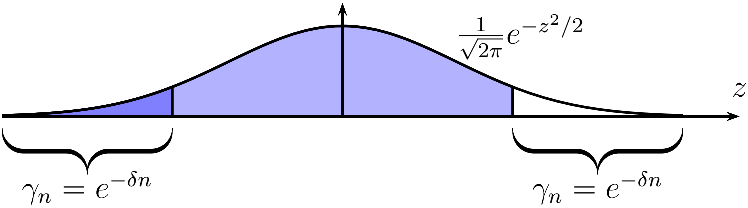

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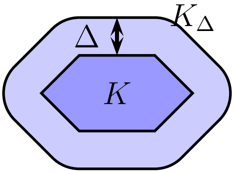

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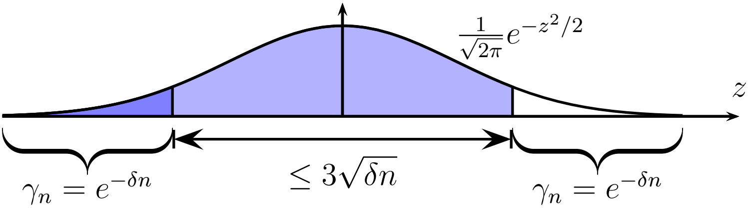

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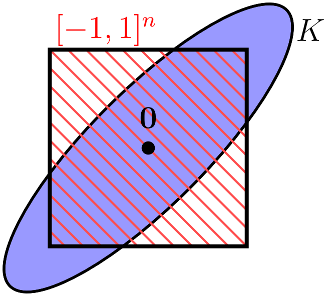

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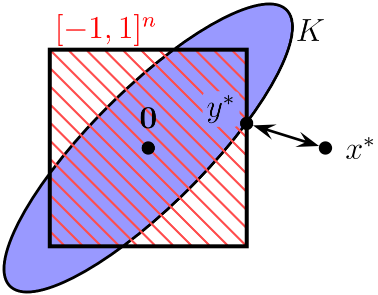

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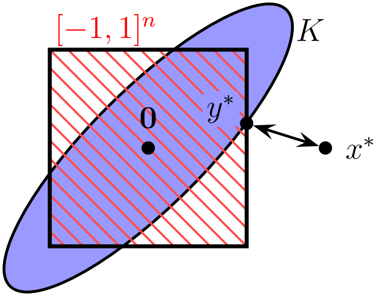

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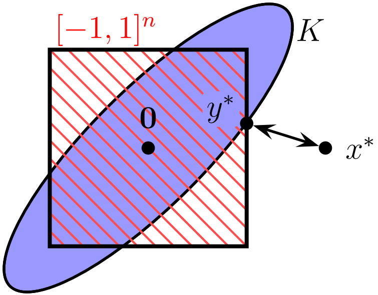

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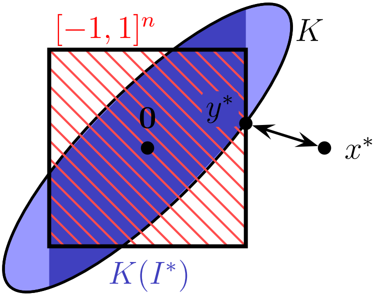

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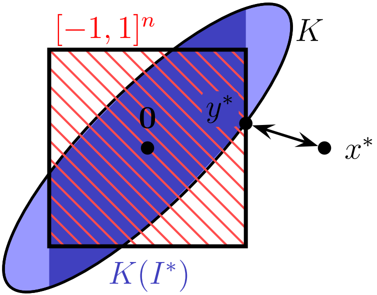

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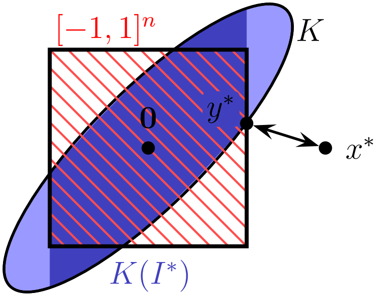

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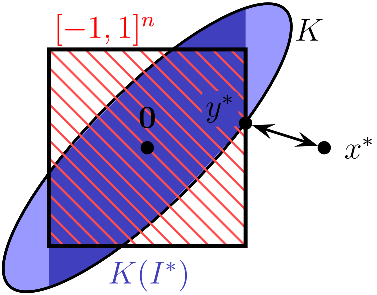

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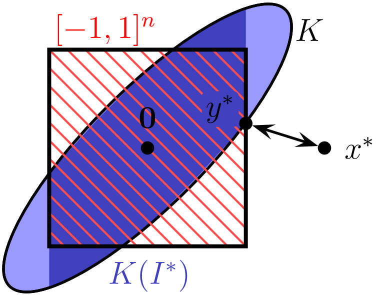

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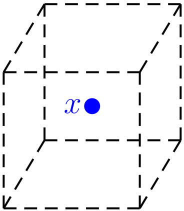
- W.h.p. $d(x^*, K(I^*)) \leq \sqrt{3\delta} \cdot \sqrt{n} \ll \frac{1}{5}\sqrt{n}$
- Union bound over all $|I| \leq \varepsilon n$:

$$\Pr\Big[\bigcup_{|I| \leq \varepsilon n} d(x^*, K(I)) < \frac{1}{5}\sqrt{n}\Big] \leq e^{-\Omega(n)} \qquad \square$$

Application to approximation algorithms

- **Classical technique:** For $K = \{x \in \mathbb{R}^n \mid Ax = b\}$ with $\frac{n}{2}$ constraints $\rightarrow$ basic solution x' has $\leq \frac{n}{2}$ fractional variables.



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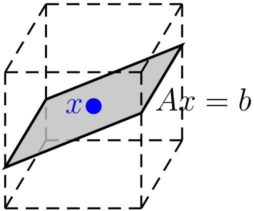

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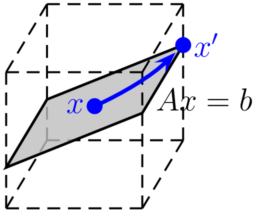

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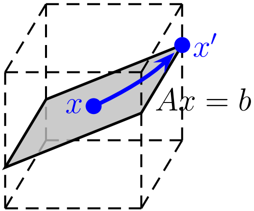

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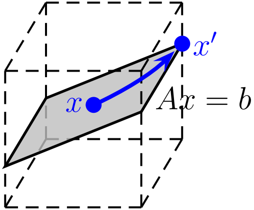

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Better approximation algorithms for

- Bin Packing [R. '13]
- Broadcast scheduling [Bansal, Charikar, Krishnaswamy, Li '14]

Open problems

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Thanks for your attention