Lecturer: Thomas Rothvoss Due date: Friday, Dec 6, 2024, 11pm, on GradeScope

Problem Set 8

CSE 521 - Design and Analysis of Algorithms

Fall 2024

Exercise 1 (10pts)
For the *Knapsack problem* we are given n items with weights $w_1,\dots,w_n \geq 0$ and profits $c_1,\dots,c_n \geq 0$. Moreover, we are given a Knapsack capacity $W \geq 0$. The goal is to select a subset $I \subseteq [n]$ of items so that $\sum_{i\in I} w_i \leq W$ and $\sum_{i\in I} c_i$ is maximized. The natural LP relaxation for this problem is

$$\begin{aligned} \max \sum_{i=1}^n c_i x_i & & \\ \sum_{i=1}^n w_i x_i &\leq W & \\ 0 \leq x_i &\leq 1 & \forall i \in [n] \end{aligned}$$

Let $x^* \in [0,1]^n$ be an optimum LP solution of value LP. Prove that there is always a feasible solution $I \subseteq [n]$ with profit $\sum_{i\in I} c_i \geq LP - c_{\max}$ where $c_{\max} := \max_{i=1,\dots,n} c_i$ is the profit of the most profitable item.
Hint: You may use the following fact without proof (this is actually a variant of the theorem you have proven on homework 7). *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m_{\geq 0}$. Then the LP*

$$\max \left\{ c^T x \mid Ax \leq b,\ 0 \leq x_i \leq 1\ \ \forall i = 1,\dots,n \right\}$$

has an optimum solution x^* with $|\{i \in [n] : 0 < x_i^* < 1\}| \leq m$ and such an optimum solution can be found in polynomial time.

Exercise 2 (3+7=10pts)
The *machine scheduling problem* is as follows: we are given a set of jobs J and a set of machines M. Each job needs to be assigned to exactly one machine. If job $j \in J$ is assigned to $i \in M$ then it will take p_{ij} time units to complete. Our goal is to find an assignment $\sigma : J \to M$ assigning jobs to machines that minimizes the *maximum load* of any machine. Here $\sum_{j:\sigma(j)=i} p_{ij}$ is the load on machine i (under the assignment σ).

(i) A natural LP relaxation is

$$\begin{aligned} \min T & & \\ \sum_{j\in J} x_{ij} p_{ij} &\leq T & \forall i \in M \\ \sum_{i\in M} x_{ij} &= 1 & \forall j \in J \\ 0 \leq x_{ij} &\leq 1 & \forall j \in J\ \forall i \in M \end{aligned}$$

Give an instance with a single job (i.e. $|J| = 1$) where the LP has a value of 1 while any feasible solution has a load of $|M|$ on some machine.

(ii) Now suppose we know the optimum objective function value T (which we can obtain by binary search or trying out). Consider the linear system (now without an objective function)

$$\begin{aligned}
\sum_{j\in J} x_{ij}p_{ij} &\leq T \quad \forall i \in M \\
\sum_{i\in M} x_{ij} &= 1 \quad \forall j \in J \\
x_{ij} &= 0 \quad \forall i,j \text{ where } p_{ij} > T \\
0 \leq x_{ij} &\leq 1 \quad \forall i \in M \,\forall j \in J
\end{aligned}$$

Describe a (natural) randomized rounding algorithm and prove that with high probability it generates an assignment σ that has a load of at most $O(\log(|M|)) \cdot T$ on any machine (say $|M| \geq 2$). **Hint:** You may use the following Chernov-type bound without proof: *Let $X_1,\ldots,X_n$ be independent random variables with $0 \leq X_i \leq 1$ and let $X := X_1 + \ldots + X_n$ be their sum. Then for any $\mu \geq \mathbb{E}[X]$ and any $\delta > 0$ one has* $\Pr[X > (1+\delta)\mu] \leq 2\exp(-\frac{1}{10}\min\{\delta,\delta^2\}\mu)$.