Lecturer: Thomas Rothvoss   Due date: Friday, Jan 19, 2024, 11pm, on GradeScope

Problem Set 2

CSE 531 - Computational Complexity

Winter 2024

Exercise 1 (not in the A&B book; 10pts)
Recall that in the lecture we have proven that the function $\texttt{HALT}$ given by

$$\texttt{HALT}(x,\alpha) := \begin{cases} 1 & \text{if } M_\alpha \text{ halts on input } x \\ 0 & \text{otherwise} \end{cases}$$

is undecidable (meaning there is no Turing machine M computing $\texttt{HALT}$). Now consider the problem $\texttt{HALT}_{\text{EMPTY}}$ where

$$\texttt{HALT}_{\text{EMPTY}}(\alpha) := \begin{cases} 1 & \text{if } M_\alpha \text{ halts on input of the empty string} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\texttt{HALT}_{\text{EMPTY}}$ is undecidable.

Exercise 2 (slightly modified from Ex 1.11 in A&B book; 10pts)
We say that a function $f : \{0,1\}^* \to (\{0,1\}^* \cup \{\texttt{UNDEF}\})$ is a *partial function* where $f(x) = \texttt{UNDEF}$ means that the function is not defined on input x. We say that a Turing machine M computes the partial function f with

$$f(x) = \begin{cases} M(x) & \text{if } M \text{ halts on } x \\ \texttt{UNDEF} & \text{if } M \text{ does not halt on } x \end{cases}$$

For a set $\mathcal{S}$ of partial functions, we define $f_{\mathcal{S}} : \{0,1\}^* \to \{0,1\}$ by letting

$$f_{\mathcal{S}}(\alpha) := \begin{cases} 1 & \text{if the partial function computed by } M_\alpha \text{ is in } \mathcal{S} \\ 0 & \text{otherwise} \end{cases}$$

We say that such a set $\mathcal{S}$ is *non-trivial* if there are Turing machines M_0 and M_1 so that the partial function computed by M_0 is not in $\mathcal{S}$ and the partial function computed by M_1 is in $\mathcal{S}$. Prove the following:

> <u>Rice's Theorem</u>. For any non-trivial set $\mathcal{S}$ of partial functions, there is no Turing machine computing $f_{\mathcal{S}}$.

Hint. By possibly replacing from $\mathcal{S}$ by its complement (which doesn't affect the claim), we may assume that $f_0 \in \mathcal{S}$, where $f_0 : \{0,1\}^* \to \{0,1\}^* \cup \{\texttt{UNDEF}\}$ is the partial function with $f_0(\alpha) = \texttt{UNDEF}$ for all α. Use this to show that an algorithm to compute $f_{\mathcal{S}}$ can compute the function $\texttt{HALT}$ considered in the first homework problem.
Comment. One can interpret this problem as proving that testing *any* kind of output behavior of Turing machines is undecidable.