Lecturer: Thomas Rothvoss Due date: Friday, Nov 1, 2024, 11pm, on GradeScope

Take Home Midterm Exam

CSE 521 - Design and Analysis of Algorithms

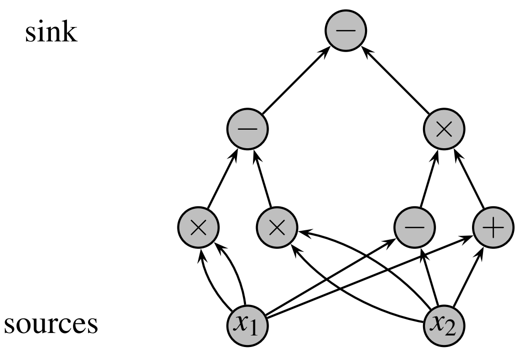
Fall 2024

Exercise 1 (4+6=10pts)
Let $p(x_1,\dots,x_n)$ and $q(x_1,\dots,x_n)$ be two multivariate polynomials.

(i) Prove that $\deg(p+q) \leq \max\{\deg(p),\deg(q)\}$.

(ii) Prove that $\deg(p\cdot q) \leq \deg(p)+\deg(q)$.

Exercise 2 (5+5=10pts)
An *arithmetic circuit* C over $\mathbb{R}$ is a directed acyclic graph where sources are labelled either with variable names $x_1,\dots,x_n$ or constants $c\in\mathbb{R}$. Non-source nodes are labelled with one of the operations $+$, $-$, $\times$ each having fan-in 2[1]. The graph contains a unique sink which given values $x_1,\dots,x_n\in\mathbb{R}$ computes the output $C(x)$ in a natural way. Consider the following example:



arithmetic circuit C

The circuit computes the function $C(x_1,x_2) = (x_1^2-x_2^2)-(x_1-x_2)(x_1+x_2) = 0$ for all $(x_1,x_2)\in\mathbb{R}^2$.

(i) We define the *depth* of a node v as the maximum length of a path (in terms of number of edges) from a source to v and denote it by $\text{depth}(v)$. For example, if v is a source then $\text{depth}(v)=0$. Let $C_v(x_1,\dots,x_n)$ be the function that is computed at a vertex v. Prove that C_v is a polynomial with degree $\deg(C_v)\leq 2^{\text{depth}(v)}$.

(ii) Design a polynomial time randomized algorithm that given an arithmetic circuit C, tests whether $C\equiv 0$. The algorithm should be correct with at least 99%. You may assume that any operation (multiplication, addition etc) with real numbers costs 1 time unit.

[1]The operation "$-$" does not commute. We agree that an order on the incoming edges is specified. In the pictures we will agree that the right input is to be subtracted from the left input.