Lecturer: Thomas Rothvoss Due date: Friday, Nov 21, 2025, 11pm, on GradeScope

Problem Set 8

Math 581A - Analysis of Boolean Functions

Fall 2025

Exercise 8.1 (20pts)
For the remainder of this exercise, let $f : \{-1,1\}^n \to \{-1,1\}$ be a boolean function. For $i \in [n]$, set $a_i := \frac{\hat{f}(i)}{\sqrt{W^1[f]}}$ and define $\ell : \{-1,1\}^n \to \mathbb{R}$ with $\ell(x) := \sum_{i=1}^n a_i x_i$. One should think of ℓ as the normalized linear part of f.

(i) Prove that $\sum_{i=1}^n a_i^2 = 1$.

(ii) Prove that $\langle f, \ell \rangle_E \geq \sqrt{W^1[f]}$.

(iii) Prove that $\mathbb{E}_{x\sim\{-1,1\}^n}[|\ell(x)|] \geq \sqrt{W^1[f]}$.

(iv) Assuming that $W^1[f] \geq \frac{1}{100}$ and $\mathrm{Inf}_i[f] \leq \varepsilon$ for all $i \in [n]$, prove that

$$\mathop{\mathbb{E}}_{x\sim\{-1,1\}^n}[|\ell(x)|] = \sqrt{\frac{2}{\pi}} \pm O(\varepsilon)$$

Hint. Use the Berry Esseen Theorem. If you draw a Gaussian $Y \sim N(0,1)$, what is $\mathbb{E}[|Y|]$? You might want to have another look at homework problem 2.1.
To be exact, in class we have seen the variant of the Berry Esseen Theorem which bounds $|\Pr[X \leq u] - \Pr[Y \leq u]| \leq 0.56 \cdot \sum_{i=1}^n \mathbb{E}[|X_i|^3]$ where $Y \sim N(0,1)$ and $X = X_1 + \ldots + X_n$ is a random variable with mean 0 and variance 1 and independent summands. What you may need here is an upper bound of the form $|\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]|$ where $h : \mathbb{R} \to \mathbb{R}$ is an L-Lipschitz function, (i.e. $|h(x) - h(y)| \leq L \cdot |x - y|$ for all $x, y \in \mathbb{R}$). In fact, the Berry Esseen Theorem in this setting is still true (constant 1 suffices, I believe), i.e.

$$|\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]| \leq L \sum_{i=1}^n \mathbb{E}[|X_i|^3]$$

You may use this fact without a proof.

(v) Prove the so called $\frac{2}{\pi}$-Theorem: For any boolean function $f : \{-1,1\}^n \to \{-1,1\}$ with $\mathrm{Inf}_i[f] \leq \varepsilon$ for all $i \in [n]$, one has $W^1[f] \leq \frac{2}{\pi} + O(\varepsilon)$.

Remark: This exercise originates from Avishay Tal's 2023 course on Analysis of Boolean functions, held at UC Berkeley. The original result is due to Talagrand (1996).