Lecturer: Thomas Rothvoss Due date: Friday, Oct 31, 2025, 11pm, on GradeScope

Problem Set 5

Math 581A - Analysis of Boolean Functions

Fall 2025

Exercise (20pts)
The distribution NAE_1 is the uniform distribution over the 6 triples

$$\{(1,1,-1),(1,-1,1),(-1,1,1),(-1,-1,1)(-1,1,-1),(1,-1,-1)\} = \{-1,1\}^3 \setminus \{(1,1,1),(-1,-1,-1)\}$$

Here NAE stands for *not-all-equal*. More generally, for $n \in \mathbb{N}$, the distribution NAE_n provides triples (x,y,z) with $x,y,z \in \{-1,1\}^n$ where $(x_i,y_i,z_i) \sim NAE_1$ independently for all $i = 1,\dots,n$. Consider the following so-called not-all-equal test:

NAE TEST
Input: Function $f : \{-1,1\}^n \to \{-1,1\}$
(1) Draw $(x,y,z) \sim NAE_n$
(2) Reject if $f(x) = f(y) = f(z)$. Accept otherwise.

Solve the following:

(i) Prove that if f is a dictatorship function then $\Pr[\text{NAE test accepts } f] = 1$.

(ii) Prove that for any function $f : \{-1,1\}^n \to \mathbb{R}$ one has

$$\mathop{\mathbb{E}}_{(x,y,z)\sim NAE_n}[f(x)\cdot f(y)] = \text{Stab}_{-1/3}[f]$$

(iii) Prove that for any function $f : \{-1,1\}^n \to \mathbb{R}$ one has $\Pr[\text{NAE test accepts } f] = \frac{3}{4} - \frac{3}{4}\text{Stab}_{-1/3}[f]$.
Hint. Define the function $T : \{-1,1\}^3 \to \{0,1\}$ with

$$T(a_1,a_2,a_3) = \begin{cases} 0 & \text{if } a_1 = a_2 = a_3 \\ 1 & \text{otherwise} \end{cases} = \frac{3}{4} - \frac{1}{4}a_1a_2 - \frac{1}{4}a_1a_3 - \frac{1}{4}a_2a_3$$

Then observe that $\Pr[\text{NAE test accepts } f] = \mathbb{E}_{(x,y,z)\sim NAE_n}[T(f(x),f(y),f(z))]$.

(iv) Let $0 \leq \varepsilon \leq 1$. Prove that if $f : \{-1,1\}^n \to \{-1,1\}$ passes the NAE test with probability $1-\varepsilon$, then $W^1[f] \geq 1 - \frac{9}{2}\varepsilon$.
Hint. Use (iii) and write the acceptance probability in terms of the weights $W^k[f]$.
Comment. Recall that $W^1[f] = \sum_{i=1}^n \hat{f}(\{i\})^2$ is the *level-1 weight* of the function and more generally $W^1[f] = \sum_{|S|=k} \hat{f}(S)^2$ is the *level-k weight*. You will reveive full points for any inequality of the form $W^1[f] \geq 1 - \text{constant}\cdot\varepsilon$.

(v) Suppose that $f : \{-1,1\}^n \to \{-1,1\}$ passes the NAE test with probability $1-\varepsilon$. Prove that there is an index $i \in [n]$ and a sign $\sigma \in \{-1,1\}$ so that $\text{dist}(f, \sigma\cdot\chi_{\{i\}}) \leq C\varepsilon$ where $C > 0$ is a universal constant.
Hint: Use the FKN Theorem.

(vi) Recall that the BLR Linearity test accepts a function f if $f(x \odot y) \cdot f(x) \cdot f(y) = 1$ for $x, y \sim \{-1,1\}^n$ uniformly. Let $f : \{-1,1\}^n \to \{-1,1\}$ be a function so that

$$\Pr[\text{NAE test accepts } f \wedge \text{BLR test accepts } f] \geq 1 - \varepsilon$$

Prove that f is close to a dictatorship function, i.e. there is an index $i \in [n]$ so that $\text{dist}(f, \chi_{\{i\}}) \leq C'\varepsilon$ for some universal constant $C' > 0$.

Comment: Recall that a dictatorship function passes both the BLR test and the NAE test with probability 1. In particular combining the BLR test and the NAE test gives a 6-query test with completeness 1 to check if a function is a dictator function.