Lecturer: Thomas Rothvoss Due date: Friday, Nov 7, 2025, 11pm, on GradeScope

Problem Set 6

Math 581A - Analysis of Boolean Functions

Fall 2025

Exercise 6.1 (10pts)
Define the functions $f,g:\{-1,1\}^n \to \mathbb{R}$ with

$$g(x) := \frac{1}{\sqrt{n}}\sum_{i=1}^n x_i \quad \text{and} \quad f(x) := \begin{cases} 1 & \text{if } g(x) > 1 \\ g(x) & \text{if } -1 \leq g(x) \leq 1 \\ -1 & \text{if } g(x) < -1 \end{cases}$$

In other words, f is the truncation of g to the interval $[-1,1]$. Prove that $\mathrm{Var}[f] \geq \Omega(1)$ and $\mathrm{Inf}_i[f] \leq O(\frac{1}{n})$ for each coordinate $i \in [n]$.
Remark. This proves that one cannot extend the KKL Theorem from boolean function $f:\{\pm 1\}^n \to \{\pm 1\}$ to bounded functions $f:\{\pm 1\}^n \to [-1,1]$. The exercise is from O'Donnell's book.

Exercise 6.2 (10pts)
The goal of this exercise is to prove *Chang's Lemma* (also called *Level-1 Inequality*) which states that for a function $f:\{\pm 1\}^n \to \{0,1\}$ with $\alpha := \mathbb{E}_{x\sim\{-1,1\}^n}[f(x)]$ one has

$$\sum_{i=1}^n \hat{f}(\{i\})^2 \leq 2\log_2\left(\frac{1}{\alpha}\right)\cdot \alpha^2$$

For the proof, let $A \subseteq \{\pm 1\}^n$ be the set with $f = \mathbf{1}_A$. Draw $Y \sim A$, i.e. Y is a uniform random point in A. Abbreviate $p_i := \Pr[Y_i = +1] \in [0,1]$.

(i) Prove that for all $i \in [n]$ one has $\hat{f}(\{i\}) = 2\alpha\cdot(p_i - \frac{1}{2})$.

(ii) Prove that $H(Y) \leq \sum_{i=1}^n (1 - 2\cdot(p_i - \frac{1}{2})^2)$.

iii) Use (i) and (ii) to prove that indeed $\sum_{i=1}^n \hat{f}(\{i\})^2 \leq 2\log_2\left(\frac{1}{\alpha}\right)\cdot\alpha^2$.

Hint. This exercise is inspired by a paper of Impagliazzo, Moore and Russell. The main tool to be used here is *entropy*. Recall that for a (discrete) random variable $X \in \Omega$, the entropy is defined as

$$H(X) := \sum_{x\in\Omega} \Pr[X = x]\cdot\log_2\left(\frac{1}{\Pr[X=x]}\right)$$

Entropy has a lot of useful properties. We mention a few here:

(I) If X is a *uniform random variable* — i.e. $\Pr[X = x] = \frac{1}{|\Omega|}$ for all $x \in \Omega$ — then the entropy is $H(X) = \log_2(|\Omega|)$.

(II) Entropy is *subadditive*. In particular if X is an n-dimensional random vector then $H(X) \leq \sum_{i=1}^{n} H(X_i)$ where $X_1, \dots, X_n$ are the coordinates of X.

(III) If X attains only two values, one with probability p and the other one with probability $1-p$, then $H(X) = h(p)$ where

$$h(p) := p \cdot \log_2\left(\frac{1}{p}\right) + (1-p) \cdot \log_2\left(\frac{1}{1-p}\right)$$

is the *binary entropy function*. An estimate that is often useful is that

$$h(p) \leq 1 - 2 \cdot \left(p - \frac{1}{2}\right)^2 \quad \forall 0 \leq p \leq 1$$