Lecturer: Thomas Rothvoss Due date: Friday, Nov 14, 2025, 11pm, on GradeScope

Problem Set 7

Math 581A - Analysis of Boolean Functions

Fall 2025

Exercise 7.1 (6pts)
Let $1 \leq k \leq n$ and consider the subcube $A := \{x \in \{\pm 1\}^n : x_1 = \ldots = x_k = 1\}$. Let $\alpha := \frac{|A|}{2^n}$ and $0 \leq \rho \leq 1$. Prove that $\Pr_{x \sim A, y \sim N_\rho(x)}[y \in A] = \alpha^{\log_2(\frac{2}{1+\rho})}$.

Exercise 7.2 (14pts)
For a function $f : \{\pm 1\}^n \to \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 0}$ we define the *projection/truncation to degree k* as the function $f^{\leq k} : \{\pm 1\}^n \to \mathbb{R}$ with $f^{\leq k} := \sum_{S \subseteq [n] : |S| \leq k} \hat{f}(S) \cdot \chi_S$. Prove the following:

(i) For $q \geq 2$ one has $\|f^{\leq k}\|_{E,q} \leq \sqrt{q-1}^k \|f\|_{E,q}$.

(ii) For $1 < q \leq 2$ one has $\|f^{\leq k}\|_{E,q} \leq (\frac{1}{\sqrt{q-1}})^k \|f\|_{E,q}$.
Hint. First prove that $\|f^{\leq k}\|_{E,2} \leq \left(\frac{1}{\sqrt{q-1}}\right)^k \cdot \|T_{\sqrt{q-1}} f^{\leq k}\|_{E,2}$.

Remark. Let us recall a few general facts that might be useful here. For any function $g : \{\pm 1\}^n \to \mathbb{R}$ and any $1 \leq p \leq q$ one has $\|g\|_{E,p} \leq \|g\|_{E,q}$. Also the projection has the property that $\|g^{\leq k}\|_{E,2} \leq \|g\|_{E,2}$. Finally, it might be worth reading (or waiting for) Chapter 5.8 which we will cover on Monday, Nov 10.