Lecturer: Thomas Rothvoss Due date: Friday, Jan 20, 2023, 11pm, on GradeScope

Problem Set 2

CSE 599S - Lattices

Winter 2023

Exercise 1.3 (10pts)
This is an application of Dirichlet's Theorem: Let $\boldsymbol{a} \in (0,1]^n$ be a real vector and consider the hyperplane $H := \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle \boldsymbol{a}, \boldsymbol{x} \rangle = 0\}$. Then there is a rational vector $\tilde{\boldsymbol{a}} \in \frac{\mathbb{Z}^n}{q}$ with $q \leq (2nR)^n$ so that $\tilde{H} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle \tilde{\boldsymbol{a}}, \boldsymbol{x} \rangle = 0\}$ satisfies the following:

$$\forall \boldsymbol{x} \in \{-R, \dots, R\}^n : \big(\boldsymbol{x} \in H \Rightarrow \boldsymbol{x} \in \tilde{H}\big).$$

Remark. You don't have to prove it but the same argument should also show that for all $\boldsymbol{x} \in \{-R, \dots, R\}^n$ one has $\boldsymbol{x} \in H_{\leq} \Rightarrow \boldsymbol{x} \in \tilde{H}_{\leq}$ where $H_{\leq} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle \boldsymbol{a}, \boldsymbol{x} \rangle \leq 0\}$.

Exercise 1.10 (10pts)
Let $\Lambda \subseteq \mathbb{R}^n$ be a full-rank lattice. Assume that $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n \in \Lambda$ are linearly-independent and minimize $|\det(\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)|$. Prove that $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$ are indeed a *basis* of Λ.