Lecturer: Thomas Rothvoss Due date: Friday, Dec 5, 2025, 11pm, on GradeScope

Problem Set 9

Math 581A - Analysis of Boolean Functions

Fall 2025

Exercise 9.1 (20pts)
In the following, for a function $f : \{-1,1\}^n \to \mathbb{R}$ and $d \geq 1$, we abbreviate $f^{\leq d} := \sum_{|S| \leq d} \hat{f}(S)\chi_S$ as the part up to degree d. We also write $W^{\leq d}[f] := \sum_{|S| \leq d} \hat{f}(S)^2$ as the Fourier weight on level at most d.

(i) Prove that for any $f : \{-1,1\}^n \to \mathbb{R}$ and any $q \geq 2$ and $d \geq 1$ one has $W^{\leq d}[f] \leq \|f\|_{E,q/(q-1)} \cdot \|f^{\leq d}\|_q$.

(ii) Extend (i) to prove that $W^{\leq d}[f] \leq \|f\|_{E,q/(q-1)} \sqrt{q-1}^d \cdot \sqrt{W^{\leq d}[f]}$

From now on we make the assumption that $f : \{-1,1\}^n \to \{-1,0,1\}$ and set $\alpha := \Pr_{x \sim \{-1,1\}^n}[|f(x)| = 1]$.

(iii) What is $\|f\|_{E,q/(q-1)}$ in terms of α?

(iv) Prove the so-called *Level-d inequality*: For $f : \{-1,1\}^n \to \{-1,0,1\}$, $d \geq 1$ and α as defined above one has

$$W^{\leq d}[f] \leq O\Big(\alpha^2 \Big(\log\Big(\frac{10}{\alpha}\Big)\Big)^d\Big)$$

Remark. This exercise is inspired by the lecture notes of Dor Minzer's Spring 2024 course on Analysis of Boolean Functions at MIT.