Lecturer: Thomas Rothvoss Due date: Friday, Oct 17, 2025, 11pm, on GradeScope

Problem Set 3

Math 581A - Analysis of Boolean Functions

Fall 2025

Exercise (20pts)

(i) Consider a function $g : \{0,1\}^n \to \mathbb{R}$. Prove that there is a unique choice of values $(a_S)_{S\subseteq[n]} \subseteq \mathbb{R}$ so that

$$g(y) = \sum_{S\subseteq[n]} a_S \prod_{i\in S} y_i \quad \forall y \in \{0,1\}^n$$

Also prove that the choice to make is in fact

$$a_S = \sum_{T\subseteq S} (-1)^{|S\setminus T|} g(\mathbf{1}_T)$$

where $\mathbf{1}_T \in \{0,1\}^n$ is the characteristic vector of set T.

(ii) Prove that for a function $g : \{0,1\}^n \to \{0,1\}$, the values a_S as defined in (i) satisfy $a_S \in \{-2^{|S|}, \ldots, +2^{|S|}\}$.

(iii) For a function $f : \{-1,1\}^n \to \mathbb{R}$, let $g : \{0,1\}^n \to \mathbb{R}$ be the function defined by $g(\frac{x+1}{2}) := f(x)$ for $x \in \{-1,1\}^n$. Again, let a_S be the coefficients so that $g(y) = \sum_{S\subseteq[n]} a_S \prod_{i\in S} y_i$ for all $y \in \{0,1\}^n$. Prove that

$$\hat{f}(S) = \sum_{T\supseteq S} 2^{-|T|} a_T \quad \forall S \subseteq [n]$$

We denote the *real degree* of g by $\mathrm{rdeg}(g) := \max\{|S| : a_S \neq \emptyset\}$ where g is represented as in (i). Combining (i) and (iii) one can derive that $\deg(f) = \mathrm{rdeg}(g)$ — you may use this fact without proving it.

(iv) Let $f : \{-1,1\}^n \to \{0,1\}$ be a function with degree $k := \deg(f)$. Prove that all Fourier coefficients are $\frac{1}{2^k}$-*granular*, i.e. $\hat{f}(S) \in \frac{\mathbb{Z}}{2^k}$ for all $S \subseteq [n]$.

(v) Let $f : \{-1,1\}^n \to \{0,1\}$ be a function with degree $k := \deg(f)$. Prove that $\hat{\|}f\hat{\|}_1 \leq 2^k$.