Lecturer: Thomas Rothvoss Due date: Friday, Oct 24, 2025, 11pm, on GradeScope

Problem Set 4

# Math 581A - Analysis of Boolean Functions

Fall 2025

**Exercise 4.1 (10 points)**
We write $\mathbf{0}$ as the function on the hypercube that is everywhere 0. In the following, let $f : \{-1,1\}^n \to \mathbb{R}$ be a function of degree $\deg(f) \leq d$.

(i) Suppose that for $a \in \{-1,1\}$ one has[1] $f_{[n-1]|a} = \mathbf{0}$. Prove that $\deg(f_{[n-1]|-a}) \leq d-1$.
**Remark:** The statement means that if one subfunction is all-zero then the other subfunction has lower degree.

(ii) Prove that if $f \neq \mathbf{0}$, then $\Pr_{x \sim \{-1,1\}^n}[f(x) \neq 0] \geq 2^{-d}$.
**Hint:** Prove this by induction over $n$. Use (i).

**Exercise 4.2 (10 points)**
A function $g : \{-1,1\}^n \to \mathbb{R}$ is called a *k-junta* if it depends on at most $k$ variables. More formally, $g$ is a $k$-junta if and only if there is a set $T \subseteq [n]$ of size $|T| \leq k$ so that $g = \sum_{S \subseteq T} \hat{g}(S) \cdot \chi_S$. Now let $f : \{-1,1\}^n \to \{-1,1\}$ be a function with $d := \deg(f)$.

(i) Prove that for each $i \in [n]$ one has either $\mathrm{Inf}_i[f] = 0$ or $\mathrm{Inf}_i[f] \geq \frac{1}{2^d}$.
**Hint.** Use the previous exercise.

(ii) Prove that $f$ is a $d2^d$-junta.

[1] Recall that $f_{[n-1]|a}$ is the restriction of $f$ on the variables $1,\ldots,n-1$ which is the function $(x_1,\ldots,x_{n-1}) \mapsto f(x_1,\ldots,x_{n-1},a)$.